A. T. CHAHLY

DESCRIPTIVE GEOMETRY

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AUTHOR'S PREFACE

Descriptive Geometry and Engineering Drawing are among the first engineering subjects taught at institutions of higher education. These subjects are essential if the students are to obtain a fundamental knowledge of the design and construction of machines and mechanisms.

Descriptive geometry, as a specific field of knowledge, like many others, arose and developed as a direct result of man's creative labour.

The remains of fortresses, palaces and other structures found in India, Egypt, Greece, and other countries where there existed ancient culture, show that they were built in accordance with representations resembling modern drawings and conforming to the laws developed by descriptive geometry. A statue discovered during the excavations of ancient Babylon shows a man holding on his knees a slab with a structural drawing on it. The representations of plans and parts of building have also been found on the walls of Egyptian pyramids.

In our times Russian scientists and engineers have played an important role in the development of descriptive geometry and engineering drawing. Descriptive geometry in Russia was taught as early as 1810 in the St. Petersburg Institute of the Road Engineering Corps, now called the Leningrad Institute of Transport Engineering. Russian scientists have written a number of books on descriptive geometry, including such comprehensive courses as *Fundamentals of Descriptive Geometry* by Prof. I. A. Sevostyanov, *Descriptive Geometry* by Prof. P. A. Galaktionov, as well as the textbooks *A Course of Descriptive Geometry, On Curved Lines and Curved Surfaces, Manual of Engineering Drawing* by Prof. A. C. Reder, the books *Complete Course of Descriptive Geometry* and *Course of Linear Perspective Drawing* by Prof. N. N. Makarov. Many other books were written by various authors during the first half of the 19th century.

The Soviet period in the history of methods of representation is marked, above all, by broad research work, better organization and improvements in the methods of teaching descriptive geometry, drawing and technical sketching in technical colleges and institutes.

Soviet scientists working in this field have contributed important theoretical works on diverse aspects of descriptive geometry. One such example is the work carried out on what are known as conditional representations by Prof. N. P. Chetveroukhin, which has led to the establishment of an entirely new branch of descriptive geometry. Others are the work on the problem of the precision of graphical constructions by Prof. D. I. Kargin, on the use of axonometric projections in architectural drawings by Prof. A. I. Viksel, on the use of methods of descriptive geometry in physical and chemical research by Academician N. S. Kournakov, Prof. V. N. Formakovsky and Prof. A. E. Anosov and, in mechanics, the work of the lecturers G. A. Ananov, E. A. Radov and many others.

Scientists who will work in the field of descriptive geometry are being trained at many institutes of the Soviet Union, in Moscow, Leningrad, Kiev, Tbilisi and other cities, where special departments of descriptive geometry and engineering graphics have been organized.

PREFACE

The geometrical properties of objects are usually studied with the help of their representations or drawings. By means of drawings the form of objects may be shown, the dimensions and relative positions of their various geometrical elements may be determined.

Descriptive geometry is devoted to the exposition of the theory and practice required for the solution of problems involving space distances and relationships and the techniques of constructing representations on a plane.

The fundamental objectives of descriptive geometry are:

1. The study of methods of constructing drawings of objects or projects.
2. The study of techniques for determining the forms and dimensions of objects with the help of drawings, i. e., the reading of drawings.
3. The study of the methods of solving problems involving forms in space by means of plane-geometry drawings.

Descriptive geometry is the theoretical foundation of engineering drawing which, in turn, is the preparatory graphical course required for studying all general engineering and special subjects.

Descriptive geometry develops the capacity for imagining objects in space which is essential for reading drawings.

Every engineer must have a good knowledge of the techniques used to represent objects by drawings. He must, so to say, be able to think in three dimensions. Creative engineering without this ability is quite impossible.

In this book descriptive geometry is dealt with step by step. Numerous examples are given to illustrate all theoretical aspects outlined and the methods of constructing projections.

In this work the geometrical elements of objects in space are denoted in the following manner:

Points — $A, B, C, \ldots$
Straight lines — $a, b, c, \ldots$
Segments or portions of straight lines — $AB, CD, EF, \ldots$
Planes — $\alpha, \beta, \gamma, \delta, \ldots$

The principal planes of projection:
the horizontal plane — Π_1;
the frontal or vertical plane — Π_2;
the profile plane — Π_3.
The axes of projection or ground lines — Ox, Oy, Oz.
The traces of oblique planes — k, l, m.
The trace-projections of projecting planes — α_1, β_2, $\gamma_3, \ldots$.
New positions of points — $\bar{A}, \bar{B}, \bar{C}, \ldots, \bar{\bar{A}}, \bar{\bar{B}}, \bar{\bar{C}}, \ldots$
Sequence of points — $A', B', \ldots, A'', B'', \ldots$
Coincidence of two elements — $a \equiv b$. $A_1 \equiv B_1, \ldots$
This system of notation is further developed:

1. On the horizontal plane of projection Π_1 the projections of points — $A_1, B_1, C_1, D_1, \ldots, A'_1, B'_1, A''_1, B''_1, \ldots$

The projections of straight lines — $a_1, b_1, c_1, d_1, \ldots$

The projections of portions of lines — A_1B_1, C_1D_1, $E_1F_1, \ldots$, A_1A , $B_1B'_1$, $A_1A''_1, \ldots$

The horizontal trace of an oblique plane — k_1.

The horizontal trace-projection of a plane $\alpha \perp \Pi_1$—α_1.

2. On the frontal plane of projection Π_2 the projections of points — $A_2, B_2, C_2, D_2, \ldots, A'_2, B'_2, A''_2, B''_2, \ldots$

The projections of straight lines — $a_2, b_2, c_2, d_2, \ldots$

The projections of portions of lines — A_2B_2, C_2D_2, $E_2F_2, \ldots$, $A_2A'_2$, $B_2B'_2$, $A'_2A''_2, \ldots$

The frontal trace of an oblique plane — l_2.

The frontal trace-projection of a plane $\alpha \perp \Pi_2$—α_2.

3. On the profile plane of projection Π_3 the projections of points — $A_3, B_3, C_3, D_3, \ldots, A'_3, B'_3, A''_3, B''_3, \ldots$

The projections of straight lines — $a_3, b_3, c_3, d_3, \ldots$

The projections of portions of lines — A_3B_3, C_3D_3, $E_3F_3, \ldots$, A A'_3, B_3B', $A'_3A''_3, \ldots$

The profile trace of an oblique plane — m_3.

The profile trace-projection of a plane $\alpha \perp \Pi_3$—α_3.

Chapter I

TYPES OF PROJECTIONS

§ 1. Central Projections

In descriptive geometry the construction of plane-geometry drawings, representing three dimensional designs of objects, is based on the method of projections*.

In Fig. 1 points A and S are shown situated in space in front of plane Π**. Through these points a straight line is passed to pierce plane Π in point A_1. This point is called the *central projection* of point A. Point A is termed the *original point*, point S — the *centre of projection*, the plane Π — the *plane of projection*, and the straight line SAA_1 — the *projecting line* or *projector*.

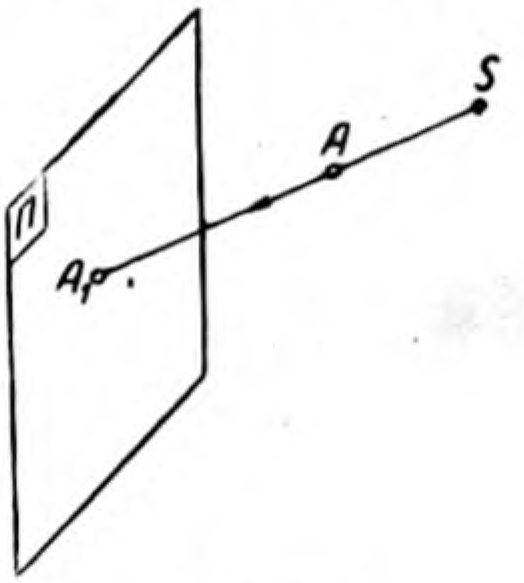

Fig. 1

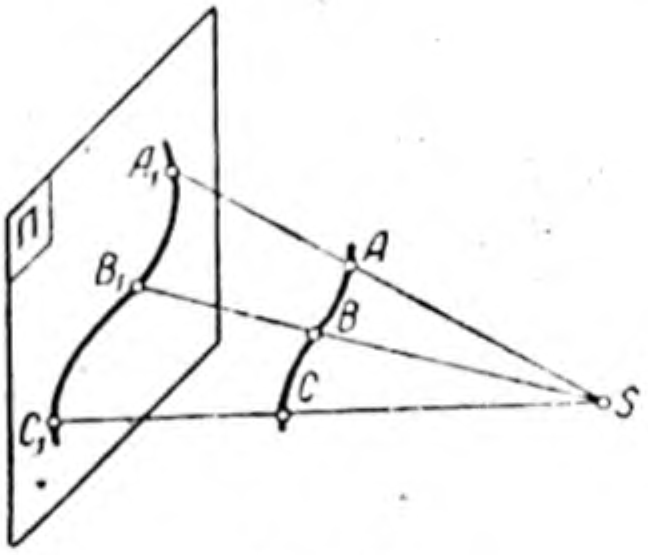

Fig. 2

The centre of projection S may be also called the *pole* and this type of projection — *polar projection*. A curve ABC and its central projection $A_1B_1C_1$ on plane Π are shown in Fig. 2. The bundle of rays originating at the centre S and projecting the curve ABC forms *a projecting cone*. Central projection may also be called *conical projection*. The curve $A_1B_1C_1$ is the intersection of the projecting cone with the plane of projection. It is necessary to understand the concept of central projection, although it is not dealt with in detail in this book.

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* The word "projection" comes from the Latin *projicere* — to hurl or throw.

** П is the first letter of the Russian word плоскость meaning plane.

§ 2. Parallel Projections

If the centre of projections S is considered to lie at infinity, it follows that all the projecting rays are parallel. In order to draw these rays it is necessary to know the direction of projection. The representations thus obtained are called *parallel projections.*

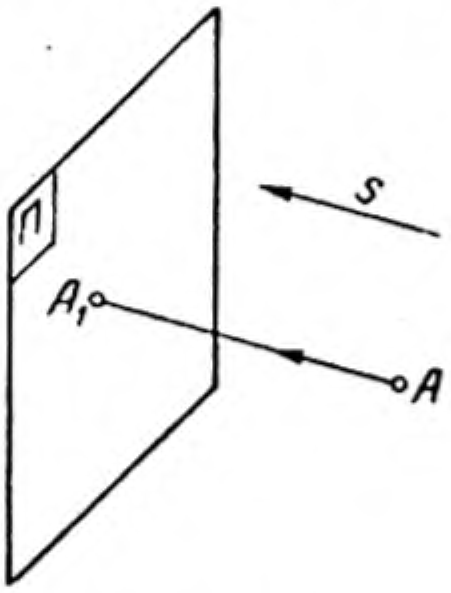


Fig. 3

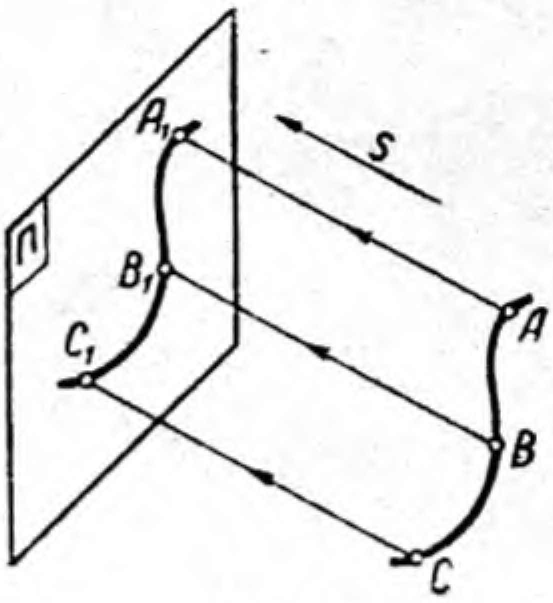


Fig. 4

Fig 3. shows the construction of the parallel projection A_1 of point A on the plane of projection Π. The lines to which the projecting ray AA_1 is parallel gives the *direction of projection.*

The parallel rays which project the points of curve ABC (Fig. 4) form a cylindrical surface, that is why parallel projections are also called *cylindrical projections.*

The given direction of projection may form an oblique angle with the plane of projection or it may be perpendicular to the plane.

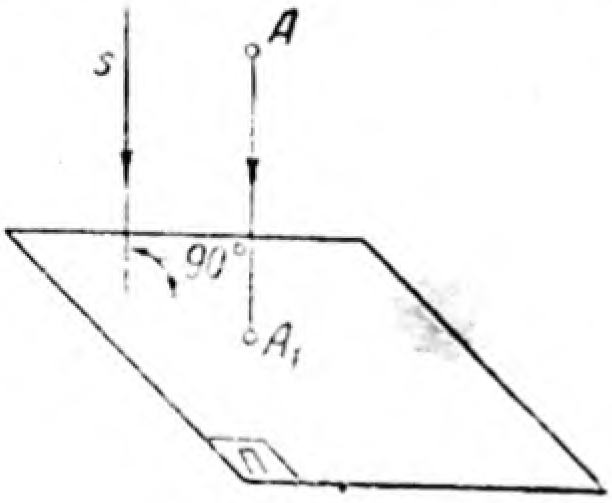


Fig. 5

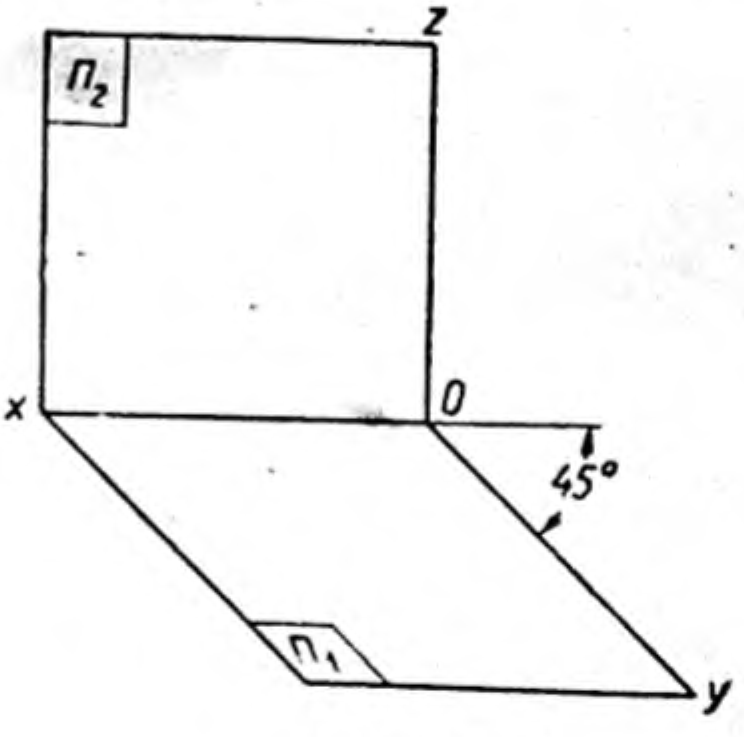


Fig. 6

In the first case the parallel projections are called *oblique projections* (Fig. 4), in the second case *right-angle projections* (Fig. 5).

An example of oblique parallel projection is the representation of a right dihedral or quadrant formed by the *horizontal* plane (Π_1) and the *vertical frontal* plane (Π_2) (Fig. 6). This drawing is constructed according to the rules of frontal axonometric projection.*

---

* Axonometric projection, which includes frontal projection, is discussed in Ch. XI.

When representing objects by means of frontal axonometric projection straight lines and their lengths, angles and figures, lying in planes parallel to the frontal plane Π_2, are projected onto that plane Π_2 without distortion. As for plane Π_1, the right angles of the quadrant are seen there as angles of 45° and 135°, and the length of

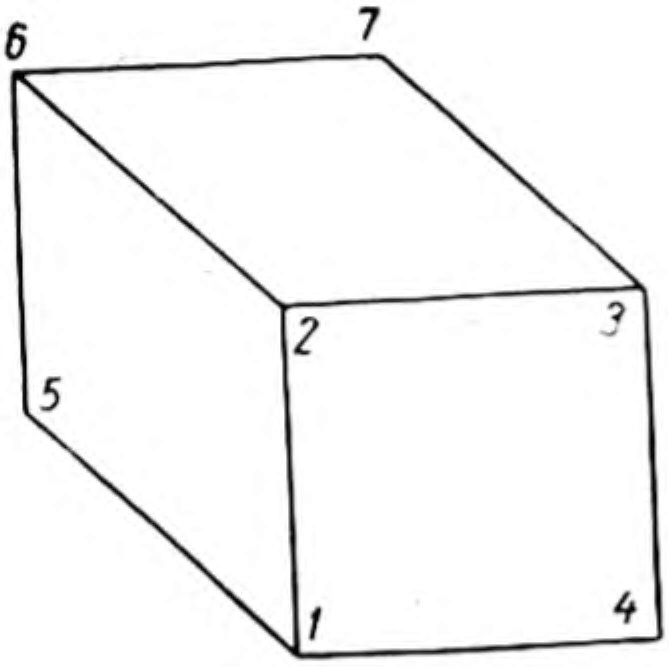


Fig. 7

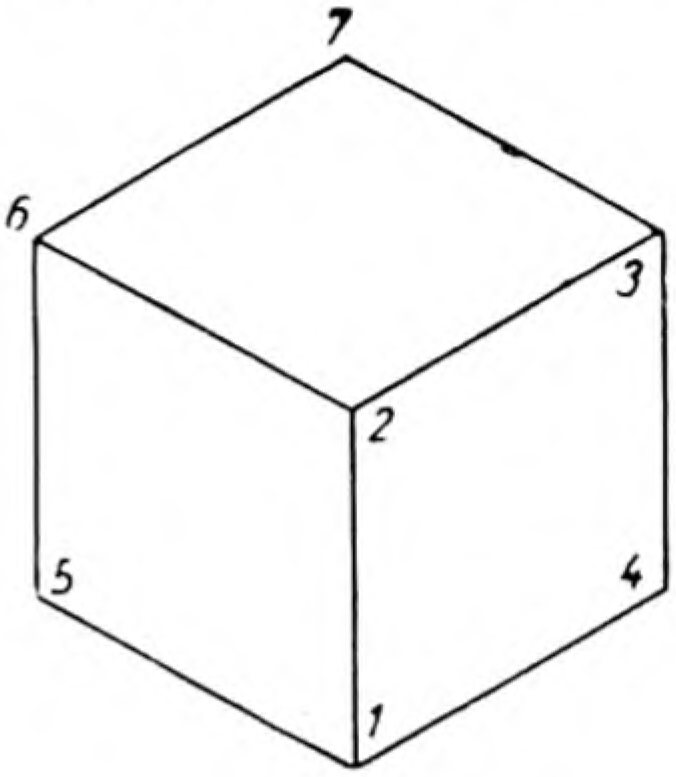


Fig. 8

lines is distorted, unless they are parallel to Ox and Oy. This drawing, Fig. 6, will frequently be referred to in this book.

Various ways of representing a cube by means of parallel projection are shown in Figs. 7, 8 and 9. First, this representation is constructed by means of oblique projection onto a plane parallel to the front face of the cube (Fig. 7), then by means of right-angle projection onto a plane inclined to all the sides of the cube (Fig. 8) and, finally, by means of right-angle projection onto a plane parallel to the front face of the cube (Fig. 9).

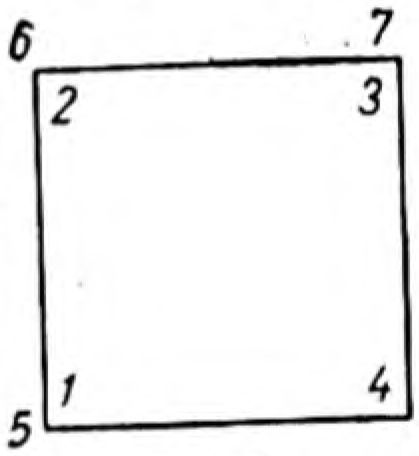


Fig. 9

In engineering practice right-angle projection is most widely used because of its comparative simplicity, the precision of construction it allows, and the possibility of obtaining dimensions without distortion. Right-angle projection is usually termed *orthographic* or *orthogonal projection.*

§ 3. Orthographic Projections

Orthographic projection is illustrated in Fig. 10. The space within the right dihedral or quadrant formed by the horizontal plane Π_1 and frontal plane Π_2 contains an object having all its faces either parallel or perpendicular to the corresponding plane of projection. For instance, the upper face, and other faces parallel to it, are parallel to plane Π_1 and at the same time perpendicular to plane Π_2, and so on.

Let the planes Π_1 and Π_2 be the planes of projection onto which the object is to be projected. Projecting lines are passed through its vertices perpendicular to the planes of projection. Then the points, where the projectors, perpendicular to plane Π_1, pierce that plane, determine the horizontal projection of the object. The points where the projectors perpendicular to plane Π_2, pierce that plane, give the vertical or frontal projection. In Fig. 10 one vertex is indicated by the letter A and its horizontal and frontal projections by A_1 and A_2, respectively.

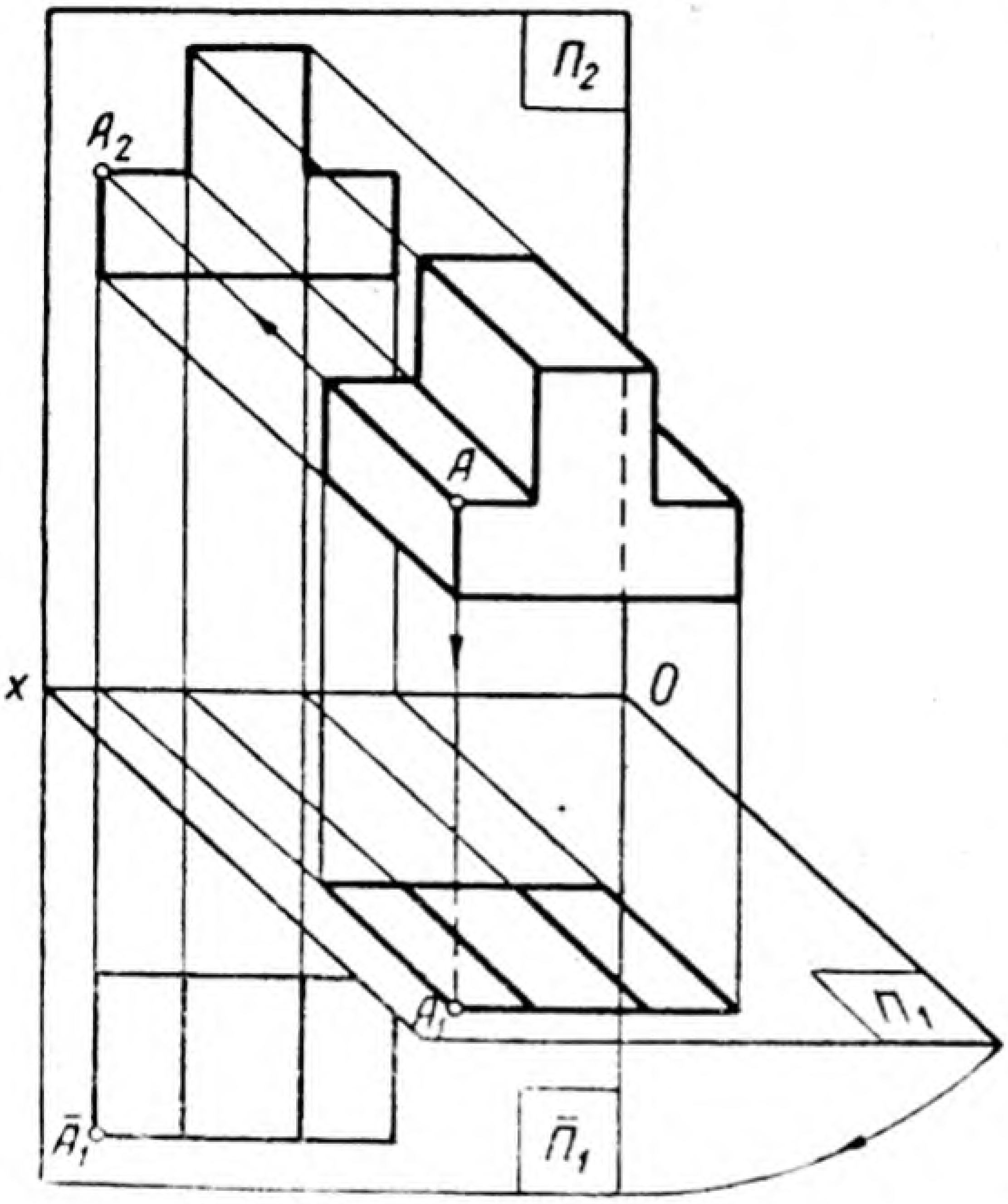


Fig. 10

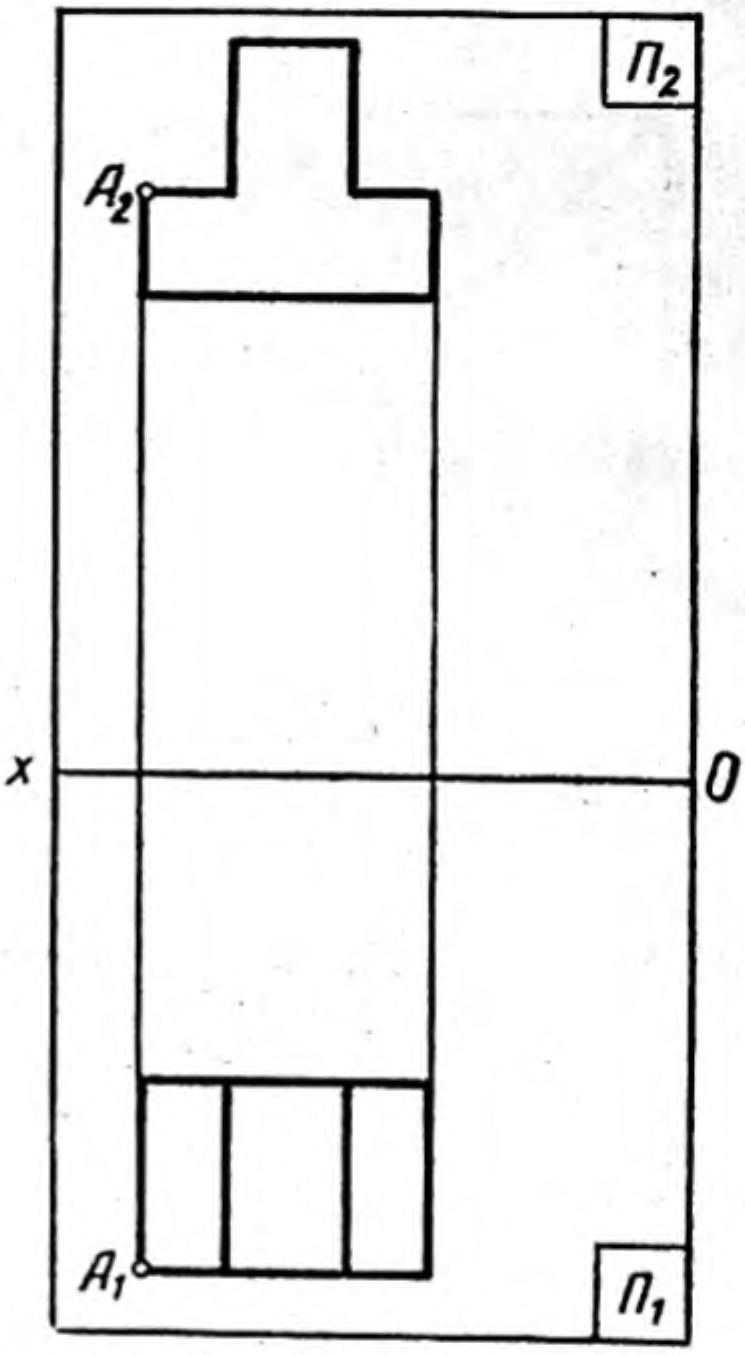


Fig. 11

The projections of the object on planes Π_1 and Π_2 make it possible to visualize the form of the object itself. In order to do this it is necessary to imagine two groups of projectors perpendicular to planes Π_1 and Π_2, respectively, and passing through the vertices of the horizontal and frontal projections of the object. Then, the form of the object and its position within the limits of the dihedral, formed by the planes Π_1 and Π_2, will become apparent as the result of the intersection of the projectors passing through the two projections of the same points. For instance, point A is the point of intersection of the projector passing from point A_1 perpendicular to plane Π_1 with the projector passing from point A_2 to plane Π_2.

From the drawing in Fig. 10 it can be seen that instead of the right angles, actually formed by the intersection of the edges of the object, acute and obtuse angles are obtained on the horizontal projection. To avoid such distortions of the right angles on the drawings the plane of projection Π_1, together with the horizontal projection of the object, is rotated about the straight line of intersection of planes Π_1 and Π_2 — line Ox — until plane Π_1 coincides with plane Π_2. The direction of rotation is indicated by an arrow. This rotation is shown in Fig. 10 as completed. The new position of projection A_1 being $\bar{A}_1$ and of plane Π_1 being $\bar{\Pi}_1$. When the horizontal projection of the object is brought into coincidence with plane Π_2 there are no distortions of the right angles.

In Fig. 11 the planes Π_1 and Π_2 are shown as coinciding, and the orthographic projections of the object in these planes of projection are given. Such a representation of an object by two or more of its orthographic projections is called a *multiview drawing*.

Chapter II

THE POINT

§ 4. The Multiview Drawing of a Point

The following important facts regarding the projections of a point should be carefully noted:

1. One projection alone does not define the position of a point in space. This can readily be understood from Fig. 12 in which the point A and its orthographic projection A_1 on plane Π_1 are shown. On the projector from point A perpendicular to plane Π_1 any number of points B, C, and so on, may be taken, all of which will be projected onto the plane as one and the same point, i. e., $A_1 \equiv B_1 \equiv C_1$. If plane Π_1, with the projection $A_1 \equiv B_1 \equiv C_1$, is brought into coincidence with the plane of the drawing (Fig. 13), then the only thing that can be determined is the distance from the base of the perpendicular passing through point A to the sides of the square that indicates part of the plane of projection Π_1. It is not possible to determine from this drawing (Fig. 13) the altitude of point A in space in relation to plane Π_1.

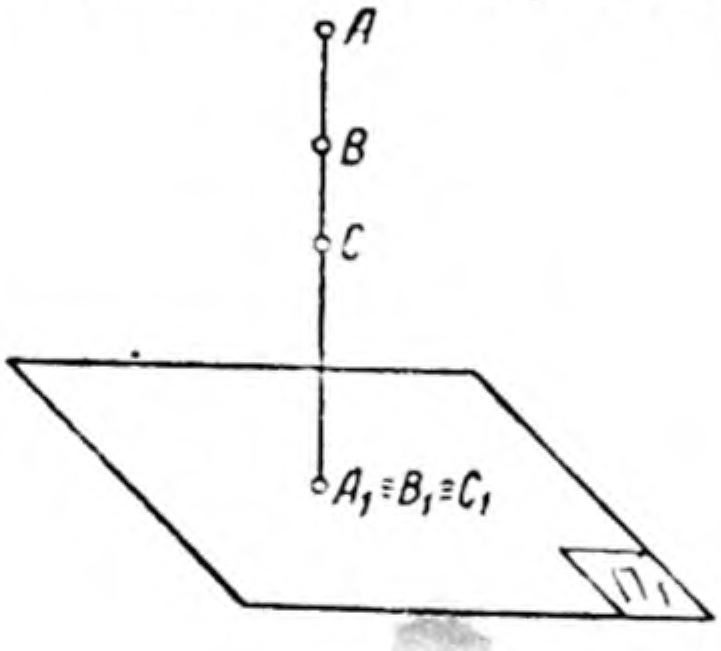


Fig. 12

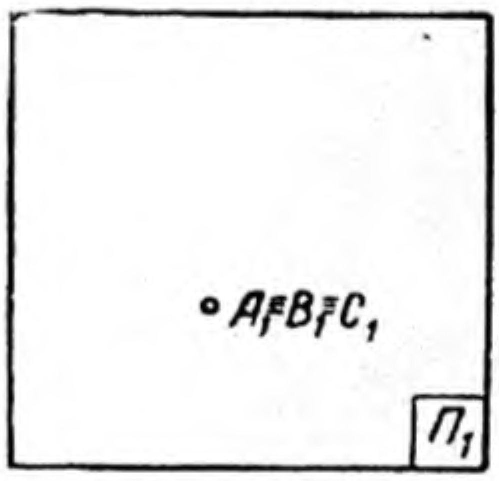
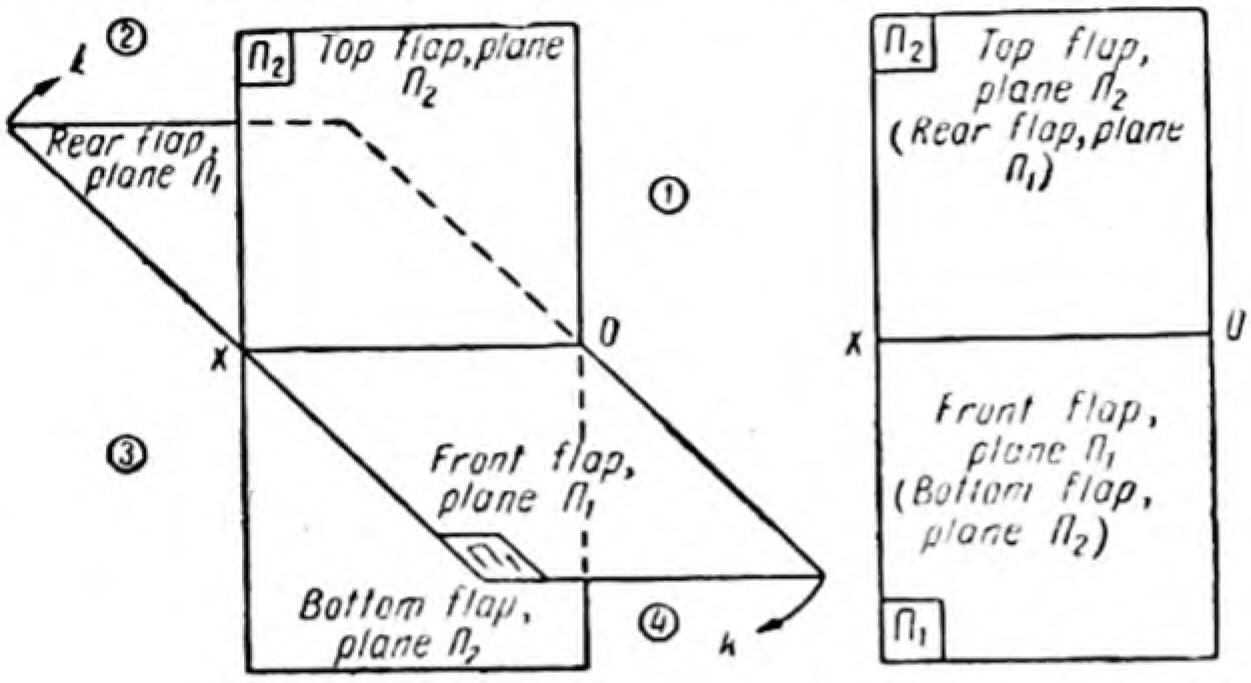


Fig. 13

2. The position of a point in space may be determined if, instead of a single projection, both projections of a point

on two planes of projection which intersect at right angles are constructed. Two such planes of projection Π_1 and Π_2 are shown in Fig. 14, *a*. The line of intersection of the horizontal plane Π_1 and the frontal plane Π_2 — line Ox — is called the *ground* or *reference line*. The ground line Ox divides each of the projection planes into two *flaps, or semi-planes*. The horizontal plane Π_1 is divided into the



a) b)

Fig. 14

front and *rear* flaps and the frontal plane Π_2 into the *top* and *bottom* flaps.

The planes Π_1 and Π_2 divide space into four *quarters* or *quadrants*. The first quadrant is limited by the front flap of plane Π_1 and the top flap of plane Π_2, the second — by the rear flap of plane Π_1 and the top flap of plane Π_2, the third — by the rear flap of plane Π_1 and the bottom flap of plane Π_2 and, finally, the fourth quadrant is limited by the front flap of plane Π_1 and the bottom flap of plane Π_2 (Fig. 14, *a*).

To construct the multiview drawing planes Π_1 and Π_2 are made to coincide with the plane of the drawing. Assuming plane Π_2 to be the plane of the drawing, plane Π_1 is rotated about the ground line Ox so that the front flap of plane Π_1 coincides with the lower flap of plane Π_2. The rotation is indicated by arrow k. The rear flap of plane Π_1 is brought into coincidence with the top flap of plane Π_2 by rotation in the direction of arrow l (Fig. 14, *b*).

3. The orthographic projection of a point onto planes Π_1 and Π_2 is illustrated in Fig. 15.

The given point A is shown in the drawing (Fig. 15, *a*) as lying within the first quadrant. By drawing a perpendicular from point A onto plane Π_1 point A_1 is obtained. This is the horizontal projection of point A. To obtain the projection of point A on plane Π_2 from point A a perpendicular is drawn to plane Π_2. The base A_2 of this perpendicular is the frontal projection of point A.

The plane determined by the projectors AA_1 and AA_2 is perpendicular both to plane Π_1 and plane Π_2. Therefore it is also perpendicular to the ground line Ox — the line of intersection of Π_1 and Π_2. The plane intersects the ground line in a point. Since this point lies in both plane Π_1 and plane Π_2, it is denoted by a double subscript as A_{12}.

The figure $AA_1A_{12}A_2$ is a rectangle. Therefore, if it is required to find the frontal projection A_2 of point A, given point A and its projection A_1, the procedure is as follows:

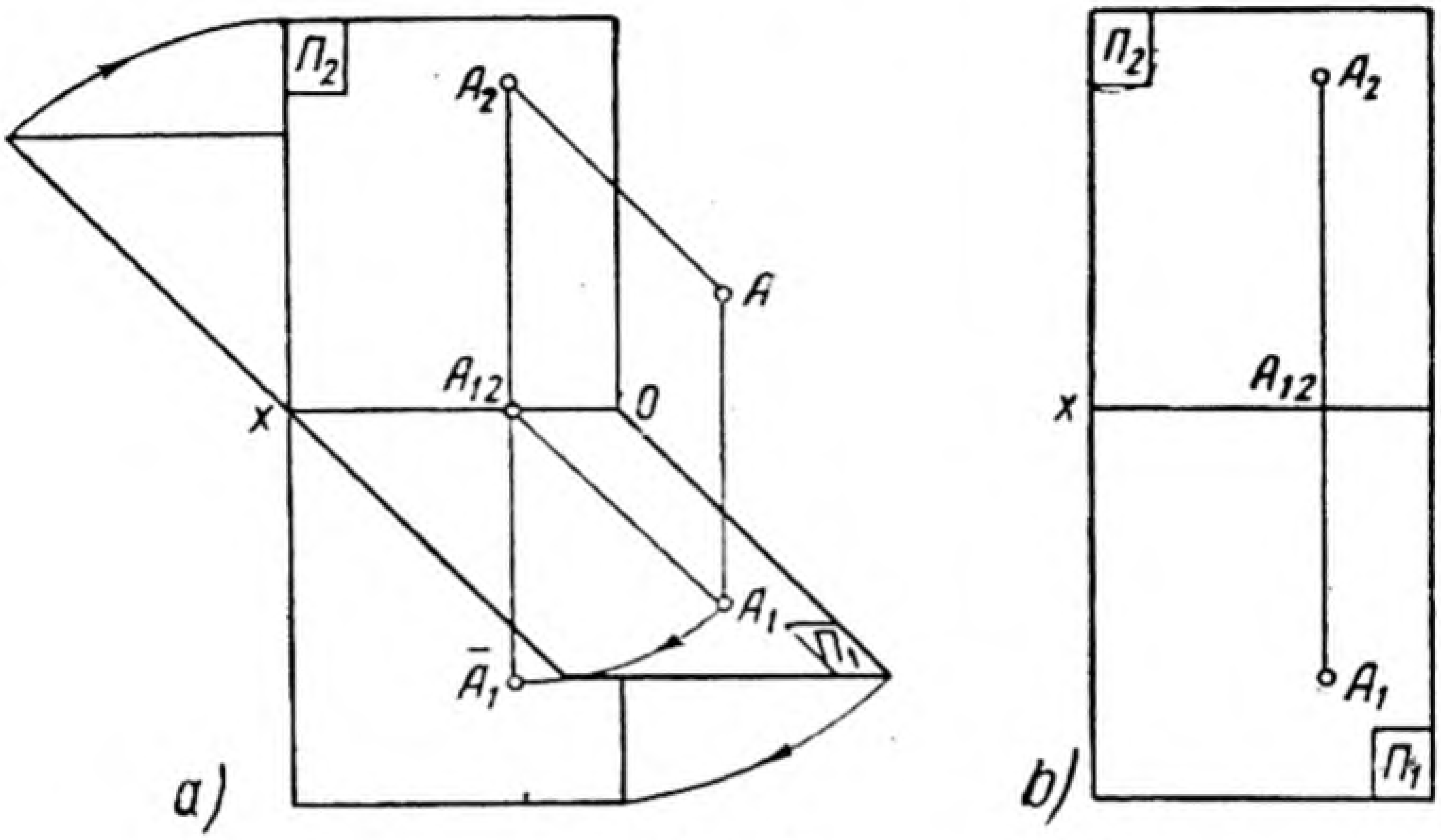


Fig. 15

1. From the horizontal projection A_1 of point A a perpendicular is drawn to the ground line Ox which intersects Ox in point A_{12}.

2. From A_{12} in plane Π_2 we erect a perpendicular to Ox to intersect with projector AA_2 in point A_2 which is the required frontal projection of point A.

To obtain the multiview drawing of point A plane Π_1, containing projection A_1, is made to coincide with plane Π_2 by rotating it about the line Ox. The new position of point A_1 (Fig. 15, *a*) is then point $\bar{A}_1$. The radius of rotation of point A_1 is the length of line A_1A_{12} and the centre of rotation is point A_{12}. Since the rotation of point A_1 takes place in a plane perpendicular to ground line Ox, then, after rotation, point A_1 will lie on the straight line passing through points A_2 and A_{12}.

It follows that in the multiview drawing (Fig. 15, *b*) the two projections A_1 and A_2 of the original point A are located on one and the same line A_1A_2 perpendicular to the ground line Ox. This line is called a *vertical projector*.

Let us take note of the fact that the portions of the straight lines A_1A_{12} and AA_2 are equal, as are A_2A_{12} and AA_1, since they are opposite sides of a rectangle (Fig. 15, *a*). The length of AA_2 determines the distance from point A to the frontal plane Π_2. On the multiview drawing (Fig. 15, *b*) the distance of point A

from plane Π_2 is given by the length of A_1A_{12}, i. e., the distance from the horizontal projection A_1 of the point to the ground line Ox.

The distance of point A from the horizontal plane of projection Π_1 — the altitude of point A — is equal to A_2A_{12} (Fig. 15, *a*). On the multiview drawing (Fig. 15, *b*) the distance of point A from plane Π_1 is given by the length of line A_2A_{12}, i. e., by the distance from the frontal projection A_2 of the point to ground line Ox.

4. In order to read the multiview drawing of point A (Fig. 15, *b*), i. e., to establish the position of the point in space in relation to the planes of projection Π_1 and Π_2, one should imagine that the drawing is folded along the ground line Ox until the planes Π_1 and Π_2 are perpendicular. The lower part of the drawing then occupies the position of the front flap of the plane Π_1, and the top part of the drawing — that of the top flap of the plane Π_2. Then imagine that through points A_1 and A_2 lines, perpendicular to the respective planes Π_1 and Π_2, are drawn. The intersection of these perpendiculars determines the position of point A in space.

§ 5. The Projection of Different Points in Relation to the Planes of Projection

1. Points situated in the *2, 3* and *4* quadrants. A point in a system of two planes of projection Π_1 and Π_2 may lie in one of the four space quadrants, on one of the plane flaps or, finally, on ground line Ox.

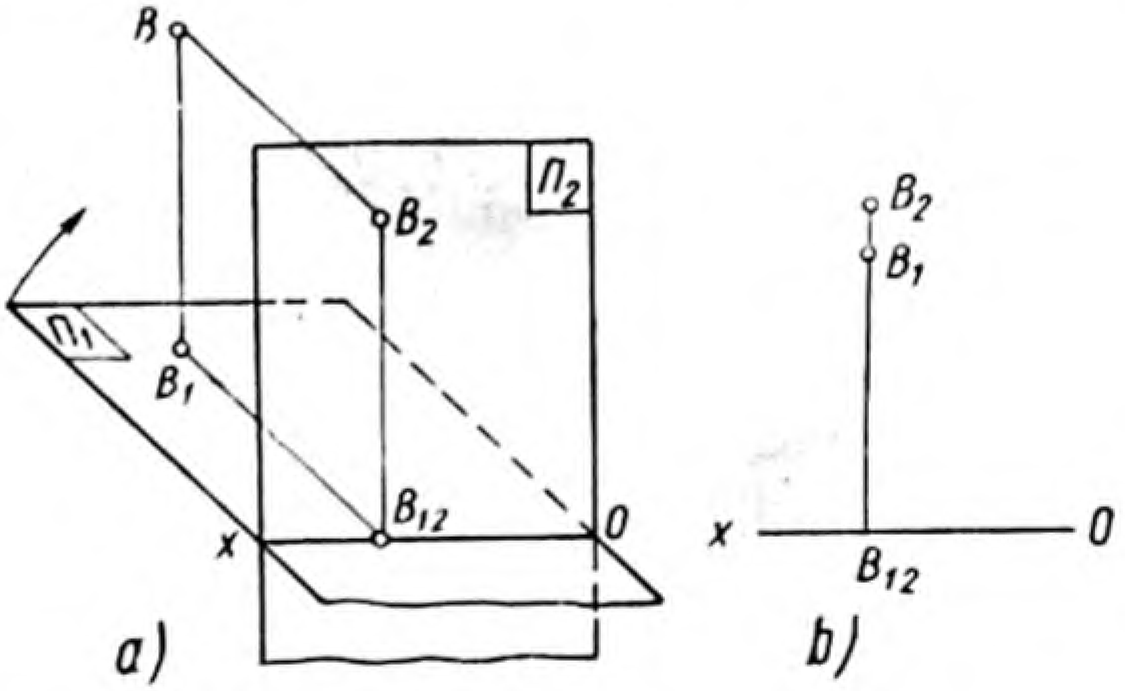


Fig. 16

We have already seen one of the possible positions of a point. Point A was located in the first quadrant (Fig. 15, *a, b*). The distinguishing feature of the multiview drawing of such a point (Fig. 15, *b*) is that its horizontal projection lies under the ground line Ox, while the frontal projection lies above this ground line.

In the case of point B, which is in the second quadrant

(Fig. 16, *a*), both projections on the multiview drawing are located above the ground line Ox (Fig. 16, *b*), because the horizontal projection B_1 of this point moves with the rear flap of plane Π_1 which is made to coincide with the top flap of plane Π_2.

Point C is located in the third space quadrant (Fig. 17, *a*). On the multiview drawing (Fig. 17, *b*) the horizontal projection C_1 of

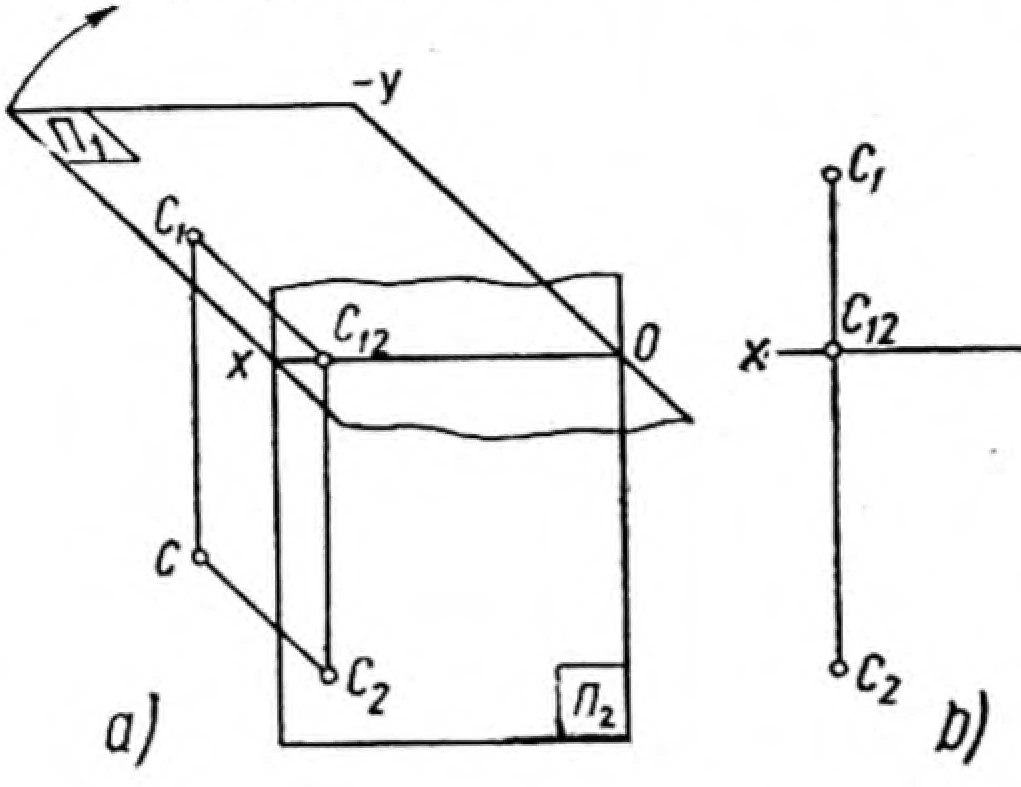


Fig. 17

point C is located above line Ox, since after rotation C_1 will lie in the top flap of plane Π_2, while the frontal projection C_2 lies under ground line Ox.

Point D (Fig. 18, *a*), which is in the fourth space quadrant, has both its projections on the multiview drawing (Fig. 18, *b*) located under the ground line Ox, since the horizontal projection D_1 of point D coincides with the lower flap of plane Π_2.

On the combined multiview drawing (Fig. 19) the projections of all the points A, B, C and D are shown for the purpose of comparison.

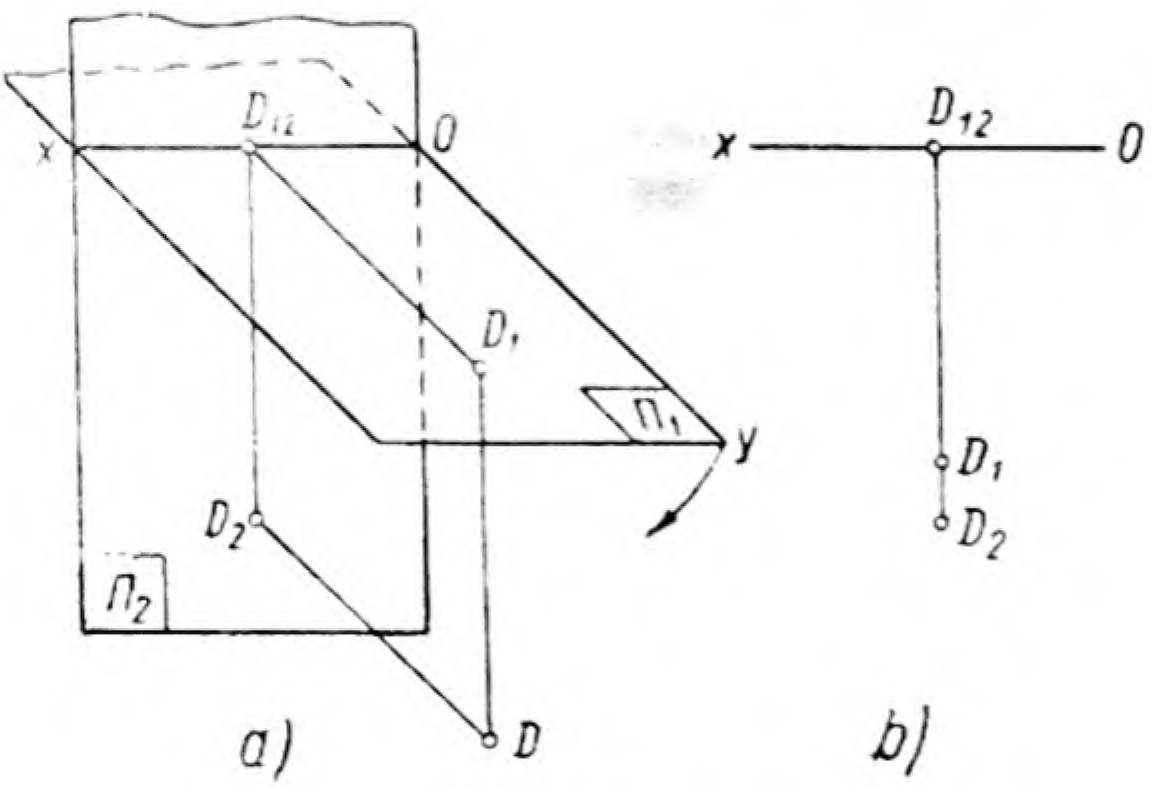


Fig. 18

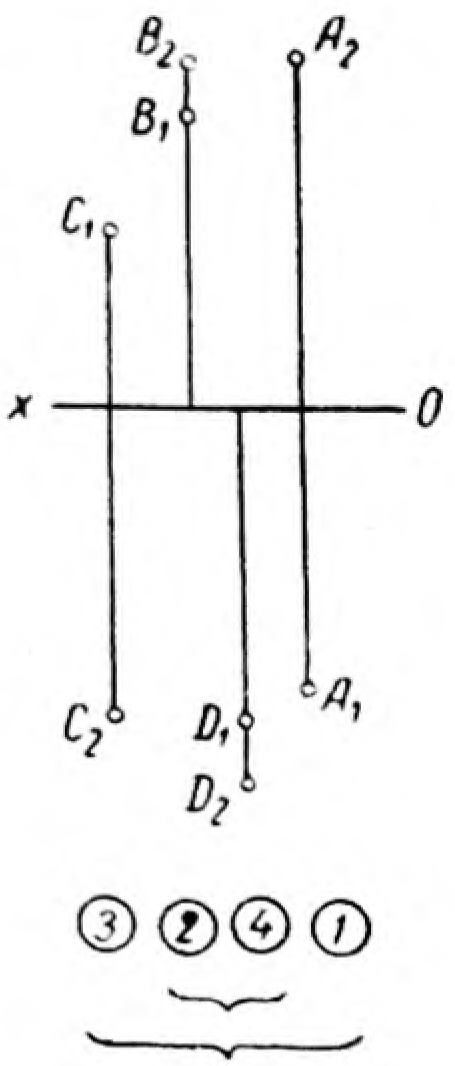


Fig. 19

The horizontal and frontal projections of points *A* and *C* in the odd-number space quadrants (*1* and *3*) are located on opposite sides of ground line *Ox*, where as both projections of each of the points *B* and *D*, in the even-number space quadrants (*2* and *4*), lie on one and the same side of the ground line *Ox*.

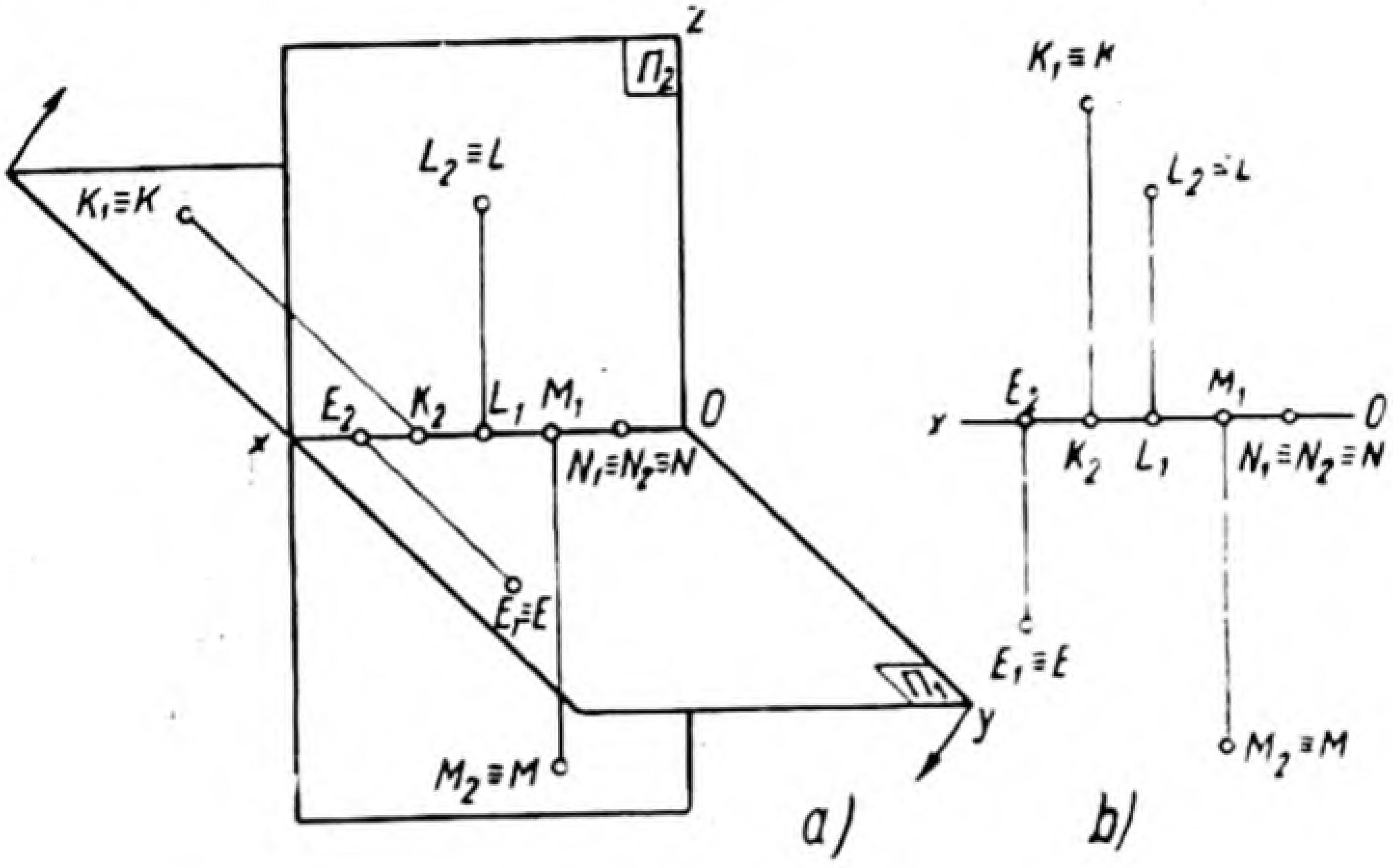


Fig 20

2. The next group of points consists of points, lying in the planes of projection Π_1 and Π_2. When their position in space is being determined it is necessary to take into account that the distance from these points to the planes on which they lie is zero. Therefore one of the two projections of these points on the multiview drawing must lie on ground line *Ox*, whereas the second projection will coincide with the original point itself.

For example, if point *E* (Fig. 20, *a*) lies on the front flap of plane Π_1, then its frontal projection E_2 in the multiview drawing (Fig. 20, *b*) is located on the ground line *Ox*, and the horizontal projection E_1 coincides with point *E* ($E_1 \equiv E$).

In Fig. 20, *a* point *K* lies in the rear flap of plane Π_1, point *L* — in the top flap of plane Π_2, and point *M* in the lower flap of plane Π_2. One of the projections of each of the points *K*, *L* and *M* coincides with the point itself, whereas the second, lies on ground line *Ox*.

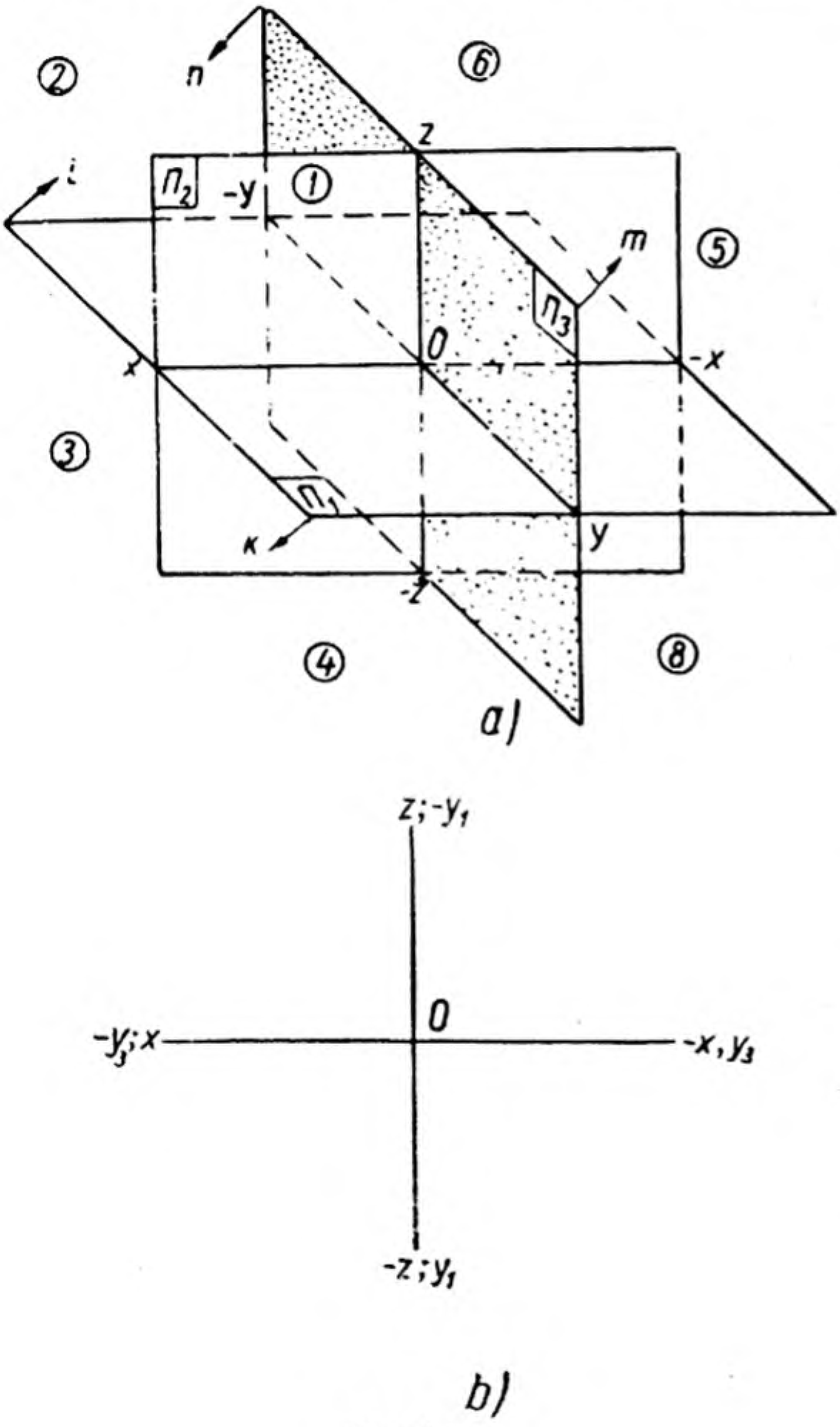
3. Finally, when point *N* is located on ground line *Ox*, both its projections N_1 and N_2 coincide with the original point *N*.

In the multiview drawing Fig. 20, *b* are shown the projections of points *E, K, L, M* and *N*.

From what has been outlined above the following conclusions may be drawn:

1. If the horizontal projection of a point in the multiview drawing lies below the ground line Ox, then the point itself is located in space in front of the frontal plane of projection Π_2. If the projection lies above the ground line, then the point itself lies behind plane Π_2.

2. If the frontal projection of a point lies above the ground line Ox, then the point itself is situated above the horizontal plane of projection Π_1. If this projection lies below the ground line, the point itself lies under this plane.



a)

b)

Fig 21

3. If one of the projections of the point lies on the ground line Ox, then the original point lies in one of the planes of projection Π_1 or Π_2.

4. If both projections of the point coincide and lie on ground line Ox, then the original point also lies on this line.

4. Projections on three planes. Let us now examine the orthographic projection of points on three mutually perpendicular planes of projection, introducing the third plane Π_3 (Fig. 21, *a*).

The plane Π_3 is called the *profile* plane of projection and the projections on it are called *profile* projections. The profile plane of projection Π_3 intersects the planes of projection Π_1 and Π_2, respectively, in the ground lines Oy and Oz, both of which are perpendicular to ground line Ox. From the drawing (Fig. 21, *a*) it may be seen, that $Ox \perp \Pi_3$, $Oy \perp \Pi_2$ and $Oz \perp \Pi_1$.

The planes of projection Π_1, Π_2, Π_3 divide space into eight parts called *octants*. There are two systems of numerating the octants —

right-hand and left-hand. In this text-book the right-hand system is used throughout, since the disposition of the views or projections then corresponds to that accepted in engineering drawing courses. The sequence of numeration using the right-hand system of octants is shown in Fig. 21, *a*. The sequence of the first four octants (*1, 2, 3* and *4*) coincides with the sequence of the four quadrants formed by the intersection of the planes of projection Π_1 and Π_2. The octants *5, 6, 7* and *8* are situated to the right of the plane Π_3 in the same order as the octants *1, 2, 3* and *4*. In Fig. 21, *a* all the octants are numbered except the seventh, situated under plane Π_1 behind plane Π_2 and to the right of plane Π_3.

Each of the ground lines is divided by the planes Π_1, Π_2 and Π_3 into two parts. Generally the letter *O* is used to indicate the point of intersection of the ground lines. The direction of the line *x* to the left of point *O* is assumed to be positive, and to the right of *O* — negative. The direction of the portion of the ground line *Oy* which is towards the observer is taken as positive and that of the opposite portion away from the observer, as negative. Similarly the direction of the upper part of *Oz*, i. e., above point *O*, is taken as positive and that of the lower part, as negative.

The transition from space representation of the three planes of projection to the multiview drawing is carried out as follows. First the horizontal projection plane Π_1 is revolved about the ground line *Ox* until it coincides with the frontal plane of projection Π_2. This rotation is carried out in a clockwise direction, viewing *x* from the left, as shown in Fig. 21, *a* by arrows *k* and *l*. Then the profile plane of projection Π_3 is rotated about ground line *Oz* until it coincides with the frontal plane Π_2. This second rotation is carried out counter-clockwise, the axis *z* viewed from the top (shown by arrows *m* and *n*). Thus we obtain the multiview drawing of the planes of projection Π_1, Π_2 and Π_3 (Fig. 21, *b*). Since the frontal plane of projection Π_2 is taken as the plane of the drawing and remains stationary, the ground lines *Ox* and *Oz* contained in it will not change their positions on the multiview drawing.

When the plane Π_1 coincides with the plane Π_2, the positive direction of the *Oy* line coincides with the negative direction of the *Oz* line, and conversely, the negative direction of the *Oy* line coincides with the positive direction of the *Oz* line.

5. It is often convenient to establish the position of points by giving their coordinates relative to the system of three coordinate planes. If the point *A* is assigned by rectangular coordinates, the planes of projection Π_1, Π_2 and Π_3 serve as the planes of the system of coordinates. The point *O* of intersection of axes *x, y* and *z* is then the point of origin (Fig. 22, *a*). The coordinates of point *A* are three values *x, y* and *z*, when: 1) the absolute magnitudes of *x, y* and *z* are respectively equal to the lengths AA_3, AA_2, AA_1, i. e., the distances of point A from the planes Π_3, Π_2 and Π_1; 2) $x>0$, if A lies to the left of Π_3; $x<0$, if A lies to the right of Π_3; 3) $y>0$, if A lies in front of Π_2; $y<0$, if A lies behind Π_2; 4) $z>0$, if A lies above Π_1; $z<0$, if A lies under Π_1. Knowing the coordinates of a point it is possible to construct the projection of the point and, conversely, by means

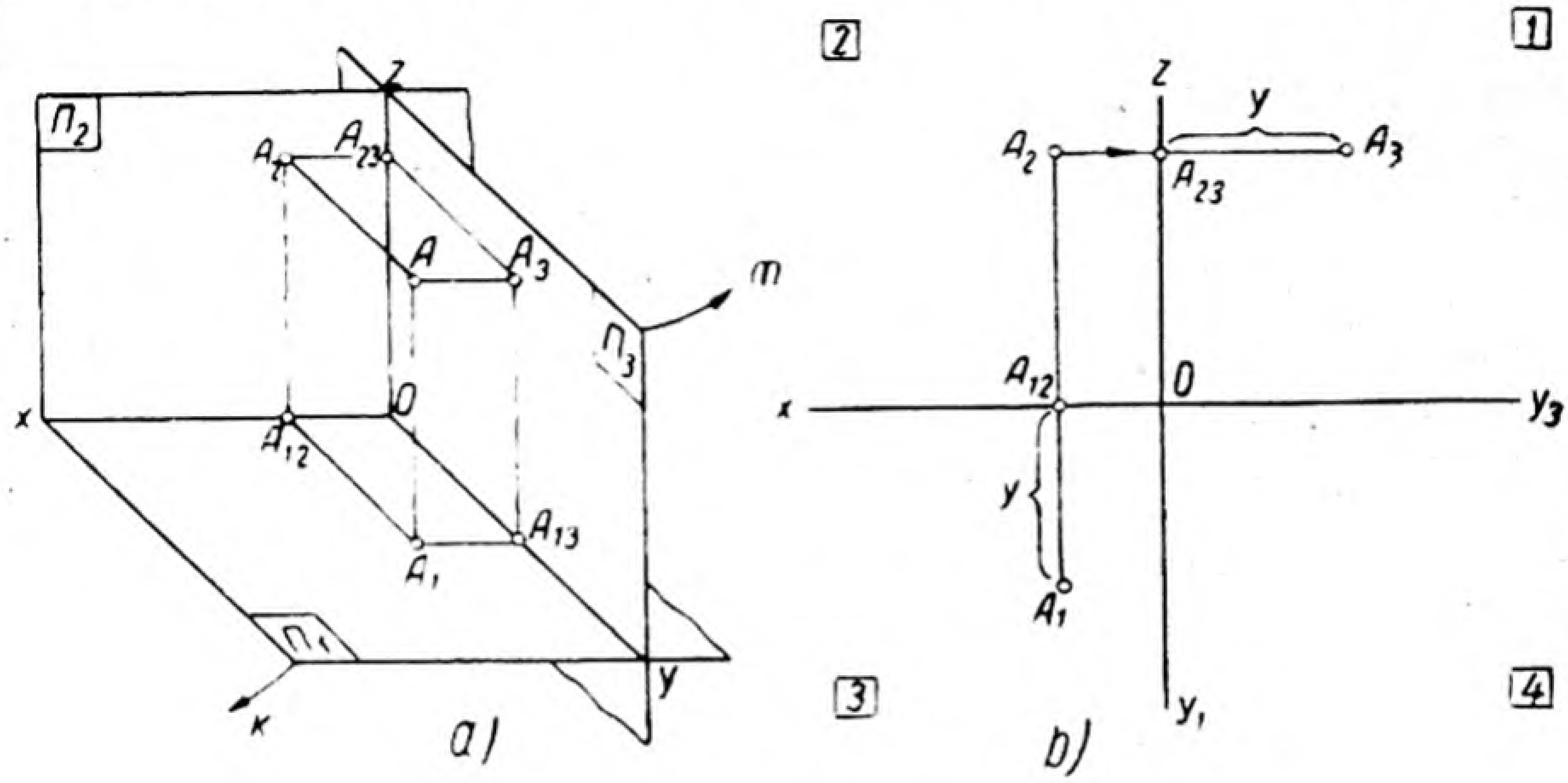


Fig. 22

of the multiview drawing the coordinates of a point may be determined.

The planes AA_1A_2, AA_2A_3, AA_3A_1, Π_1, Π_2 and Π_3 form the rectangular *coordinate parallelepiped.*

The coordinate signs for all the octants are given in the table.

Octants	Coordinates		
	x	y	z
	Signs of the Coordinates		
1	+	+	+
2	+	−	+
3	+	−	−
4	+	+	−
5	−	+	+
6	−	−	+
7	−	−	−
8	−	+	−

6. In Fig. 22, *a* point A is situated in the first octant. Its coordinates x, y, z are respectively equal to 7, 13 and 18 mm. This should be checked from the drawing. The construction of the coordinate parallelepiped of this point consists in marking the points A_{12}, A_{13} and A_{23} on the lines Ox, Oy and Oz. This is done by stepping off a distance of 7 mm from point O on the ground line Ox, a distance of 13 mm on the ground line Oy and a distance of 18 mm on Oz. The other edges of the parallelepiped are drawn parallel to these segments.

It should be noted that any two projections of point A are contained in each of the faces of the coordinate parallelepiped, when this face does not coincide with the plane of projection. For example, the horizontal and frontal projections, A_1 and A_2, lie in the profile face $AA_1A_{12}A_2$. The horizontal and profile projections, A_1 and A_3, lie in the frontal face $AA_1A_{13}A_3$ and, finally, the frontal and profile projections, A_2 and A_3, lie in the horizontal face $AA_2A_{23}A_3$ of the parallelepiped. Therefore, to obtain two projections of a point, given its coordinates x, y and z, it is necessary to construct only the corresponding face of the coordinate parallelepiped. The projections A_1 and A_2 of point A (7; 13; 18) are shown in Fig. 23. First from point O on the line Ox mark off the distance $x=7$ mm ($OA_{12}=7$ mm). After that on a line, drawn from A_{12} parallel to the line Oy, the distance $A_{12}A_1 = 13$ mm is marked off. Finally, from point A_{12} a line is drawn parallel to axis Oz, and the distance $A_{12}A_2 = 18$ mm is marked off on it.

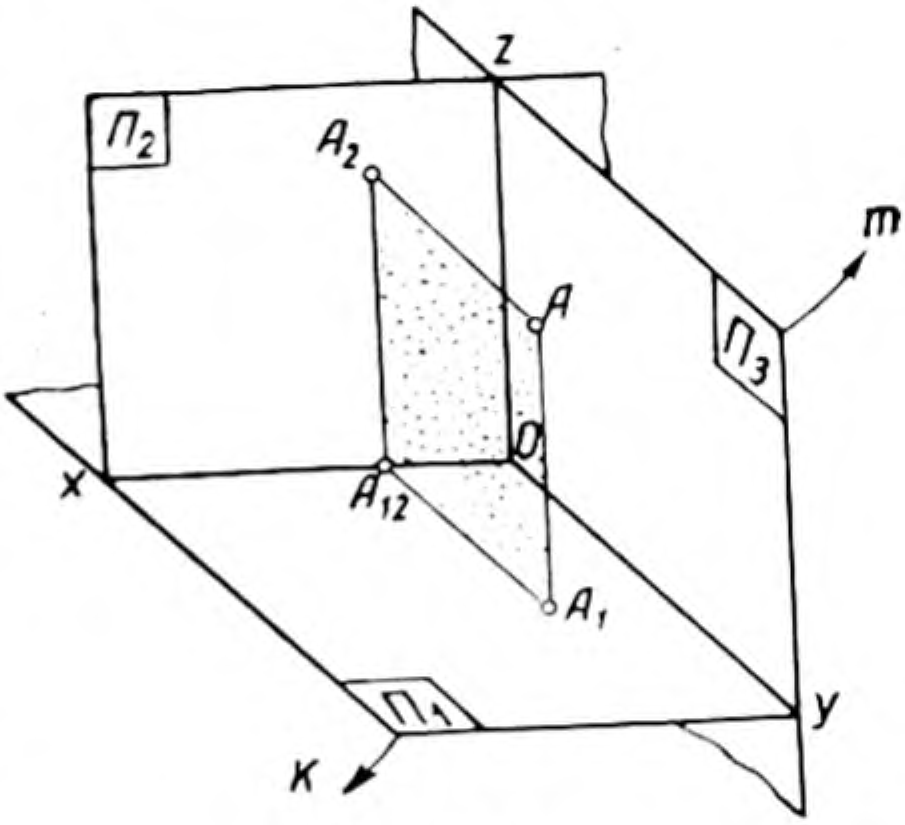


Fig. 23

If it is necessary to construct a point by its assigned coordinates, this may be done by constructing one *coordinate open polygon*. Thus in Fig. 23 point A and its projection A_1 are obtained by drawing the coordinate open polygon $OA_{12}A_1A$.

The construction of the horizontal and frontal projections of point A in the multiview drawing (Fig. 22, *b*) is the same as for any point located in the first space quadrant (Fig. 15, *b*). The construction of the profile projection A_3 of point A is carried out as follows: 1) through point A_2 draw a horizontal projector, which is the line of altitude, and 2) from point A_{23} lay off along this line, to the right of Oz, distance $A_{23}A_3 = 13$ mm ($y=13$).

Point A_3 is the profile projection of point A. This is so because 1) the distance of the profile projection A_3 from the ground line Oy_3, which coincides with the x axis, is the same as the distance from frontal projection A_2 to the x axis, and 2) $A_{23}A_3$ and A_1A_{12} are equal to y, given by the coordinates of point A.

7. Let us now consider the construction of the multiview drawings of points located in the other seven octants. Point B lies in the second octant (Fig. 24). Its projections B_1B_2 and B_3 in Fig. 24, *a* are obtained by constructing the profile face ($BB_1B_{12}B_2$) and the horizontal face ($BB_2B_{23}B_3$) of the coordinate parallelepiped.

In the multiview drawing (Fig. 24, *b*) the two principal projections B_1 and B_2 of point B are located to the left of point O and above ground line Ox. The construction of these projections corresponds to that of points B_1 and B_2 in Fig. 16, *b*.

The profile projection B_3 is located to the left of Oz since, after the plane Π_3 has been revolved about the ground line Oz, the rear half of this plane coincides with the left half of plane Π_2. The construction of the profile projection B_3 of point B is carried out as follows: 1) through the frontal projection B_2 a horizontal projector is drawn parallel to Ox; 2) on this projector, to the left of Oz, lay off the distance y equal to the distance (B_1B_{12}) at which projection B_1 lies from the ground line Ox.

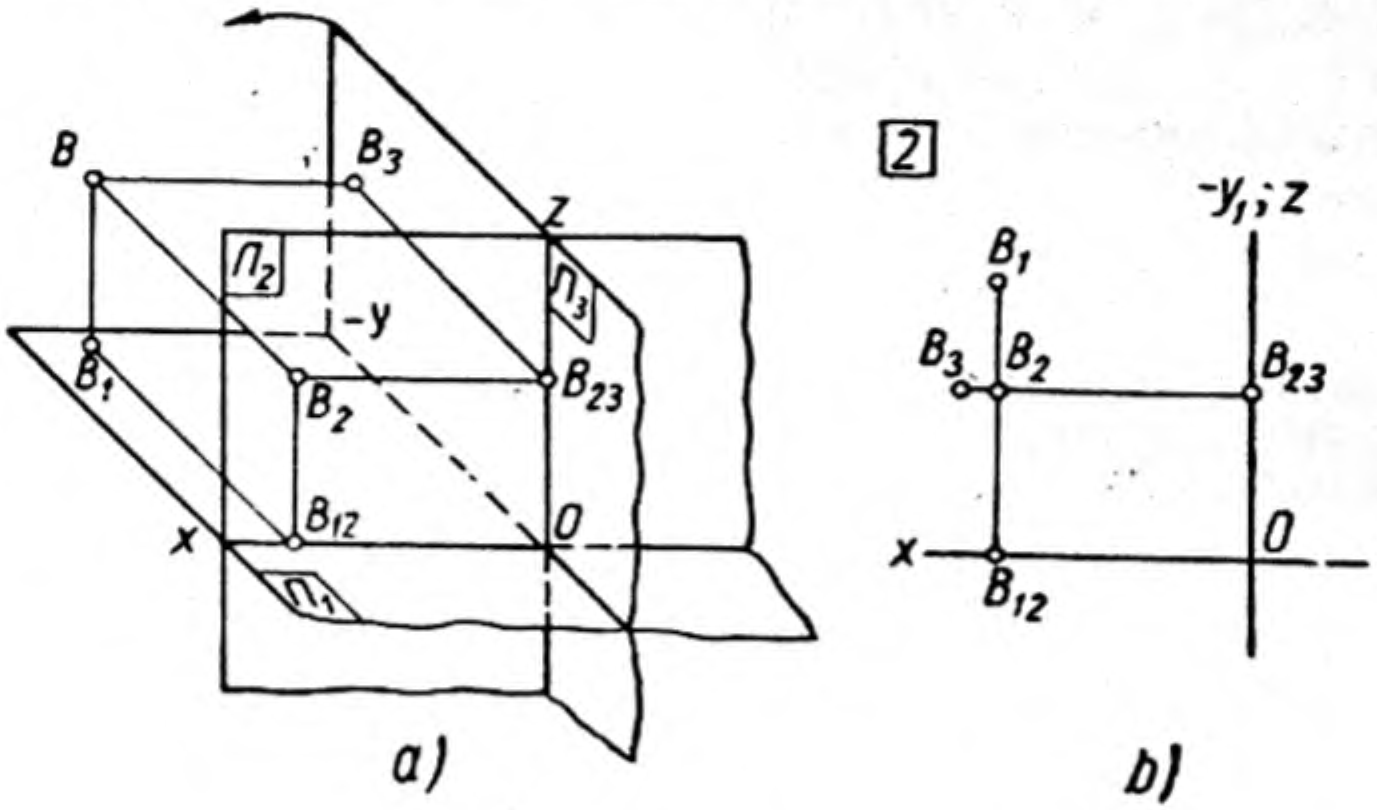


Fig. 24

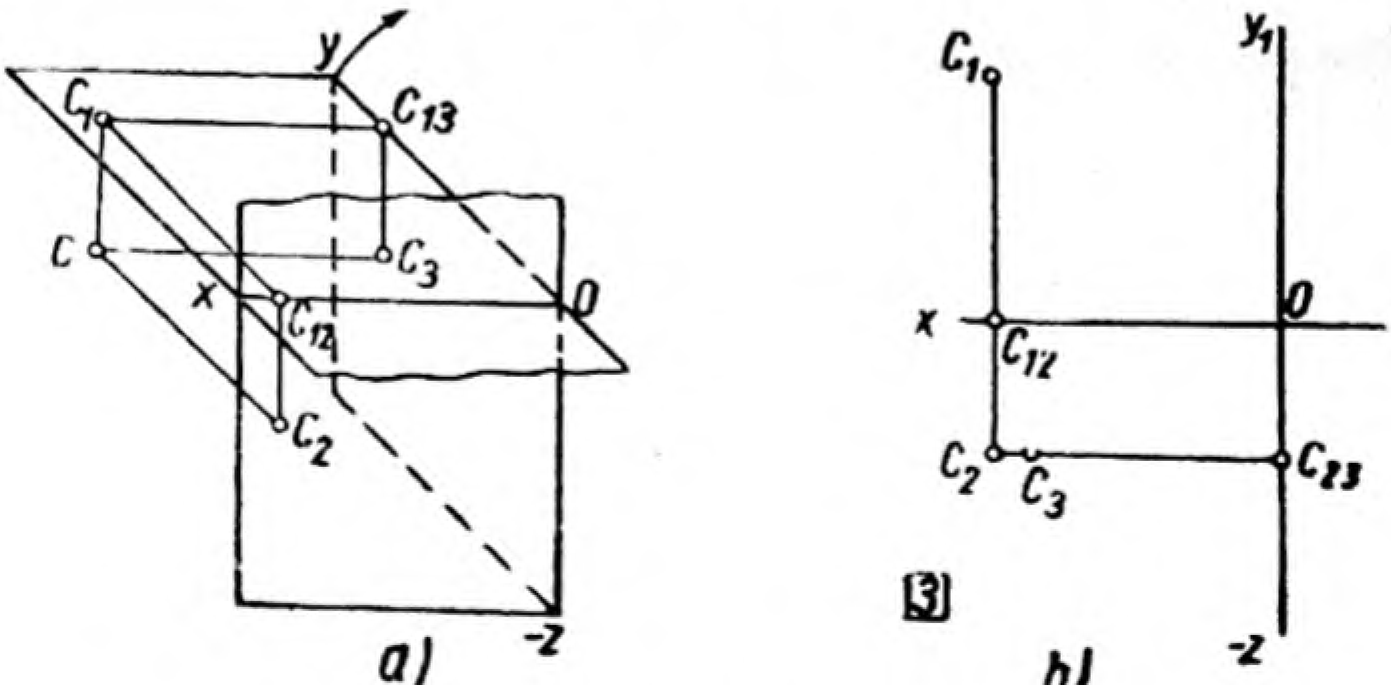


Fig. 25

Point C lies in the third octant (Fig. 25, *a*) and in the multiview drawing (Fig. 25, *b*) the positions of its principal projections C_1 and C_2 are the same as those of the projections of a point in the third quadrant (Fig. 17, *b*), namely, the horizontal projection lies above the ground line Ox, and the frontal projection lies under Ox.

To construct the profile projection C_3 by means of these two projections the horizontal projector of point C is drawn through point C_2. On it, to the left of Oz, mark off the negative distance — y, which is equal to the length of C_1C_{12}.

Point D is situated in the fourth octant (Fig. 26). Its projections D_1, D_2 and D_3 in the drawing (Fig. 26, *a*) are constructed with

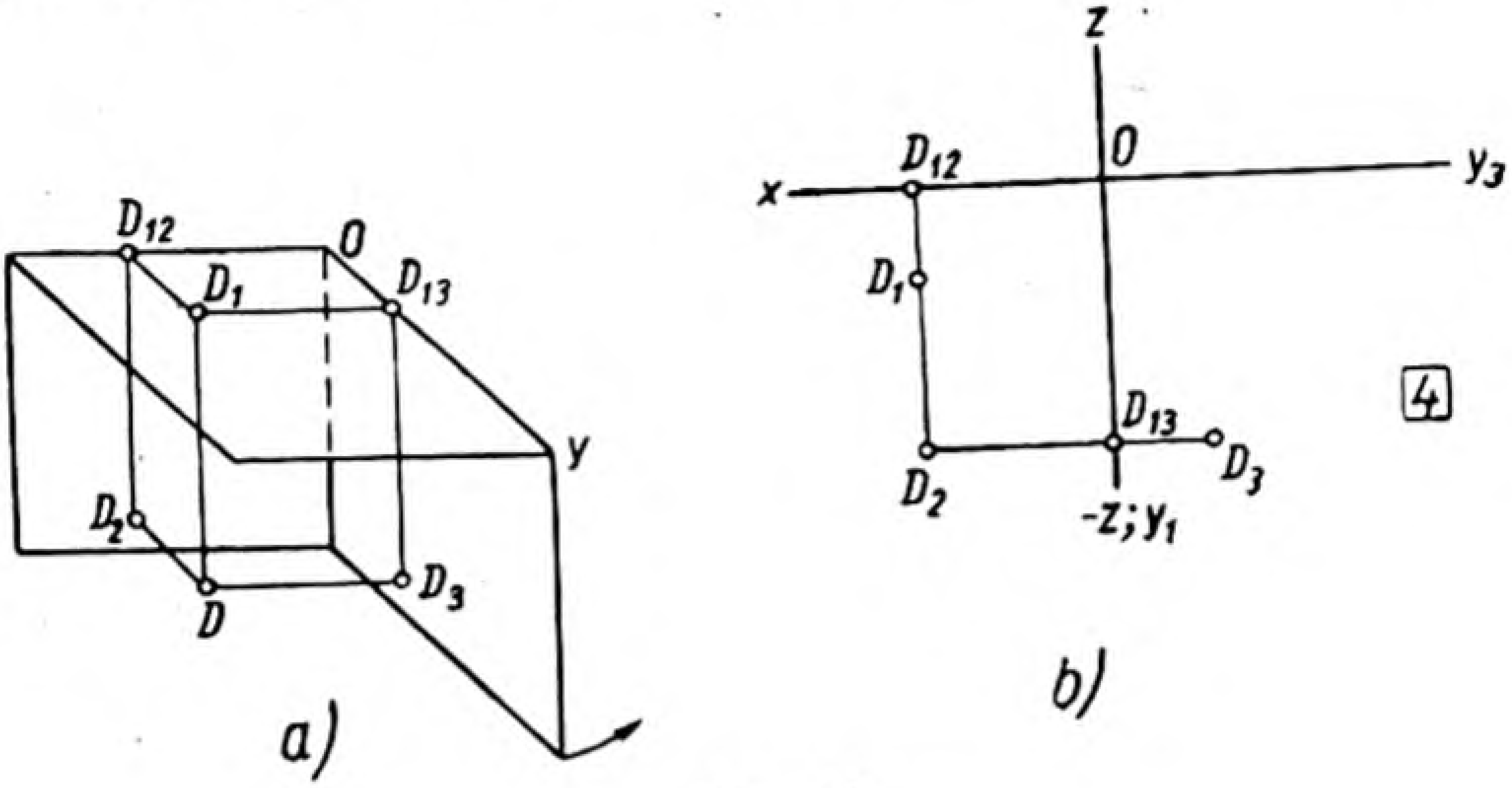


Fig. 26

the help of the profile face ($DD_1D_{12}D_2$) of the coordinate parallelepiped and the frontal face ($DD_1D_{13}D_3$).

The principal projections D_1 and D_2 of point D in the multiview drawing (Fig. 26, *b*) are constructed in full accordance with the construction of a point situated in the fourth quadrant (Fig. 18, *b*). To construct the profile projections D_3 of point D by means of these projections, through the frontal projection D_2 a horizontal projector is drawn and on it, to the right of Oz, the positive coordinate y, which is equal to D_1D_{12}, is laid off. From the multiview drawing of points A, B, C and D situated in the *1*, *2*, *3* and *4* octants, respectively, it can easily be seen that the profile projections of these points are located in the parts of the multiview drawing (divided by the x, y, z axes, as shown in Fig. 22, *b*) the numbers of which $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ and $\boxed{4}$ coincide with the numbers of the octants. So projection A_3 of point A of the first octant lies in the top right part $\boxed{1}$ of the multiview drawing (Fig. 22, *b*). The projection B_3 of point B of the second octant lies in the top left part $\boxed{2}$ of the drawing (Fig. 24, *b*). The projection C_3 of point C of the third octant lies in the lower left part $\boxed{3}$ of the drawing (Fig. 25, *b*). Finally, the projection D_3 of point D of the fourth octant lies in the lower right part $\boxed{4}$ of the drawing (Fig. 26, *b*). This will help the student to orientate himself more quickly when constructing the three projections of points situated in the first four octants. It will also

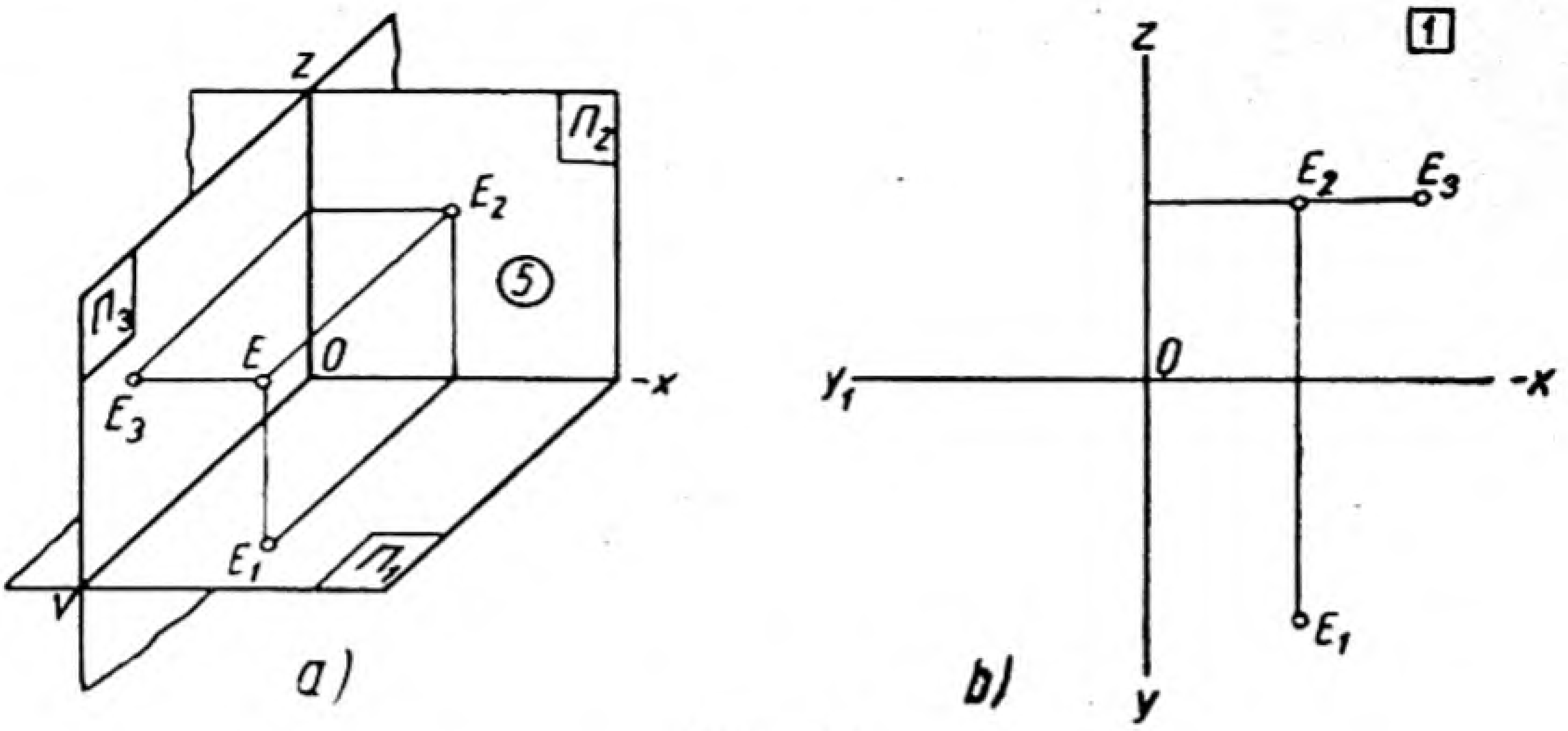

z
Π2
E2
Π3
5
E
0
-x
E3
E1
Π1
y
a)
z
1
E2
E3
0
y1
-x
E1
b)
y

Fig. 27

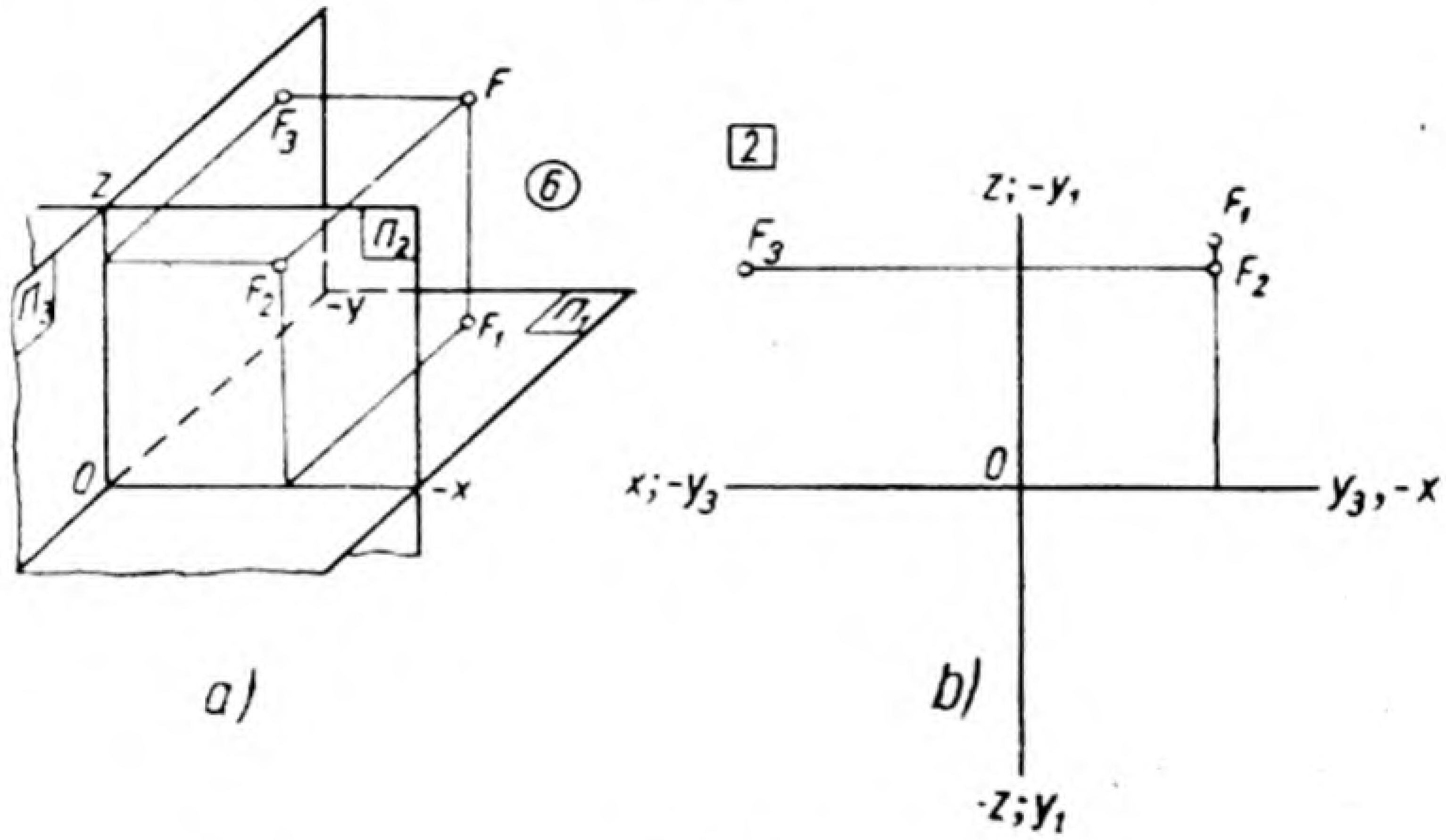

F
F3
6
z
Π2
Π3
F2
-y
F1
Π1
0
-x
a)
2
z; -y1
F1
F3
F2
0
x; -y3
y3, -x
b)
-z; y1

Fig. 28

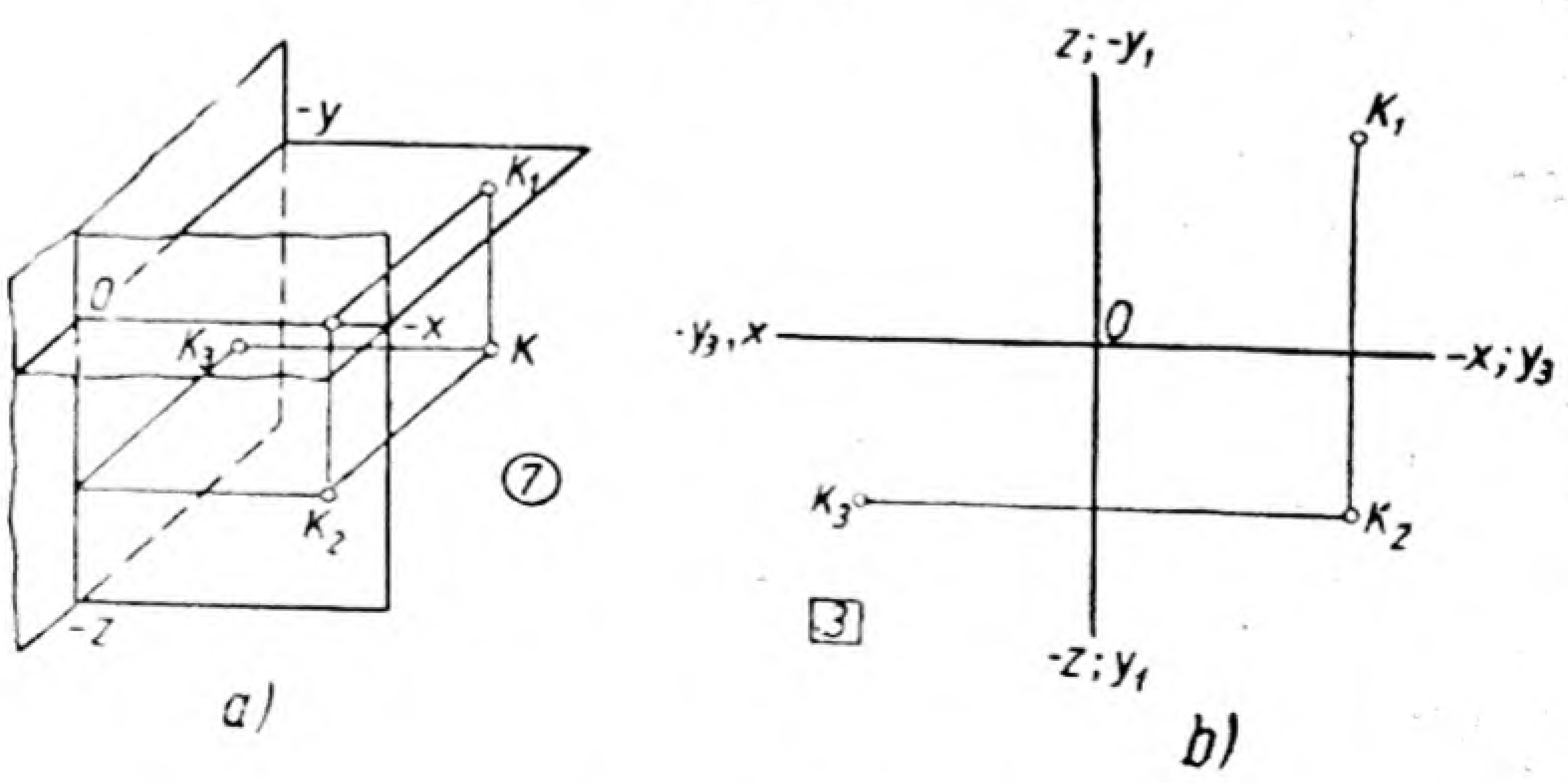

-y
K1
0
-x
K3
K
7
K2
-z
a)
z; -y1
K1
0
-y3, x
-x; y3
K3
K2
3
-z; y1
b)

Fig. 29

help him to determine, by means of the multiview drawing, in which octant a point is located in space.

The points E, F, K and L, situated respectively in the fifth, sixth, seventh and eighth octants, are shown in Figs. 27 to 30. The sequence of the construction of their multiview drawings is the same as for points A, B, C and D, i. e., the principal projections of these points on the planes Π_1 and Π_2 are constructed first, and then their projections on plane Π_3 are found using the method already outlined.

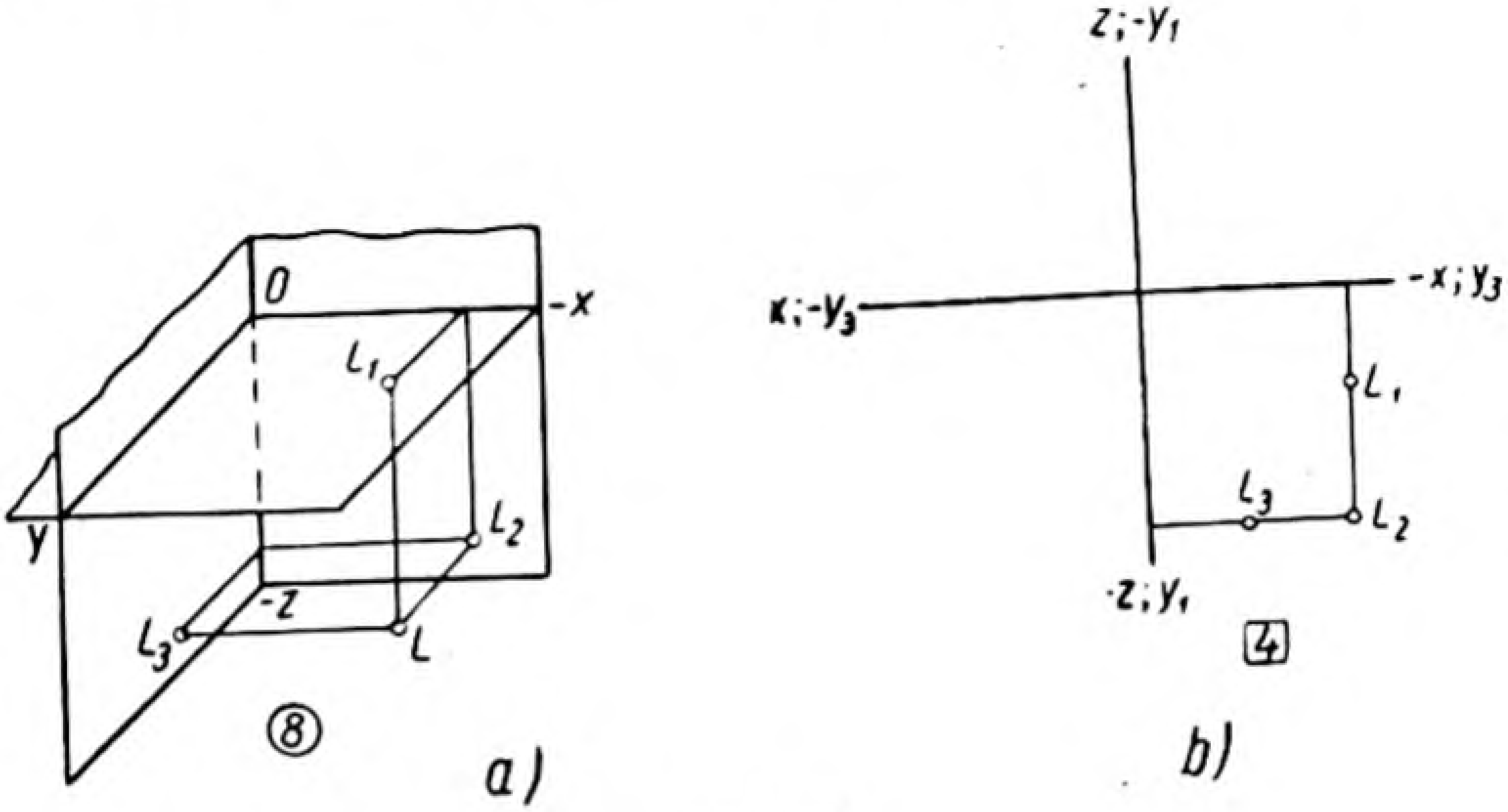


Fig. 30

It should be noted that:

1. The horizontal and frontal projections of points which are located in the *5, 6, 7* and *8* octants are seen in the multiview drawing in the same order as the points of the *1, 2, 3* and *4* octants, correspondingly located in space. In other words, the projections of points in the odd-numbered octants (*5* and *7*) are located on both sides of the ground line Ox, the projections of points in the even-numbered octants (*6* and *8*) are both on one side of Ox.

2. The profile projections of points located in the *5, 6,* 7 and *8* octants are situated in those parts of the multiview drawing (divided by lines x, y, z, as shown in Fig. 22, *b*), the numbers of which are obtained by subtracting 4 from the number of the corresponding octant. In this way the projection E_3 of point E in the fifth octant (Fig. 27, *b*) is situated in the top right-hand part $\boxed{1}$ of the multiview drawing. The projection F_3 of point F in the sixth octant (Fig. 28, *b*) lies in the top left part $\boxed{2}$ of the drawing. The projection K_3 of point K in the seventh octant (Fig. 29, *b*) lies in the lower left side $\boxed{3}$ of the drawing and, finally, the projection L_3 of point L in the eighth octant (Fig. 30, *b*) is situated in the lower right part $\boxed{4}$ of the drawing.

3. Since the values of x for points located in the *5, 6, 7* and *8* octants are negative, the horizontal and frontal projections of these points in the multiview drawing are situated to the right of Oz.

Problems and Exercises

1. Does one projection of a point determine the position of that point in space?

2. Prove that two projections of a point completely determine its position in space.

3. Prove that in the multiview drawing two projections of a point are located on one line perpendicular to the ground line.

4. How is the distance from plane Π_1 of the projection of a point determined in the multiview drawing?

5. How is the distance from plane Π_2 of the projection of a point determined in the multiview drawing?

6. What is the position in space, in relation to plane Π_1, of a point, the frontal projection of which is a) above the ground line Ox, and b) below it?

7. What is the position in space relative to plane Π_2 of a point, the horizontal projection of which is a) below the ground line Ox, and b) above it?

8. How are the principal projections situated in the multiview drawing of a point which is a) in the first quadrant, b) in the second quadrant, c) in the third quadrant, and d) in the fourth quadrant?

9. How are the principal projections located in relation to the ground line Ox in the multiview drawing of a point which lies a) in the front flap of plane Π_1, b) in the top flap of plane Π_2, c) in the bottom flap of plane Π_2, and d) on the ground line Ox?

10. How is the profile projection of a point determined in the multiview drawing by means of its two principal projections on planes Π_1 and Π_2?

11. In which octant is a point located if a) its coordinates x, y and z are positive, if b) x and z are positive and y negative, and if c) x is negative and y and z positive?

12. Which are the positive and which are the negative coordinates of points located in the a) second, b) fifth, c) seventh, d) eighth octant?

13. Construct the three-dimensional drawing and the multiview drawing of points A (5; 15; 7), B (10; — 15; 20), C (15; — 10; — 12) and D (20; 8; — 16). The coordinates are given in mm.

Hint. First determine in which octant the given points are located, imagining the broken coordinate line for each of the points (see Fig. 23 and the corresponding explanation).

14. Write down the coordinates of the points, given in Fig. 31, and state in which space quadrants these points are located.

Hint. It is necessary to imagine that the drawing is folded along ground line Ox as in Fig. 15, *b* and that through the projections of each point perpendiculars are drawn to the planes of projections Π_1 and Π_2 respectively.

15. Given point A (20; 10; 15), construct point B symmetrical to the given point A in relation to plane Π_2, point C symmetrical to point A in relation to plane Π_1, and point D symmetrical to point A in relation to the ground line Ox.

Hint. Point A is located in the first quadrant (all the three coordinates x, y and z are positive). Therefore point B lies in the second quadrant, point C in the fourth and point D in the third.

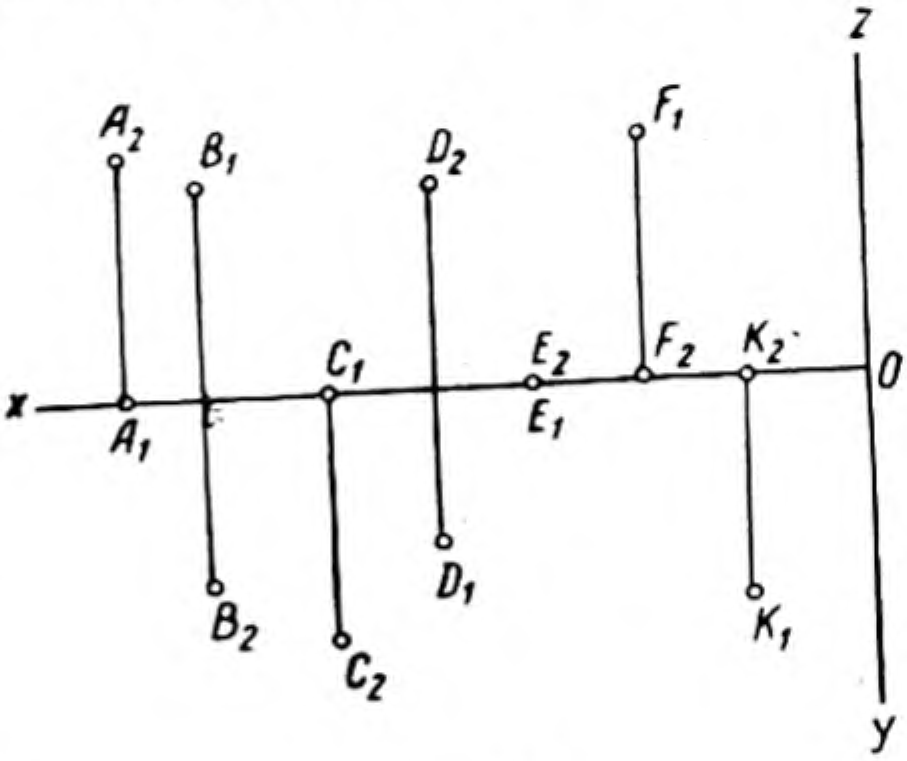


Fig. 31

16. Construct the three-dimensional view and the multiview drawing of points A, B and C, point A being in the second octant, point B in the fifth and point C in the seventh.

17. Write down the coordinates of the points A, B and C and state in which octants these points are located (Fig. 32).

Hint. The text explaining Figs. 22, 24, 25 and 26 should be studied.

18. By means of the pairs of projections of points A and B, construct their third projection and indicate the octants in which these points lie (Fig. 33).

Hint. It should be borne in mind that in the multiview drawing the horizontal and frontal projections of any point lie on the vertical

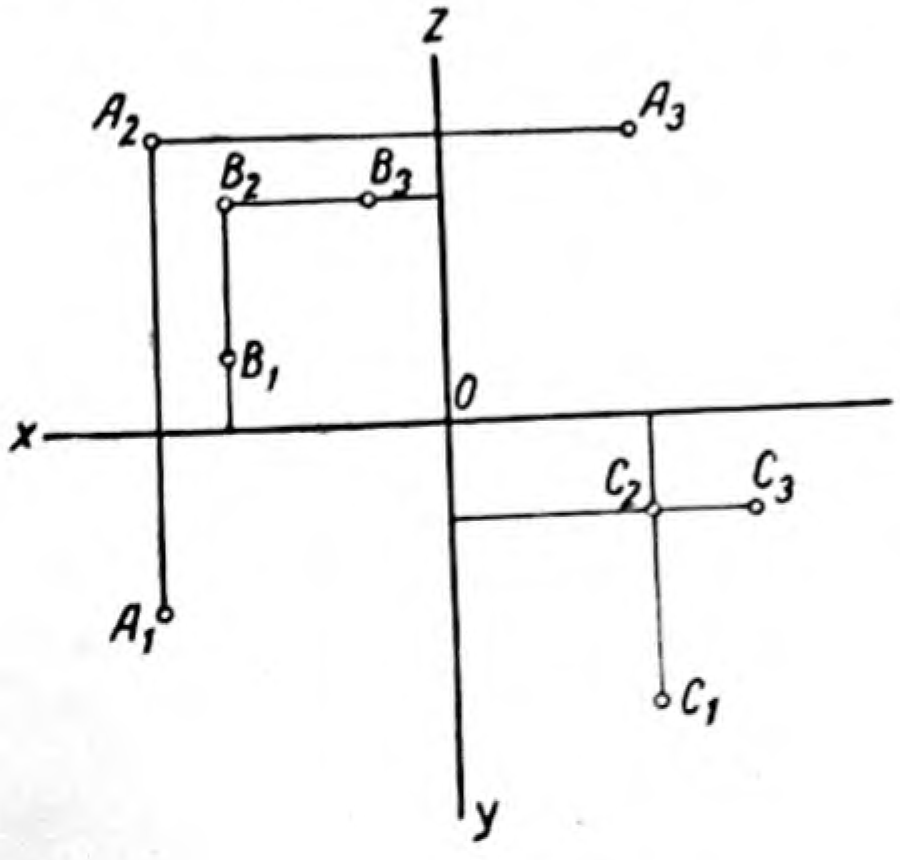


Fig. 32

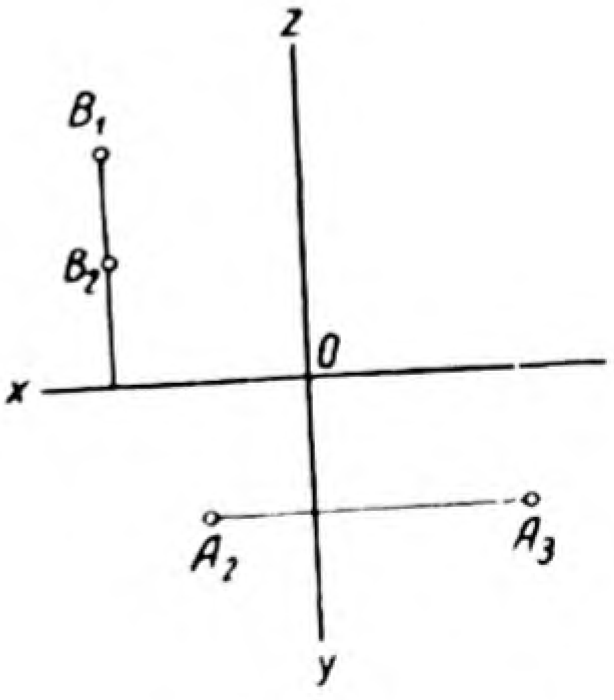


Fig. 33

projector, i. e., on the line perpendicular to ground line Ox, and the frontal and profile projections on the horizontal line of altitude on a line perpendicular to Oz.

It should also be recalled that $+y$ on the vertical projector is laid off from line Ox below that line and $-y$ above it. On the horizontal projector that same distance $+y$ is laid off to the right of line Oz, and $-y$ to the left of that line.

Chapter III

THE STRAIGHT LINE

§ 6. The Multiview Drawing of a Straight Line

1. In order to obtain the orthographic projection of a straight line on a plane, through the line a plane perpendicular to the plane of projection is passed. The line of intersection of this plane with the plane of projection is then the orthographic projection of the line.

In Fig. 34 through a given line AB a plane α perpendicular to plane Π_1 is passed. The intersection A_1B_1 of plane α with plane Π_1 determines the only possible projection of line AB* on plane Π_1.

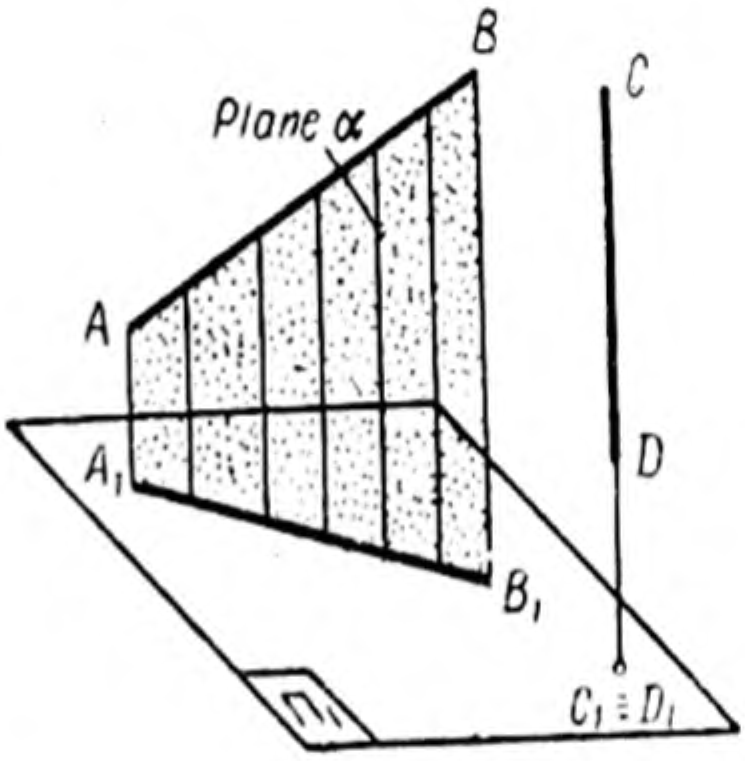


Fig. 34

Since a line is completely determined by any two of its points, instead of drawing a plane, it is sufficient to find the orthographic projections A_1 and B_1 of points A and B on plane Π_1. The line joining points A_1 and B_1 is the orthographic projection of AB on plane Π_1.

In Fig. 34 a line CD perpendicular to plane Π_1 is also shown. Perpendiculars to plane Π_1, passing through points C and D, will coincide with the line itself and intersect plane Π_1 in one point $C_1 \equiv D_1$.

It follows that the orthographic projection of a straight line on a plane is a straight line except when this line is perpendicular to the plane of projection.

Similarly any straight line in a plane may be considered as being the projection of a straight line.

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* The word line in this book may be assumed to imply a straight line, unless otherwise stated.

Let a plane α perpendicular to Π_1 (Fig. 34) be passed through the straight line A_1B_1 which lies in plane Π_1. Then AB will be the projection on plane Π_1 of a line, not perpendicular to Π_1, for example AB, drawn in plane α.

2. A single projection of a line does not determine the position of that line in space, since it is

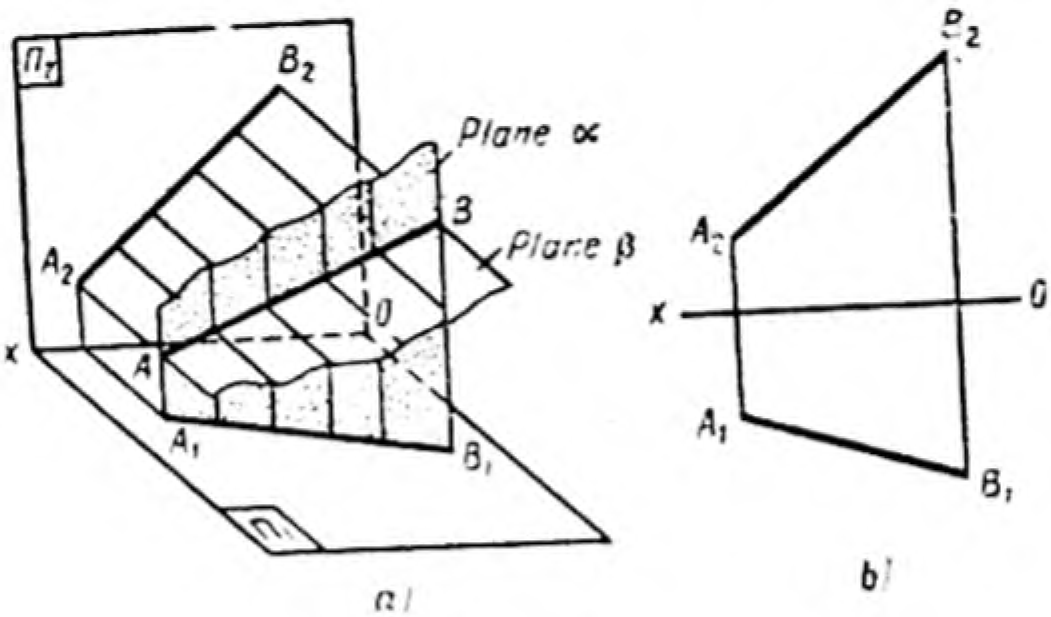


Fig. 35

possible to draw in the plane α (Fig. 34) any number of lines both parallel to Π_1 or inclined to that plane, while all their projections on plane Π_1 coincide with one line, i. e., A_1B_1.

3. Two projections of a line fully determine its position in space.

Knowing the position of two projections of a line — the horizontal A_1B_1 and the frontal A_2B_2 (Fig. 35, *a*) it is possible to pass through A_1B_1 a plane α perpendicular to plane Π_1 and through A_2B_2 a plane β perpendicular to plane Π_2. The line of intersection AB of the planes α and β determines the one and only position in space of the line, the projections of which were given.

Since two points fully determine the position of a line in space, the construction of the multiview drawing of a line is therefore reduced to the construction of the projection of two of its points. It is common practice to assume the extreme points as these points, if the line represented is of a given length.

To construct the multiview drawing of the line AB (Fig. 35, *a*) the coordinates x, y and z of points A and B are determined and transferred to Fig. 35, *b*. The pairs of projections obtained, i. e., A_1 and B_1 and A_2 and B_2 are then joined up. Lines A_1B_1 and A_2B_2 are, respectively, the horizontal and frontal projections of line AB.

Example. It is required to construct the multiview and the three-dimensional drawings of line AB by means of the coordinates of its points A (20; 15; 7) and B (6; 22; 23) (Fig. 36).

First let us construct the multiview drawing (Fig. 36, *a*):

1. Draw a horizontal line which is taken as the ground line Ox.

2. From point O lay off coordinates $x=20$ mm and $x=6$ mm, given by the coordinates of points A and B.

3. Through the points A_{12} and B_{12} so obtained draw perpendiculars to line Ox.

4. On these perpendiculars from points A_{12} and B_{12}, below Ox, lay off distances y, given by the coordinates of points A and B. These are respectively 15 and 22 mm, and above Ox lay off $z=7$ mm and $z=23$ mm. Note, the coordinates y and z are positive. The pairs of points A_1 and B_1 and A_2 and B_2 so obtained, are respectively the horizontal and frontal projections of points A and B.

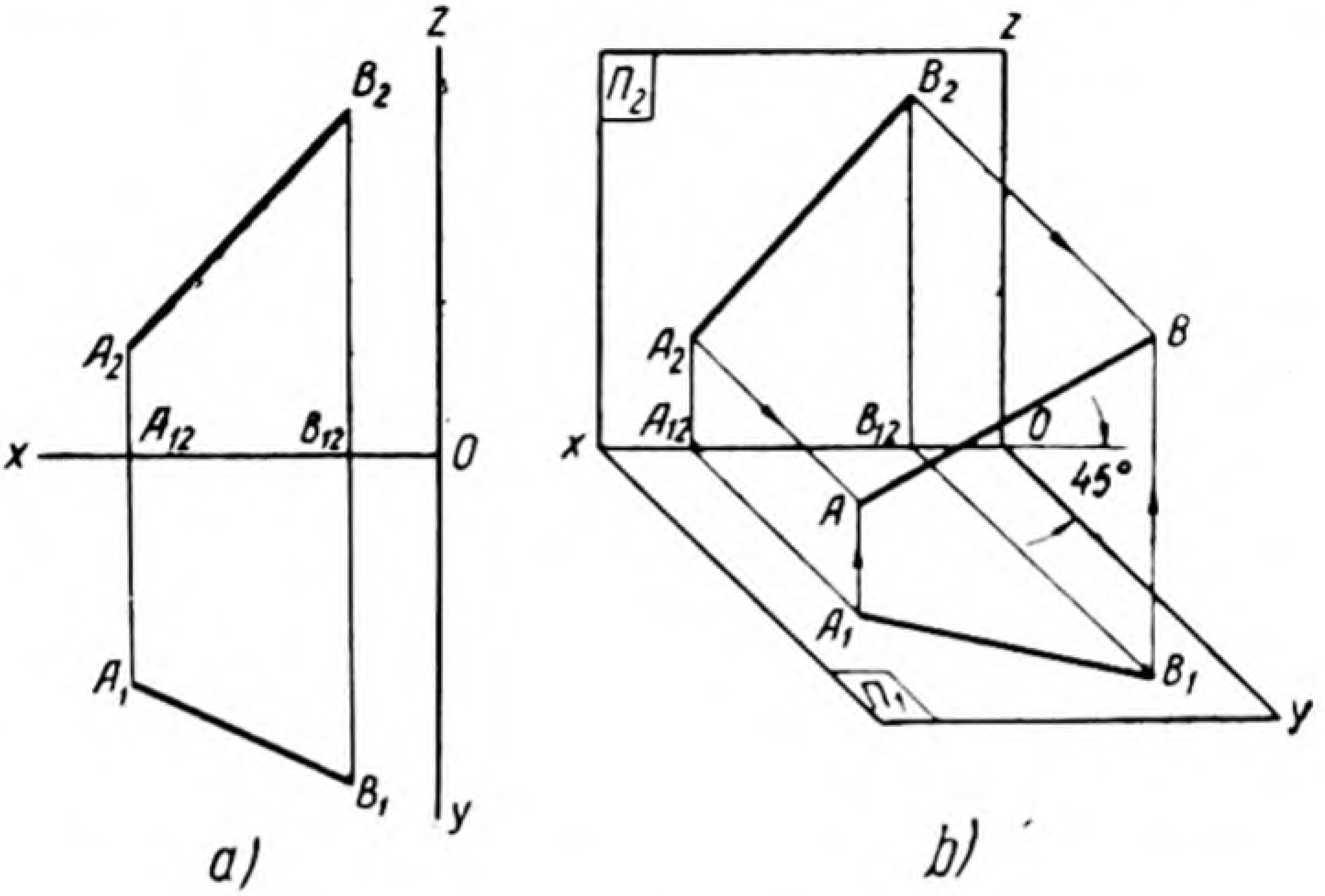


Fig. 36

5. By joining the corresponding projections of the points A and B the horizontal projection A_1B_1 and the frontal projection A_2B_2 of line AB are obtained.

In order to construct a space drawing of line AB (Fig. 36, *b*), in plane Π_1 straight line A_1B_1 is constructed using the coordinates x and y of points A and B. In plane Π_2 line A_2B_2 is constructed using the coordinates x and z of points A and B. To obtain the representation of the point A and B in space, projectors, respectively parallel to Oz and Oy, are drawn through points A_1 and A_2 and through points B_1 and B_2. The intersection of the projectors passing through A_1 and A_2 gives the position of point A in space. The intersection of the projectors passed through B_1 and B_2 gives the position of point B. Joining points A and B we obtain the three-dimensional representation of line AB.

§ 7. The Projections and Positions of Straight Lines Relative to the Planes of Projection

1. An *oblique line* is inclined to all three planes of projection. In Fig. 37, *a* the representation in space of an oblique line is given and in Fig. 37, *b*, its multiview drawing.

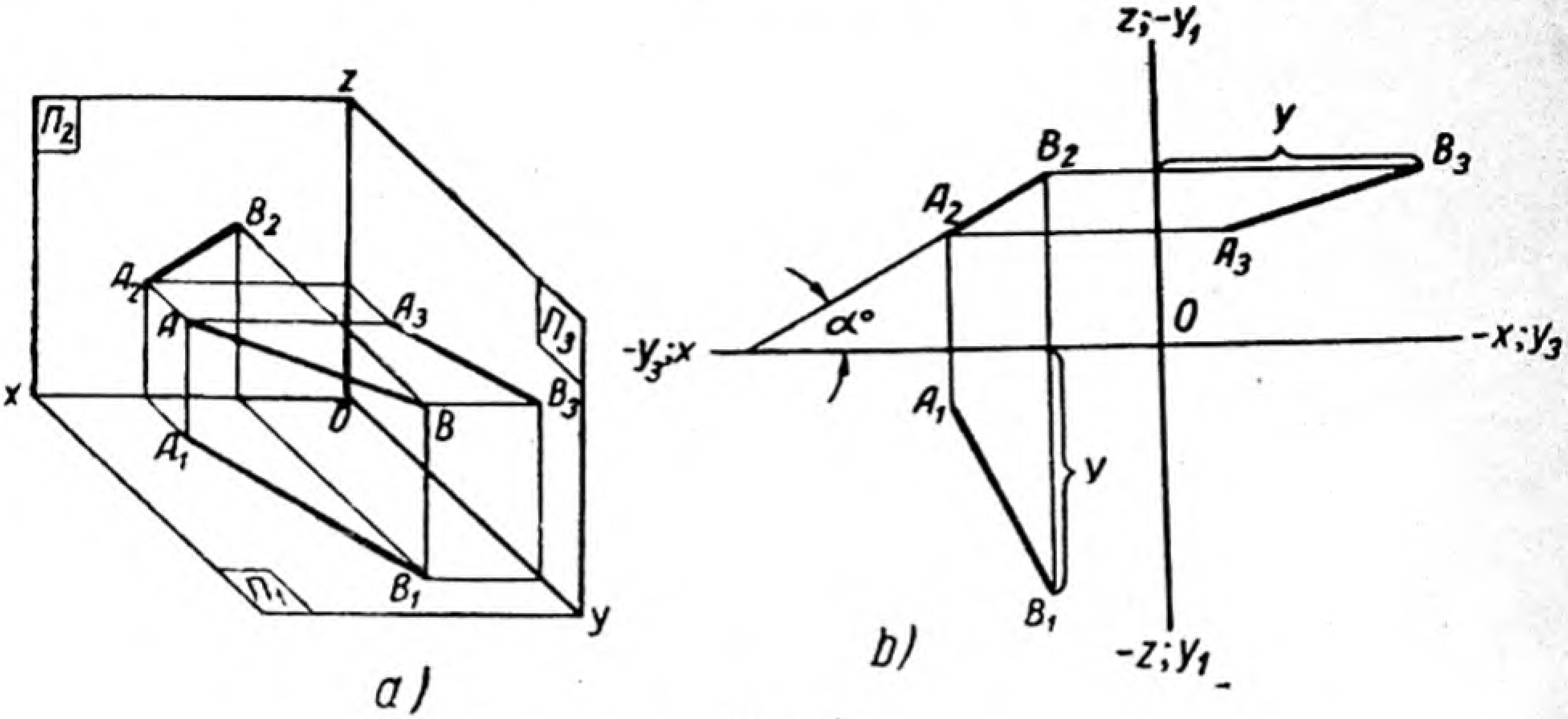


Fig. 37

On the multiview drawing it is seen that all three projections of the line are inclined to the ground lines, i. e., to the *x*, *y* and *z* axes.

Each of the projections A_1B_1, A_2B_2 and A_3B_3 of the straight line AB on the multiview drawing is shorter than AB itself. This can easily be proved. In Fig. 38 the line AB, its projection A_1B_1 on plane Π_1 and the projectors AA_1 and BB_1 are shown.

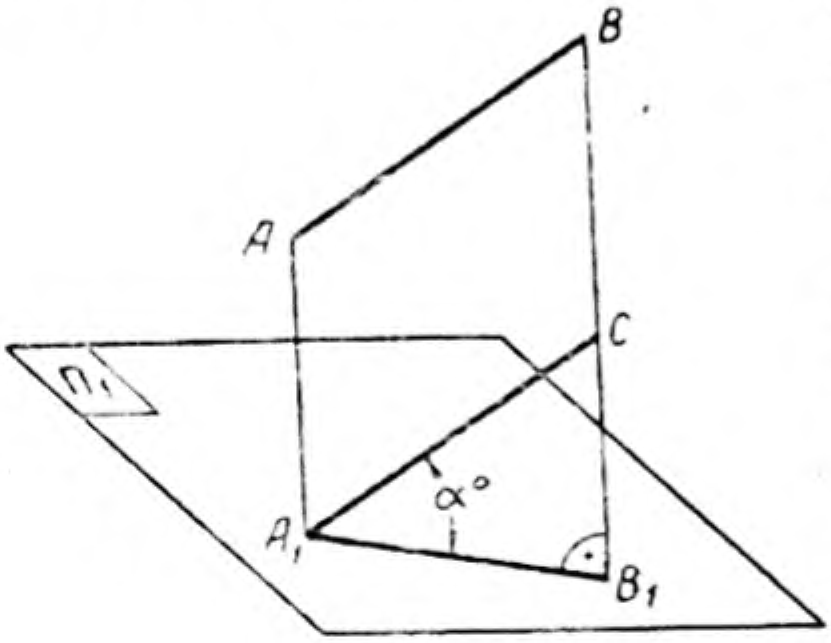


Fig. 38

Through point A_1 draw a line A_1C parallel to AB. Angle $A_1B_1C_1$ of the triangle obtained is a right angle ($B_1B \perp \text{plane } \Pi_1$). If angle B_1A_1C is $\alpha°$, then $A_1B_1 = A_1C \cos \alpha°$, but $A_1C = AB$. Therefore $A_1B_1 = AB \cos \alpha°$, and it follows that all projections of AB such as A_1B_1 are shorter than the line AB itself. If $\alpha° = 0$, then $A_1B_1 = AB$. If $\alpha° = 90°$, then $A_1B_1 = 0$ and line AB is projected onto plane Π_1 as a point.

The angles α, β and γ formed by the projections of line AB with the ground lines *x*, *y* and *z*, are not equal to the angle of inclination of the line itself to the planes of projection Π_1, Π_2 and Π_3.

For example, since the horizontal projecting plane ABB_1A_1 (in which lines projecting AB onto the horizontal plane Π_1 lie) is not parallel to the plane Π_2, the angle at which the line AB meets plane Π_1 is not equal to the angle α between the ground line Ox and projection A_2B_2 (Fig. 37, *b*).

2. Lines in special positions. A straight line parallel to the horizontal projection plane Π_1 is called a *horizontal line* or simply a *horizontal* (Fig. 39, *a*). Since all points of a horizontal lie at the same distance from the plane Π_1, the frontal projection of a horizontal is parallel to the ground line Ox ($C_2D_2 \parallel Ox$). The profile projection C_3D_3 is parallel to the line Oy_3. The horizontal projection of the horizontal line gives its true length, i. e., $C_1D_1 = CD$ (Fig. 39, *b*).

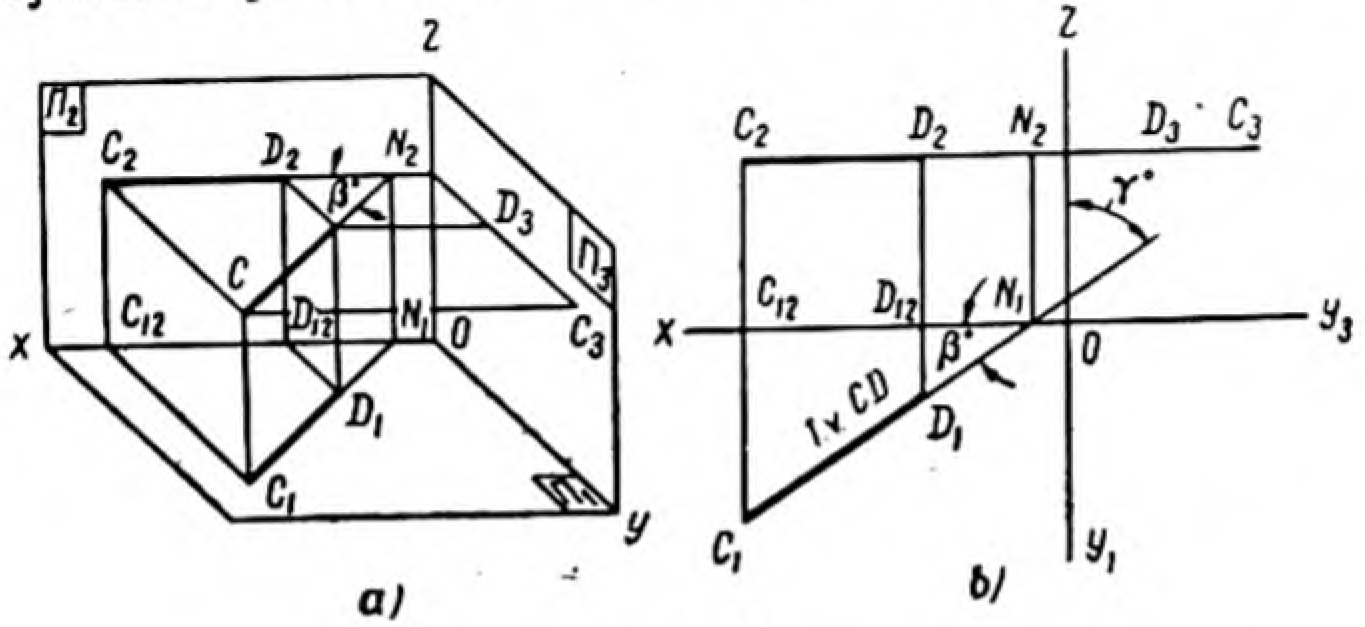


Fig. 39

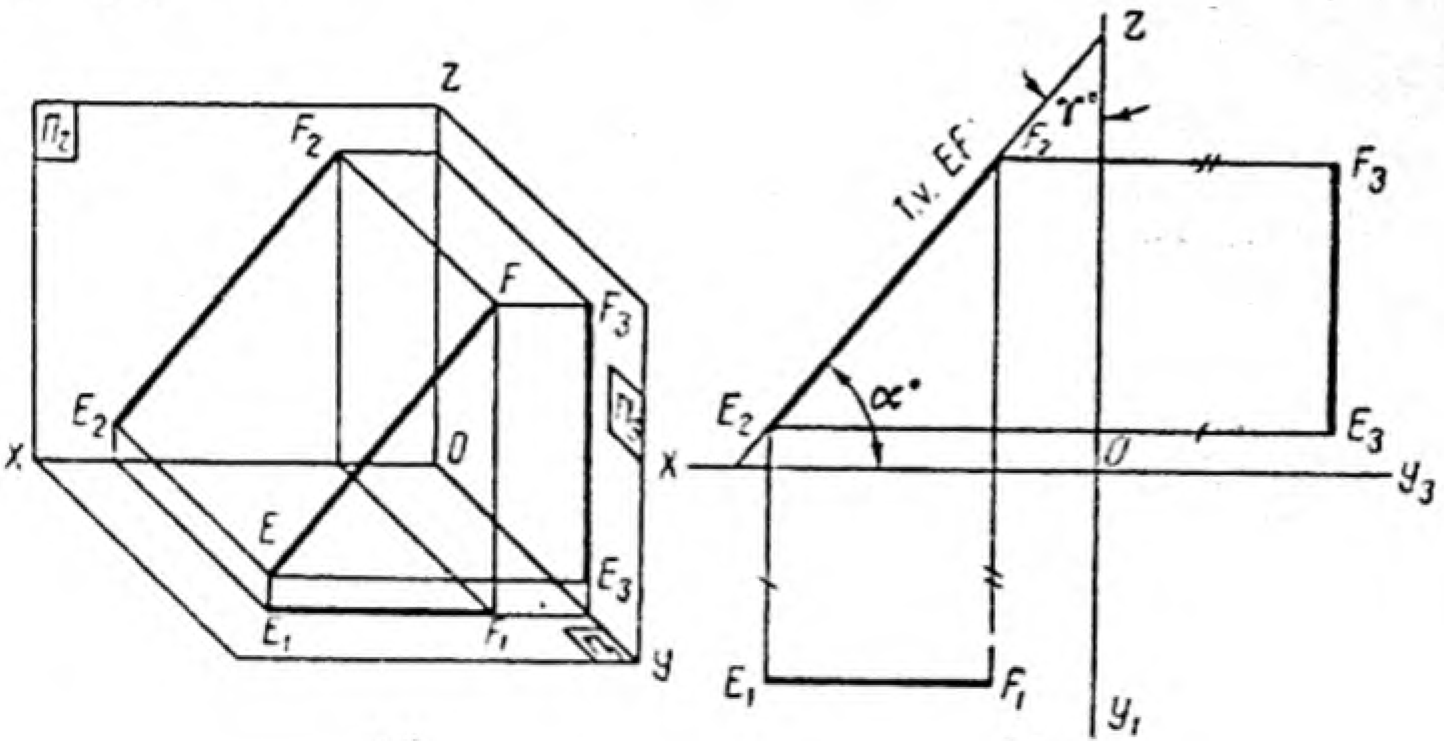
In the drawings the true length, or true value, is indicated by the letters *T. V.*

The angle between a straight line and a plane is the angle formed by the line itself and its orthographic projection on the given plane.

The angle β between the ground line Ox and the horizontal projection C_1D_1 of the horizontal CD in the multiview drawing is equal to the angle of inclination of the horizontal to the frontal plane of projection Π_2 and the angle γ between the line Oy_1, and the horizontal projection C_1D_1 of the horizontal CD in the multiview drawing is the angle of inclination of the horizontal to the profile plane of projection Π_3 (Fig. 39, *b*) and $<\beta^\circ + <\gamma^\circ = 90^\circ$.

3. A line parallel to the frontal plane of projection Π_2 is called a *frontal line* or simply a *frontal* (Fig. 40). Since all points of a frontal lie at the same distance from plane Π_2, the horizontal projection E_1F_1 of the frontal is parallel to the line Ox, the profile projection E_3F_3 is parallel to the line Oz, and the frontal projection is equal in length to the projected frontal ($E_2F_2 = EF$, Fig. 40).

Angle α between the ground line Ox and the frontal projection E_2F_2 of the frontal (Fig. 40, *b*) gives the true inclination of the frontal to the horizontal plane of projection Π_1. The angle γ between the line Oz and frontal projection E_2F_2 of the frontal gives the true inclination of the frontal to the profile plane of projection Π_3 and $\sphericalangle\alpha^\circ + \sphericalangle\gamma^\circ = 90^\circ$.



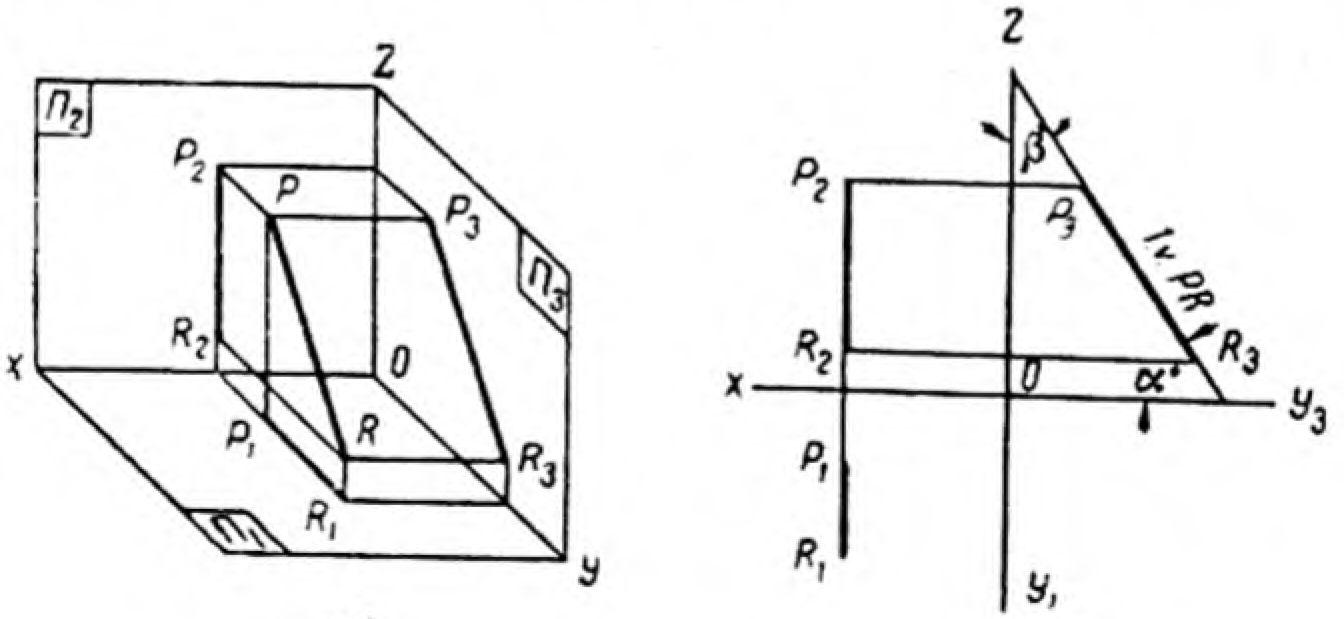
Π2
Z
F2
F
F3
Π3
E2
X
0
E
E3
E1
F1
Π1
y
z
T.v. EF
F2
γ°
F3
E2
α°
x
0
E3
y3
E1
F1
y1

a) b)

Fig. 40

4. A line parallel to the profile plane of projection Π_3 is called a *profile line* (Fig. 41).

Since all points of a profile line lie at the same distance from the plane Π_3 the horizontal projection P_1R_1 and the frontal projection P_2R_2 of the profile line are perpendicular to the ground line Ox, and the length of the profile projection P_3R_3 is the true length of the line PR (Fig. 41, *b*).



Π2
Z
P2
P
P3
Π3
R2
X
0
P1
R
R3
R1
Π1
y
z
P2
β
P3
T.v. PR
R2
R3
x
0
α°
y3
P1
R1
y1

a) b)

Fig. 41

The angles α and β (Fig. 41, *b*) give the true values of the angles of inclinations of the line PR to planes Π_1 and Π_2, respectively. The sum of the angles α and β is 90°.

The profile line, shown in Fig. 41, is called an *ascending line* and that, shown in Fig. 42, a *descending line.*

The side pieces of a ladder leaning against a wall illustrate the former. The sides of a picture frame hanging at an angle on the wall illustrate the latter.

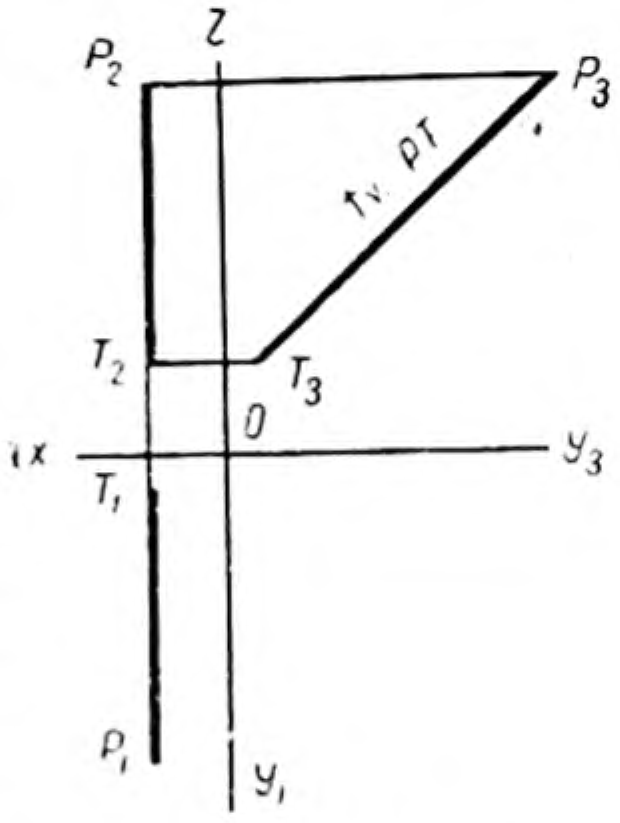


Fig. 42

5. Lines parallel to two of the three planes of projection. Any line parallel to two planes of projection is obviously perpendicular to the third plane of projection and therefore is projected onto it as a point. In Fig. 43 lines *AB*, *CD* and *EF* are shown. *AB* is parallel to the planes of projection Π_2 and Π_3, therefore the length of this line when projected onto planes Π_2 and Π_3 is the true length of *AB*, and its projection on plane Π_1 is a point. Line *CD* is parallel to the planes Π_1 and Π_3, so its projections on those planes give the true length of the line. Its projection on plane Π_2 is a point. *EF* is parallel to planes Π_1 and Π_2, and its projection on those planes gives the true length of *EF* itself. On plane Π_3 its projection is a point.

6. Lines lying in the planes of projection. In Fig. 44 is shown a line *AB*, lying in the plane Π_1. Its horizontal projection A_1B_1 coincides with the line itself $(A_1B_1 \equiv AB)$.

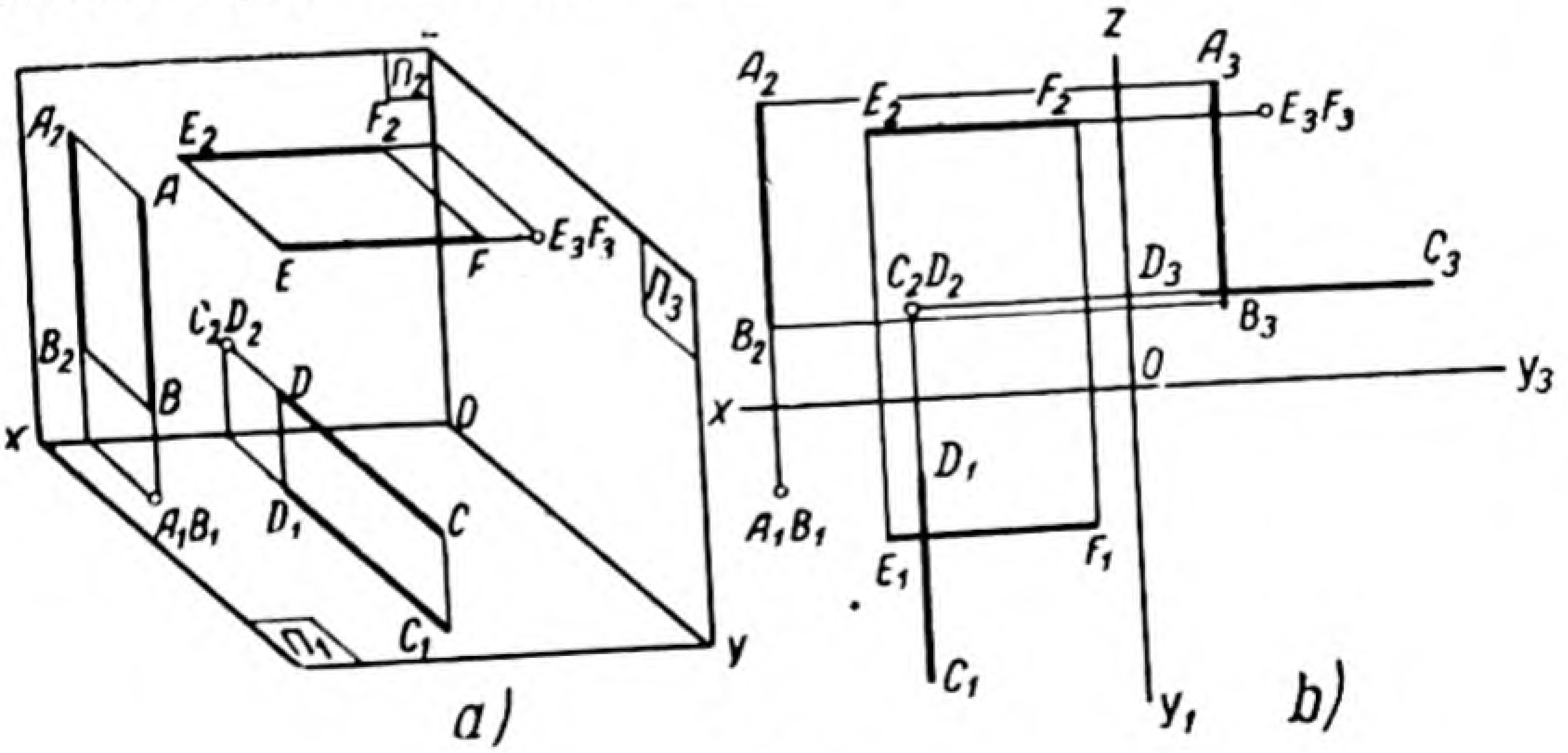


Fig. 43

The frontal projection A_2B_2 lies on the *Ox* ground line and the profile projection A_3B_3 lies on the *Oy* ground line.

In Fig. 45 the line *CD* lies in the plane Π_2. Its frontal projection

C_2D_2 coincides with the line itself ($C_2D_2 \equiv CD$), the horizontal projection C_1D_1 is situated on the ground line Ox and the profile projection C_3D_3 lies on the ground line Oz.

If a line lies in the Π_3 plane, its profile projection coincides with the line itself, the horizontal projection lies on the line Oy and the frontal projection, on the line Oz.

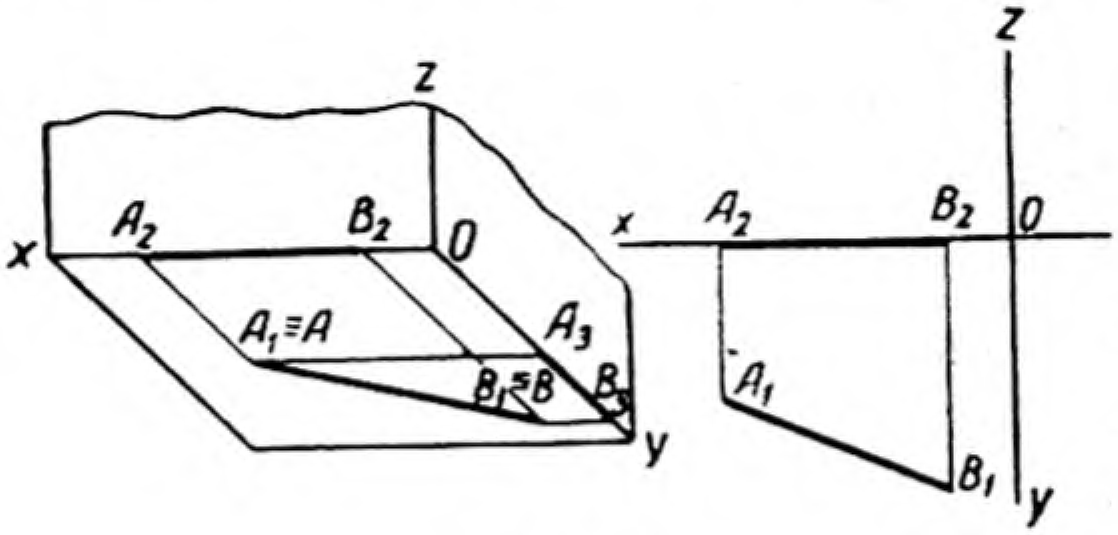


Fig. 44

7. Straight lines of limited length situated in the first quadrant or in the first octant have already been examined above. If a line lies in any other quadrant or octant, the construction of two or three of its projections consists in obtaining the orthographic projections on the planes of projection of the points limiting the line and joining up the corresponding pairs of projection. The following three-dimensional and multiview drawings serve as examples:

1. Line EF is a horizontal located in the second quadrant (Fig. 46).

2. Line KL is a frontal located in the third quadrant (Fig. 47).

3. Line MN is an oblique line located in the fourth quadrant (Fig. 48).

Problems and Exercises

1. When is the projection of a straight line itself a straight line, and when does it become a point?

2. How many projections of a line are required to determine its position in space? Why is one projection insufficient for this purpose?

3. How are the projections of a horizontal located?

4. How are the projections of a frontal located?

5. What is the position of a line in space if its two principal projections, i. e., on Π_1 and Π_2, are perpendicular to the ground line Ox?

6. Describe the projections on planes Π_1, Π_2 and Π_3 of a horizontal which is perpendicular to plane Π_2.

7. Describe the projections of a line which lies in the planes Π_1, Π_2, Π_3.

8. In which space quadrants are the lines AB, CD and EF situated

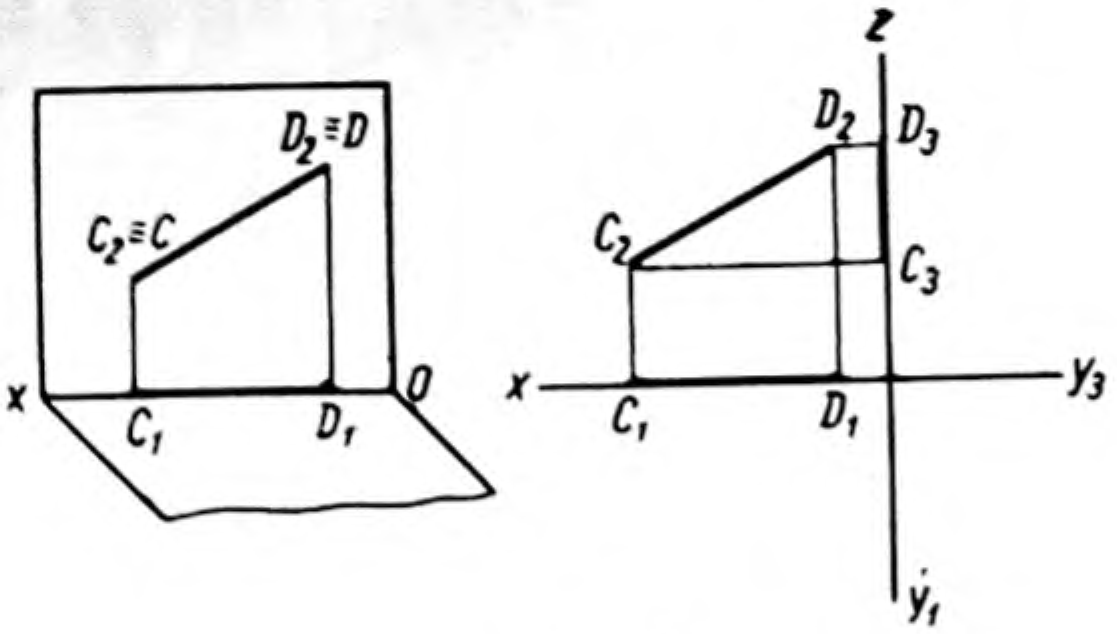

$C_2 \equiv C$
$D_2 \equiv D$
x
C_1
D_1
0
z
D_2
D_3
C_2
C_3
x
C_1
D_1
y_3
y_1

Fig. 45

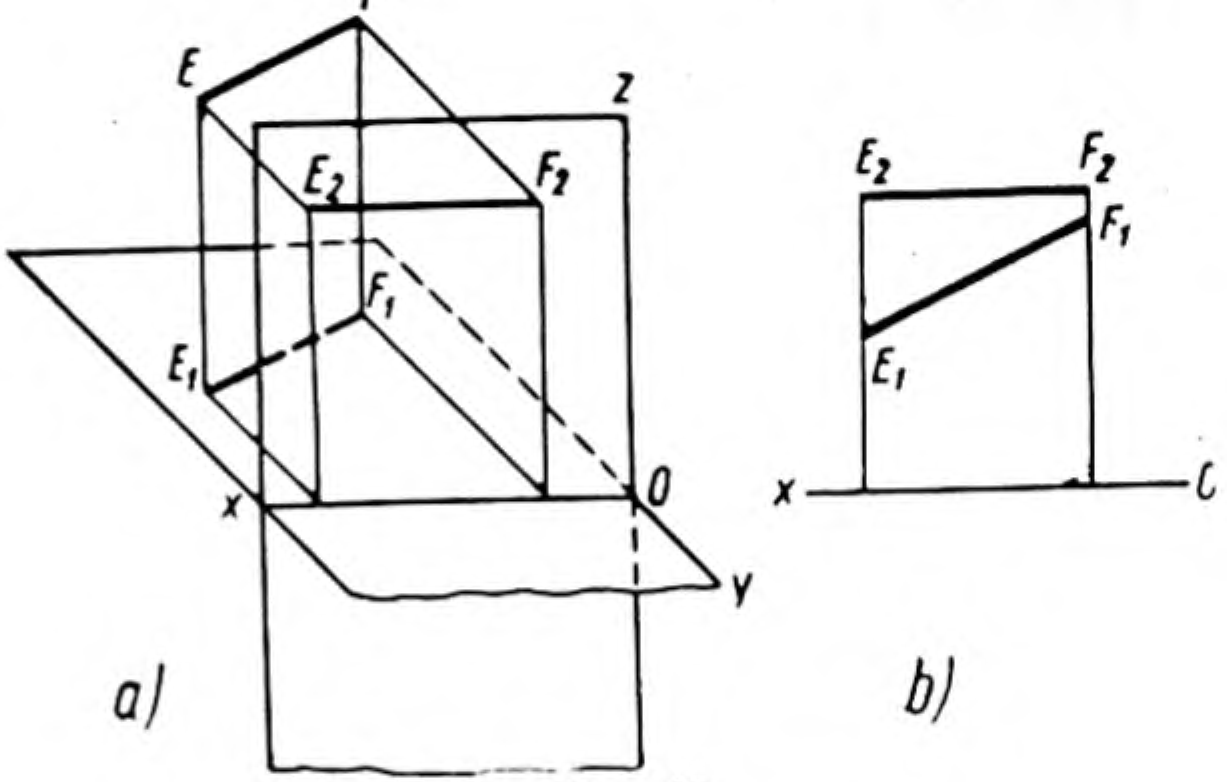

F
E
z
E_2
F_2
F_1
E_1
x
0
y
a)
E_2
F_2
F_1
E_1
x
0
b)

Fig. 46

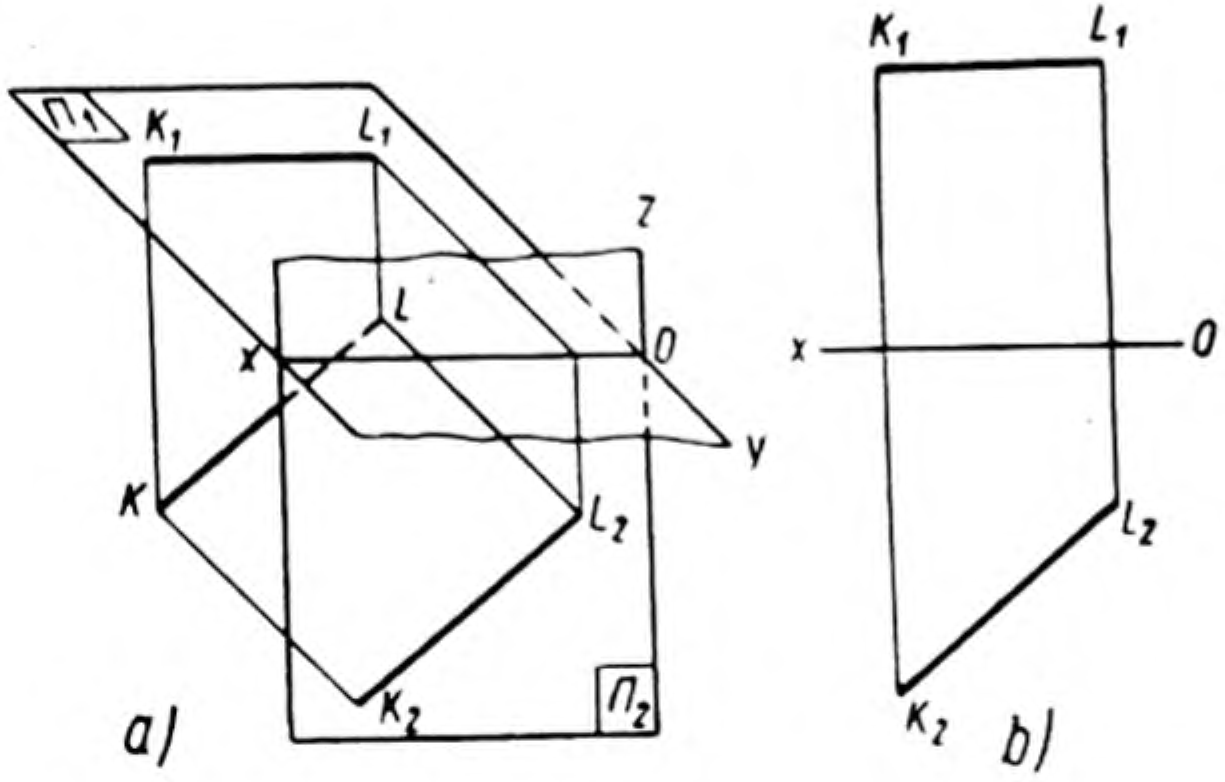

Π_1
K_1
L_1
z
L
x
0
y
K
L_2
K_2
Π_2
a)
K_1
L_1
x
0
L_2
K_2
b)

Fig. 47

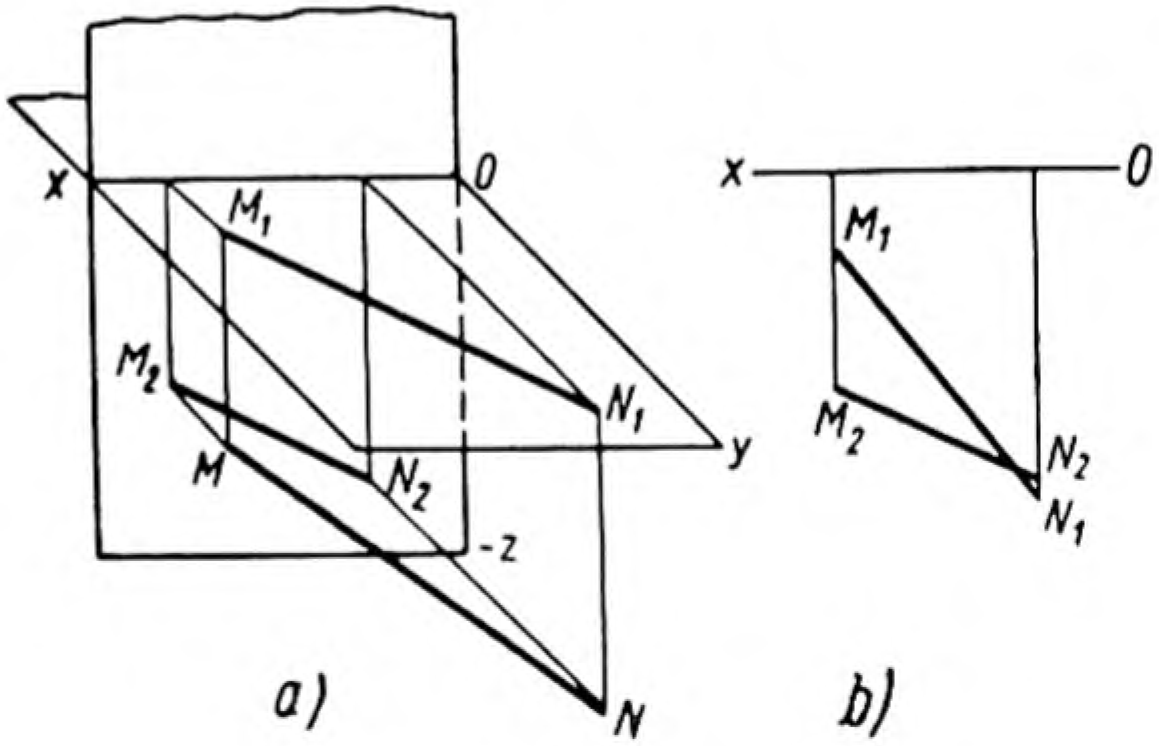

X
0
M_1
M_2
M
N_1
N_2
y
-z
N
a)
x
0
M_1
M_2
N_2
N_1
b)

Fig. 48

if their coordinates are A (13; —5; 10), B (5; —9; 10), C (5; —10, 20), D (18; —16; 4), E (14; 4; —6), F (18; 8; —9)?

Hint. In this case two planes of projections Π_1 and Π_2 should be visualized. Having determined the quadrant in which the limiting points of the line lie, imagine them joined by a straight line.

9. Construct the multiview drawing, consisting of three projections, and the space drawing of line AB by means of two given projections and determine the coordinates of the ends of the given line (Fig. 49).

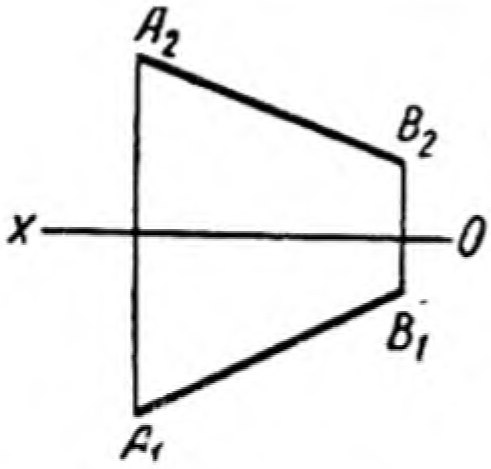


Fig. 49

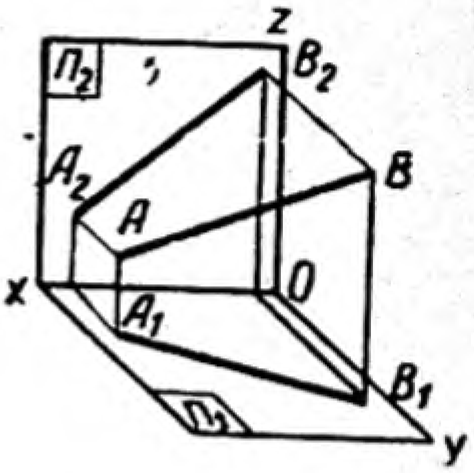


Fig. 50

10. Construct the multiview drawing, consisting of three projections of the line AB using the space drawing. Determine the coordinates of the end points of the line (Fig. 50).

11. Construct the space drawing and the multiview drawing, two projections, of line AB using the coordinates of its limiting points A (20; 30; 40) and B (5; 5; 10).

§ 8. The True Length of a Line and Its Angle of Inclination to the Planes of Projection*

1. The line AB (Fig. 51) may be considered as the hypotenuse of a right triangle ABC in space, the right angle ACB is formed by the projector AA_1 and line BC which is drawn parallel to the horizontal projection A_1B_1 of the line AB. One of the sides **(BC)** of the triangle ABC is equal to the horizontal projection A_1B_1, since parts of parallel lines limited by other parallel lines are equal in length. The length of the second side (AC) is equal to the difference $z—z'$ determined from the coordinates of points A and B.

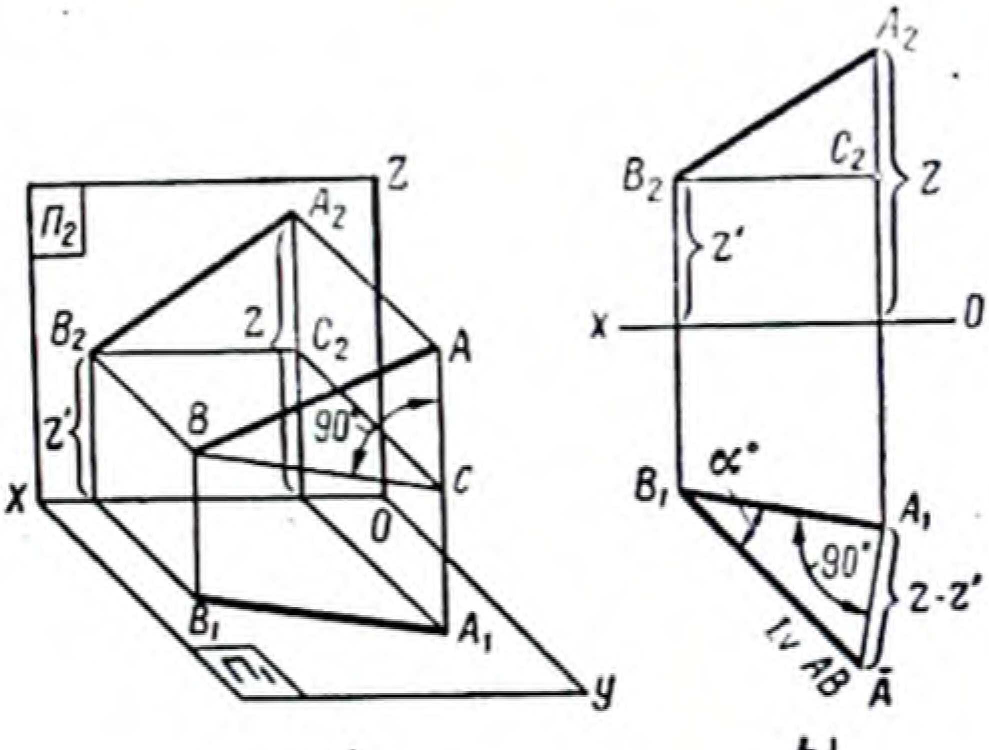
On this basis the construction of the true length of AB in the multiview drawing (Fig. 51, *b*) is carried out as follows. Taking the horizontal projection A_1B_1 as one side of a right triangle, through point A_1 draw a perpendicular to A_1B_1, lay off on it from point A_1 the length $A_1\bar{A}$ equal to the difference coordinates z and

---

* The method of finding the true length of a line by means of conversion of its projections is given in Chapter VI.

z', using the given coordinates of points A and B. The point $\bar{A}$ obtained is joined to point B_1. The hypotenuse $\bar{A}B_1$ of the right triangle constructed gives the true length of AB.

The true length of a line may be determined by constructing a right triangle if the frontal projection of the line serves as the first side. In this case the length of the second side will be equal to



a) b)

Fig. 51

the difference $y - y'$ determined by means of the given coordinates of A and B. This construction is shown in Fig. 55.

Thus the true length of a line in the multiview drawing is constructed as the hypotenuse of a right triangle, one side of which is equal in length to one of the projections of the line and the second side of which is equal to the difference between the distances of the extreme points of the line from the plane of projection.

2. The true angle of inclination of a line to the plane of projection may also be determined by constructing a right triangle. In Fig. 51, *a* angle ABC is the angle between line AB and plane Π_1. Its true value is given in Fig. 51, *b* by angle $A_1B_1\bar{A}$, i. e., by the acute angle α, formed by the horizontal projection A_1B_1, and the hypotenuse of the right triangle $A_1B_1\bar{A}$, with the help of which the true length of line AB was found.

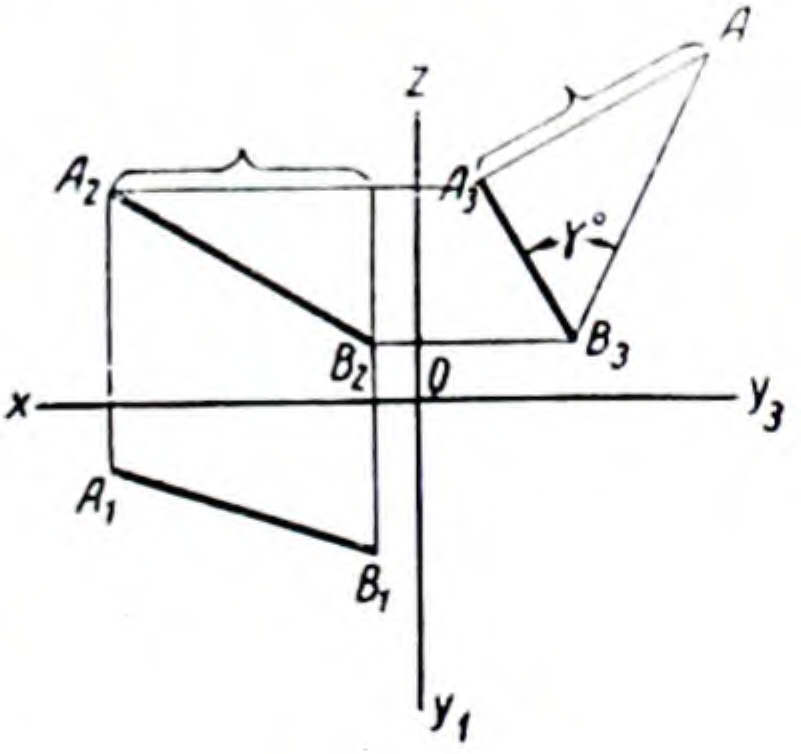


Fig. 52

The true values of angles β and γ, i. e., the angles of inclination of line AB to the planes of projection Π_2 and Π_3 respectively, can be determined by constructing right triangles. To find $\beta°$, the angle of inclination of AB to Π_2, the first side should be the frontal projection. To find γ, the angle of inclination of AB to Π_3, the side should be the profile projection.

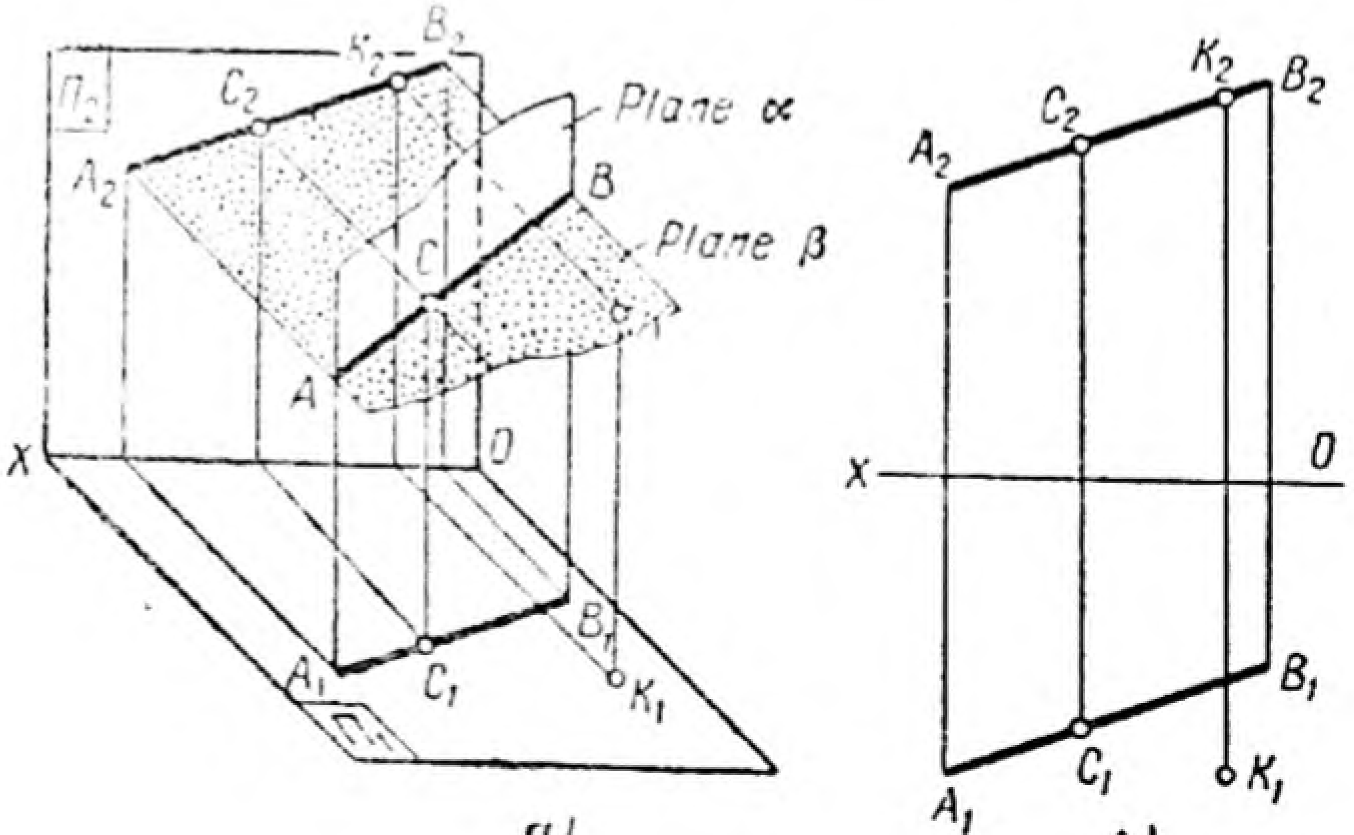
Example. It is required to determine $\gamma°$, the angle of inclination of line AB to the profile plane of projection Π_3 (Fig. 52).

Construct the right triangle $A_3B_3\bar{A}$, the first side of which will be the profile projection of line AB. The second side will be given by the difference between the distances of the extreme points of A_2 and B_2. The angle γ formed between projection A_3B_3 and hypotenuse $B_3\bar{A}$ is the required angle.

§ 9. Relative Position of Points and Lines

If a point in space lies on a line, the projections of that point also lie on the corresponding projections of the line. To demonstrate this let there be given the line AB, its projections A_1B_1 and A_2B_2 and a point C on line AB (Fig. 53). To obtain the projections of point C draw projectors CC_1 and CC_2. The projectors lie respectively in planes α and β, which project line AB onto planes Π_1 and Π_2. It follows that the projectors pierce the planes of projection Π_1 and Π_2 in points C_1 and C_2, and these points lie on the corresponding lines of intersection of planes α and β with the planes of projection Π_1 and Π_2, in other words they lie on the projections A_1B_1 and A_2B_2, which is what had to be proved.

If one projection of the point lies on the corresponding projection of a line, while another projection of the point does not lie on the corresponding projection of the line, then the point does not



a)   b)

Fig. 53

lie on the given line in space. Such a case is also shown in Fig. 53, where the frontal projection K_2 of point K lies on the frontal projection A_2B_2 of line AB, while the point K itself does not lie on the line AB, since the horizontal projection K_1 of point K does not lie on the horizontal projection A_1B_1 of the line AB.

Example. It is required to mark on the given line AB a point C at an altitude of 15 mm from plane Π_1 (Fig. 54).

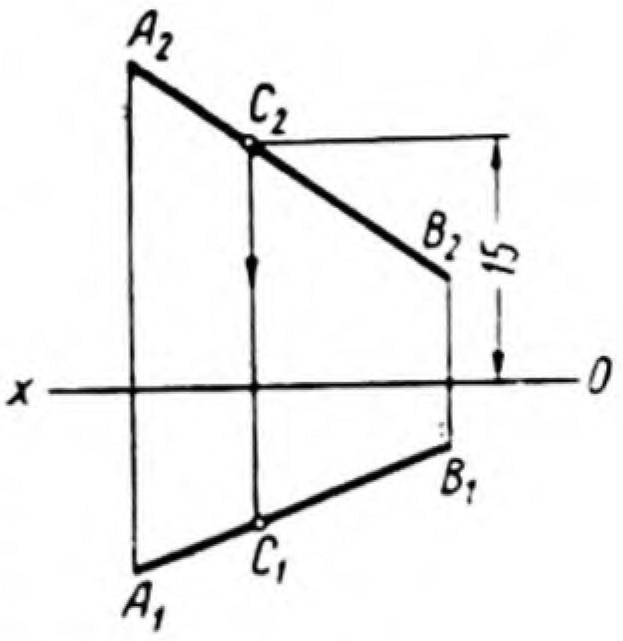


Fig. 54

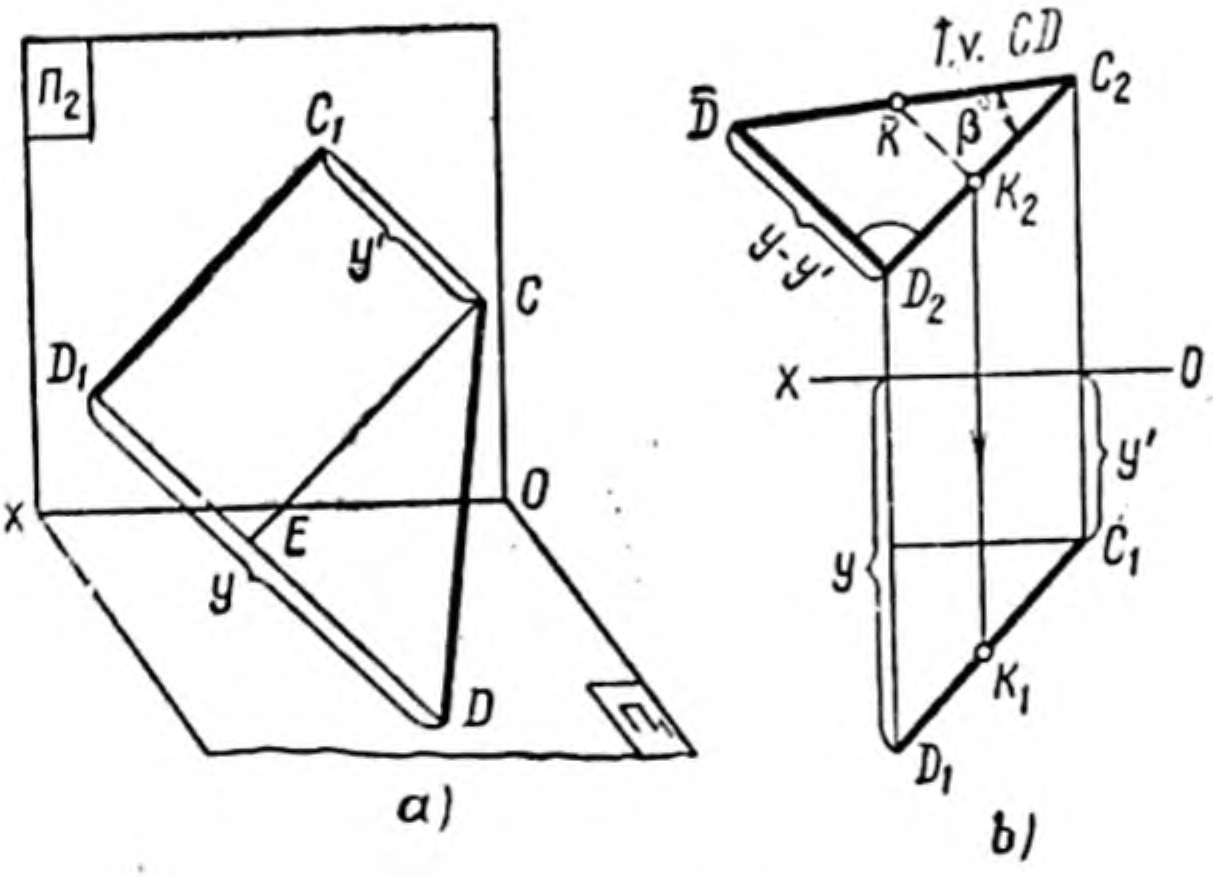
At a distance of 15 mm from the ground line Ox draw a line parallel to it to intersect the frontal projection A_2B_2 of line AB.

The point of intersection C_2 is the frontal projection of the required point C. The horizontal projection C_1 of point C is given by the intersection of a projector drawn through C_2, perpendicular to the ground line Ox, with the horizontal projection A_1B_1 of line AB.

§ 10. The Division of a Line in a Given Ratio

If a line in space is divided by a point in the ratio $m:n$, the projections of this line are also divided by the corresponding projections of that point in that same ratio $m:n$.

In the triangle $C_2D_2\bar{D}$ (Fig. 55, *b*), with the help of which the true length $C_2\bar{D}$ of line CD has been found, draw a straight line $K_2\bar{K}$ parallel to the side $D_2\bar{D}$. This line $K_2\bar{K}$ cuts off a triangle $C_2K_2\bar{K}$. This triangle is similar to triangle $C_2D_2\bar{D}$, and it follows



a) b)

Fig. 55

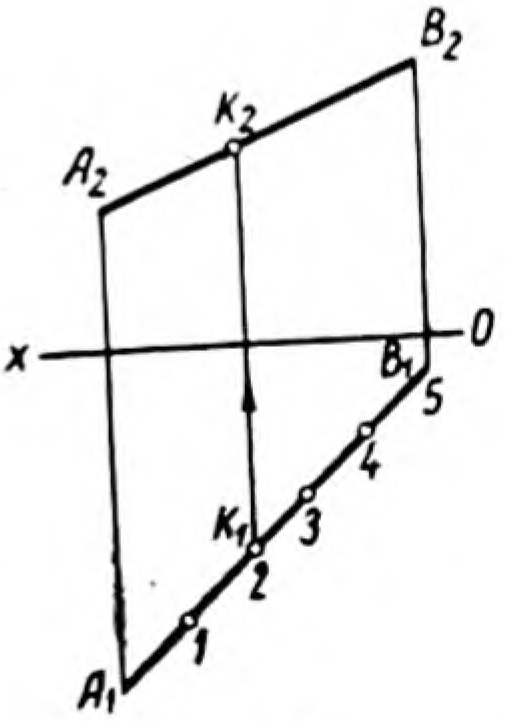


Fig. 56

that $D_2K_2 : K_2C_2 = \overline{DK} : \overline{KC}_2$; in other words, the projection K_2 of the point K divides the projection C_2D_2 of the line CD in the same ratio as point K divides line CD.

Example. It is required to divide line AB in the ratio $2:3$ (Fig. 56).

Divide one of the projections, for example A_1B_1, into five equal parts. From point A_1 step off two parts, thus obtaining point K_1 and through K_1 perpendicular to the ground line Ox, draw a projector to cut A_2B_2 in point K_2. The points K_1 and K_2 are projections of point K dividing the line AB in the ratio $2:3$ as was required.

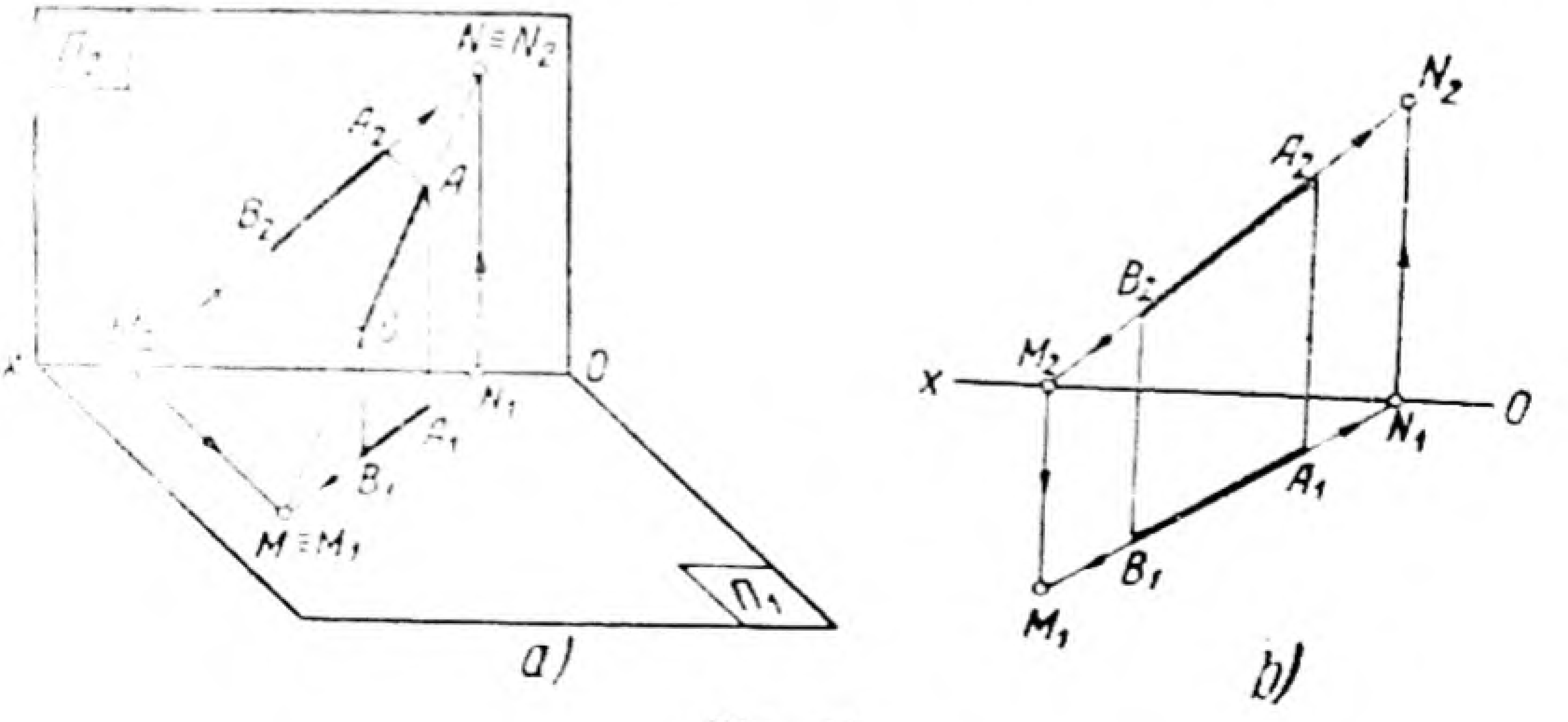
§ 11. Traces of Lines

The point in which a line pierces a plane of projections is called a *trace of that line*.

In a system containing three planes of projection Π_1, Π_2 and Π_3 an oblique line has three traces — the horizontal, frontal and profile. A line parallel to one of the planes of projection has two traces, and a line parallel to two planes of projection has one trace.

In Fig. 57 an oblique line AB is shown. It pierces plane Π_1 in point M and plane Π_2 in point N. Point M is therefore the horizontal trace of the line AB and point N, its frontal trace.

In the multiview drawing (Fig. 57, *b*) the horizontal trace of line AB can be found by determining the projection of the point contained both in the given line and in plane Π_1. In the frontal projection plane Π_2 the projection of such a point is the point M_2, in which the frontal projection A_2B_2 of the line AB intersects the ground line Ox on which lie the frontal projections of all points contained in plane Π_1, including point M. The horizontal projection M_1 must lie on the projector drawn from point M_2 perpendicular to the ground line Ox. It must, at the same time, lie on the projec-



a) b)

Fig. 57

tion A_1B_1. Point M_1 is therefore obviously the point of intersection of those two lines.

It follows that the projections of the horizontal trace M of line AB may be found as follows:

1. Extend the frontal projection A_2B_2 to the ground line Ox. The point M_2 obtained is the frontal projection of the required trace M.

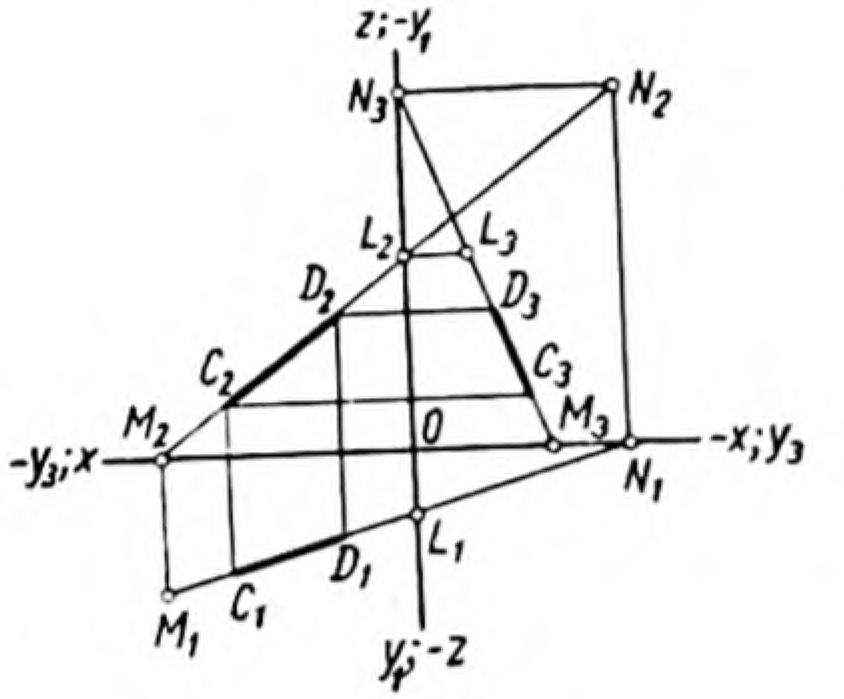


Fig. 58

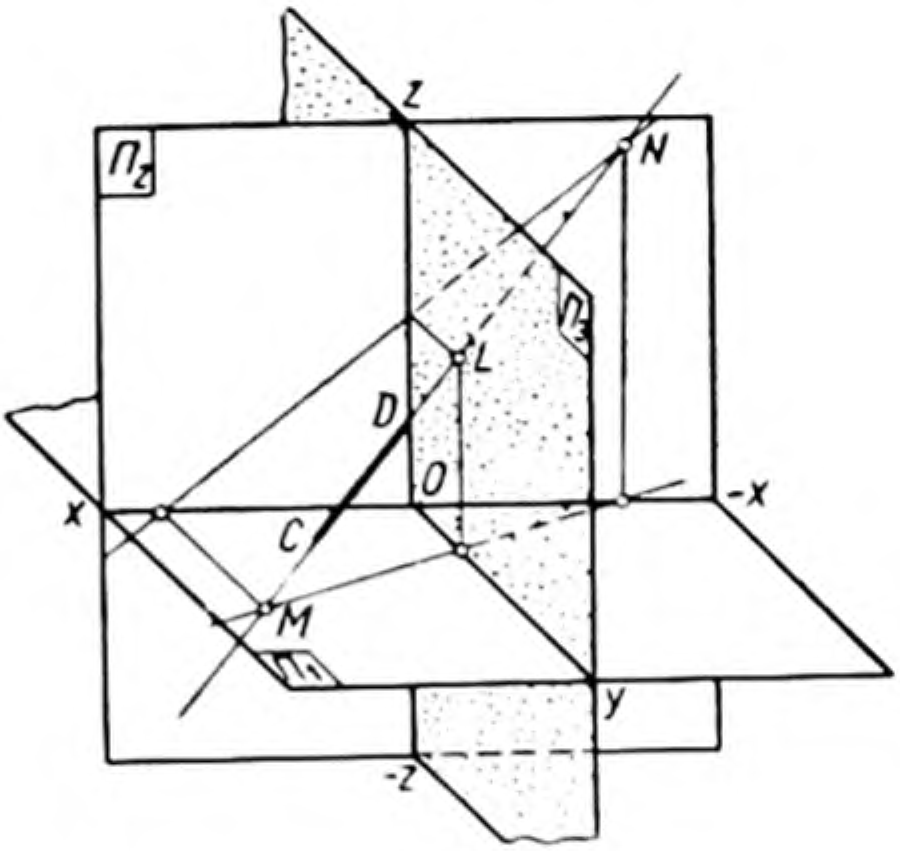


Fig. 59

2. Through point M_2 draw a perpendicular to the ground line Ox.

3. Produce the horizontal projection A_1B_1 of the line to intersect the perpendicular in point M_1. This point is the horizontal projection of the required trace M and it coincides with the trace M itself ($M_1 \equiv M$).

The frontal trace N of line AB is found in a similar manner. First its horizontal projection N_1 is found. This is the point of intersection of the horizontal projection A_1B_1 of line AB with the ground line Ox, since line Ox contains the horizontal projections of all points of plane Π_2. Then, through N_1 draw a line perpendicular to Ox to intersect the frontal projection A_2B_2 of line AB in N_2.

The construction of the profile traces of straight lines is carried out by similar constructions. The multiview drawing (Fig. 58) shows three projections of each of the traces, M, N and L, of the line CD. To construct the profile trace L the point L_2 is found. This is the point where the frontal projection C_2D_2 cuts line Oz. This point L_2 is the frontal projection of the profile trace of line CD.

A perpendicular to the line Oz drawn through point L_2 cuts the profile projection C_3D_3 of the line in point L_3 which is the profile trace of line CD and coincides with the profile trace itself ($L_3 \equiv L$).

The horizontal projection L_1 of the profile trace is the point of intersection of the horizontal projection C_1D_1 of the line with the ground line Oy_1.

Attention should be paid to the fact that two of the three projections of each trace are situated on the respective ground lines (Ox, Oy, Oz), while the third coincides with the trace itself.

The traces M, N and L divide the line into parts situated in different octants. If the line CD, the projections of which are shown in Fig. 58, is considered as moving from left to right, it passes consecutively through the fourth, first, fifth and sixth octants.

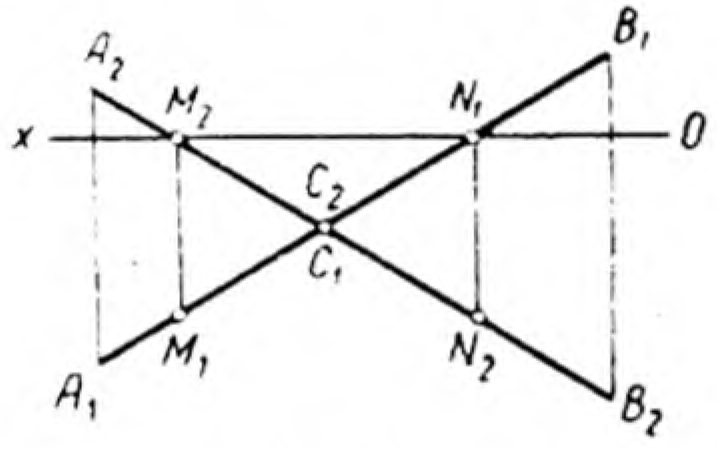


Fig. 60

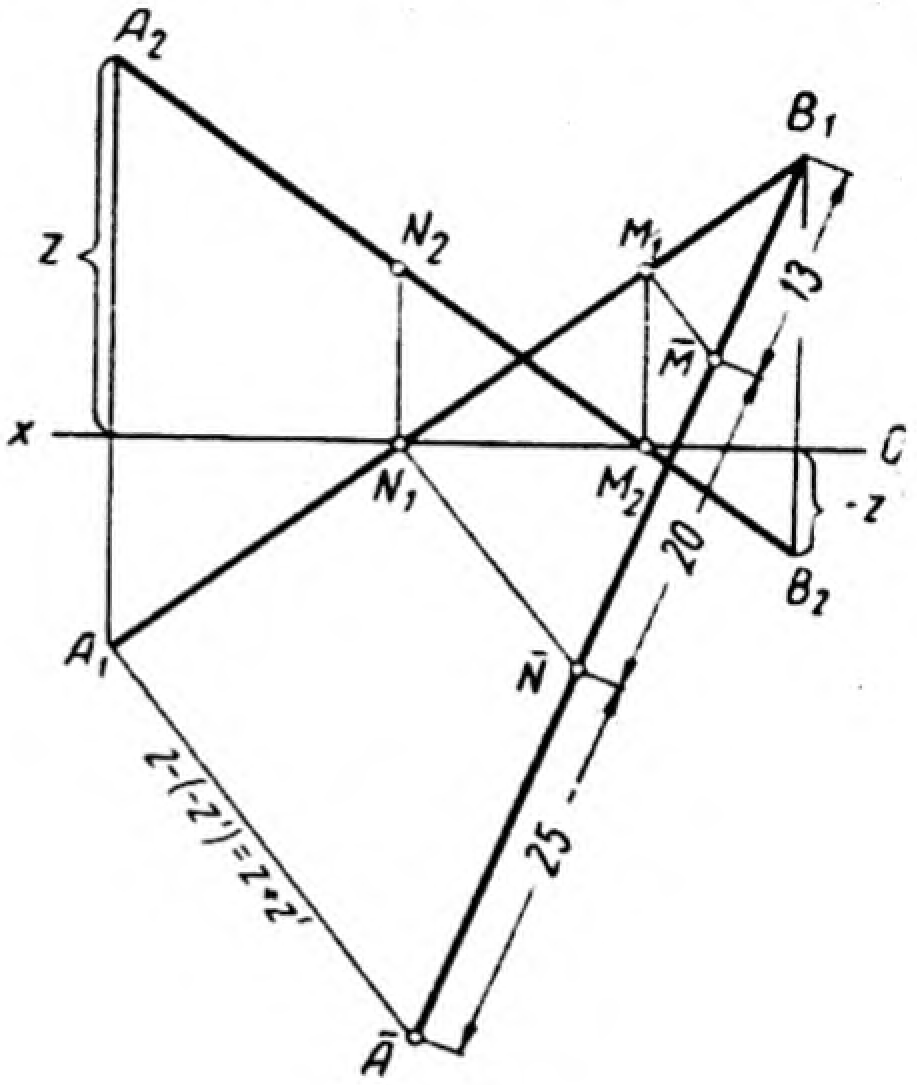


Fig. 61

In Fig. 59 the space drawing of line CD is shown with its traces M, N and L.

Example 1. Determine through which quadrants line AB passes (Fig. 60).

Find the traces of line AB. The intersection of the frontal projection A_2B_2 with the ground line Ox gives the frontal projection M_2 of the horizontal trace M. By drawing a perpendicular from point M_2 to the ground line Ox to intersect A_1B_1, point M_1, the horizontal projection of the trace, is obtained. It coincides with the trace M itself. Point N_1, where the horizontal projection A_1B_1 cuts the ground line Ox, is the horizontal projection of the frontal trace. A perpendicular to the ground line Ox from point N_1 intersects A_2B_2 in point N_2, which is frontal projection of the frontal trace of line AB and coincides with the frontal trace N itself.

Point A lies in the first quadrant, since the projection A_1 lies under the ground line Ox and the projection A_2 lies above that line. Point B lies in the third quadrant. The intermediate point C of line AB is located in the fourth quadrant, since its projections C_1 and C_2 lie below the ground line Ox. The part AM of line AB is located in the first quadrant, part MN lies in the fourth and part

NB — in the third quadrant. So line AB passes through the first, fourth and third space quadrants.

Example 2. Determine the true length of the parts of line AB, located in the various quadrants (Fig. 61).

The true lengths of AB and its parts will be determined by the right triangle method. Let the projection A_1B_1 be the first side of the triangle. From point A_1 draw a perpendicular to A_1B_1 and from point A_1 lay off on it a distance equal to the difference between the distances of points A_2 and B_2 from Ox, i. e., $z - (-z') = z + z'$. Point $\overline{A}$ thus obtained is joined to point B_1. The length of $\overline{A}B_1$ is the true length of line AB.

To determine the lengths of the parts of line AB lying in the different space quadrants, through the horizontal projections M_1 and N_1 of the traces M and N draw perpendiculars to the horizontal projection A_1B_1. These perpendiculars intersect line $\overline{A}B_1$ in points $\overline{M}$ and $\overline{N}$, respectively, and divide $\overline{A}B_1$ in the same ratio as points M_1 and N_1 divide the projection A_1B_1 of line AB. Measuring the distance between points $\overline{A}$, $\overline{N}$, $\overline{M}$ and B_1 the length of the part AN lying in the first quadrant is found to be 25 mm. The length of part NM in the second quadrant is 20 mm and, finally, the length of part MB located in the third space quadrant is 13 mm.

Problems and Exercises

1. How is the true length of a line determined by means of the right triangle method?

2. How can a line given in the multiview drawing by two projections be divided in the ratio $m:n$?

3. How are the horizontal and profile traces of a line constructed?

4. Line AB is given by the coordinates of its extremities A (15; 15; 12) and B (5; 8; 4): a) find the true length of AB and the angles of inclination of line AB to planes Π_1, Π_2 and Π_3, i. e., angles α, β and γ; b) indicate through which octants line AB passes.

Hint. To answer this question (b) construct the traces M, N and L of the assigned line AB.

5. Through point A (5; 8; 6) draw a line, the traces of which on planes Π_1 and Π_2 lie at the same distances from the ground line Ox.

Hint. The required line in the profile plane of projection Π_3 will pass through point A_3 making an angle of 45° with the axes Oz and Oy_3.

§ 12. Projections of Parallel, Intersecting and Non-intersecting Lines

Two lines may occupy the following relative positions in space: 1) they may be parallel, 2) they may intersect, or 3) not intersect.

1. The corresponding projections of parallel lines are parallel to each other.

If through the given parallel lines AB and CD projecting planes are passed, they are parallel and their intersections with the plane of projection Π_1 give two parallel lines A_1B_1 and C_1D_1. These are the orthographic projections of the given lines AB and CD on plane Π_1 (Fig. 62). In the multiview drawing (Fig. 63) it can be seen that corresponding projections of parallel lines AB and CD are themselves parallel, i. e., $A_1B_1 \| C_1D_1$, $A_2B_2 \| C_2D_2$ and $A_3B_3 \| C_3D_3$.

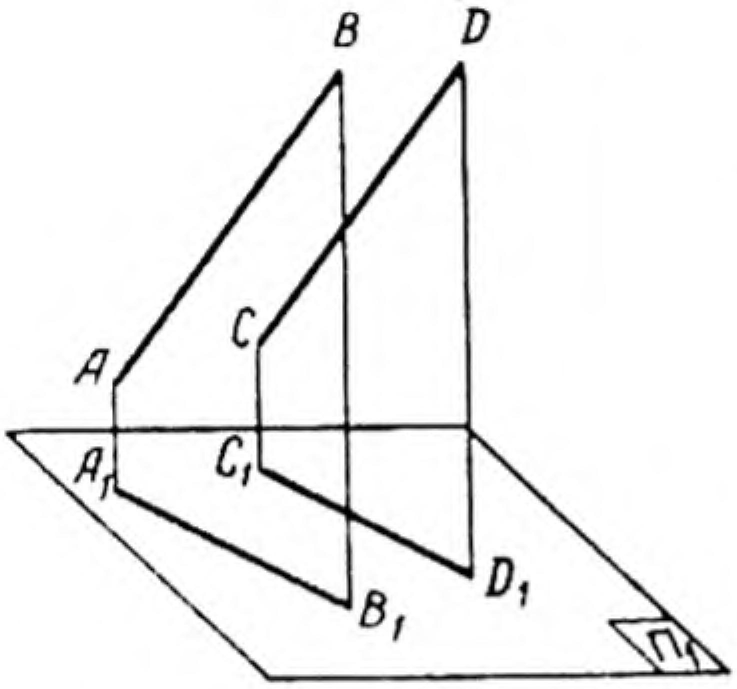

B
D
C
A
A_1
C_1
D_1
B_1
Π_1

Fig. 62

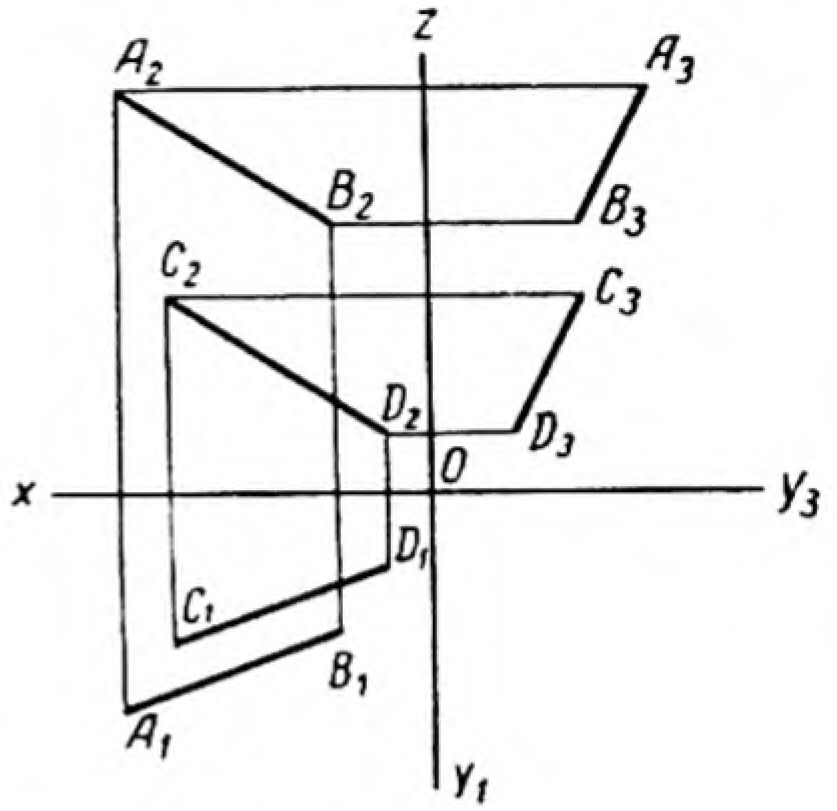
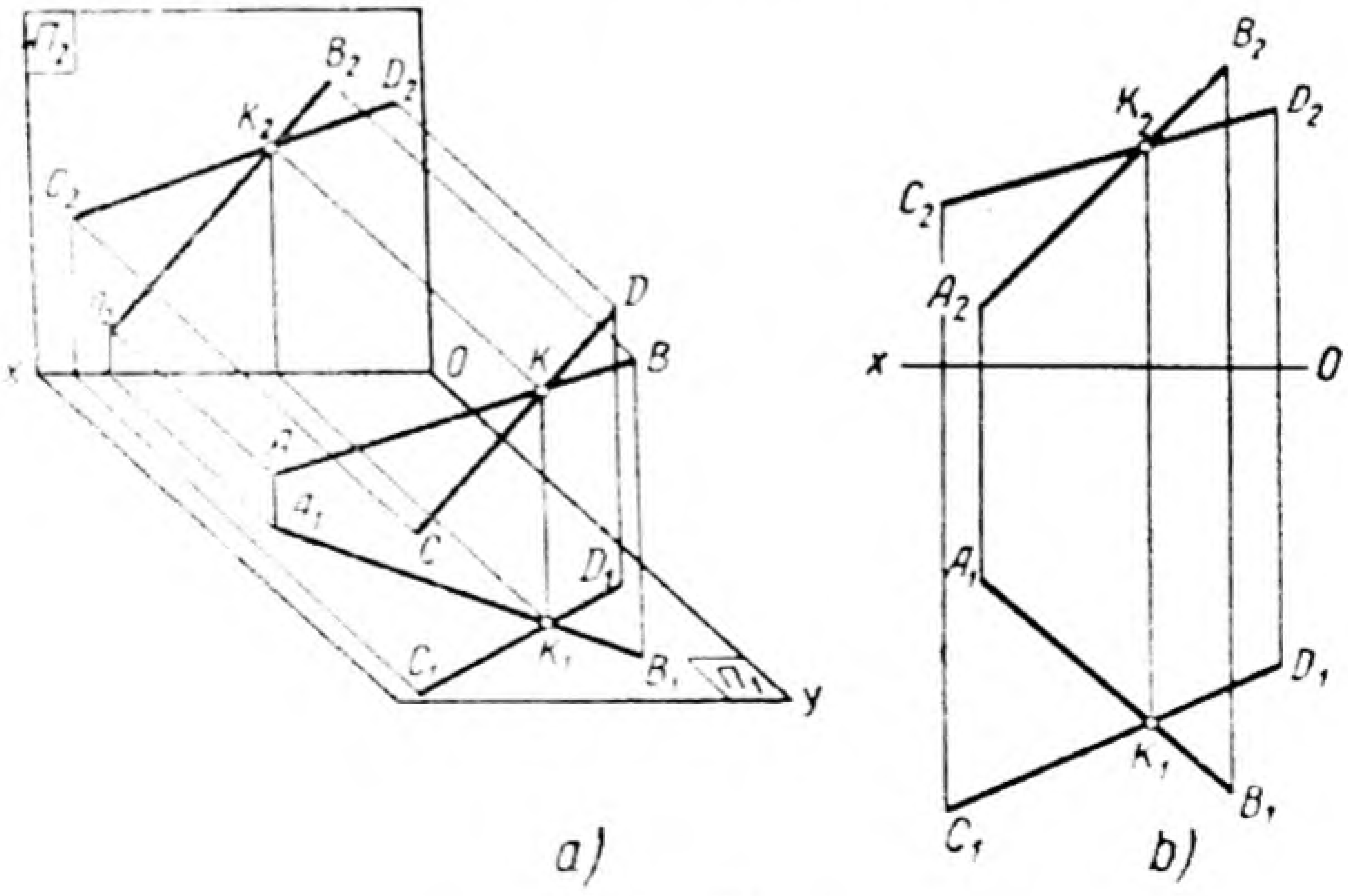

Z
A_2
A_3
B_2
B_3
C_2
C_3
D_2
D_3
O
x
y_3
D_1
C_1
B_1
A_1
y_1

Fig. 63

2. Intersecting lines, for example AB and CD (Fig. 64), have a common point K. Therefore the horizontal projection K_1 and frontal


Z
Π_2
B_2
D_2
K_2
C_2
A_2
x
O
D
K
B
A
A_1
C
D_1
C_1
K_1
B_1
Π_1
y
B_2
K_2
D_2
C_2
A_2
x
O
A_1
D_1
K_1
C_1
B_1

a)

b)

Fig. 64

projection K_2 of this point must lie at the intersection of the corresponding projections of the given lines.

The points K_1 and K_2, being projections of one and the same point K in space, in the multiview drawing (Fig. 64, *b*) are situated on one and the same perpendicular to the ground line.

This property of the intersecting lines is also true for a combination of other projections, for example, their frontal and profile projections.

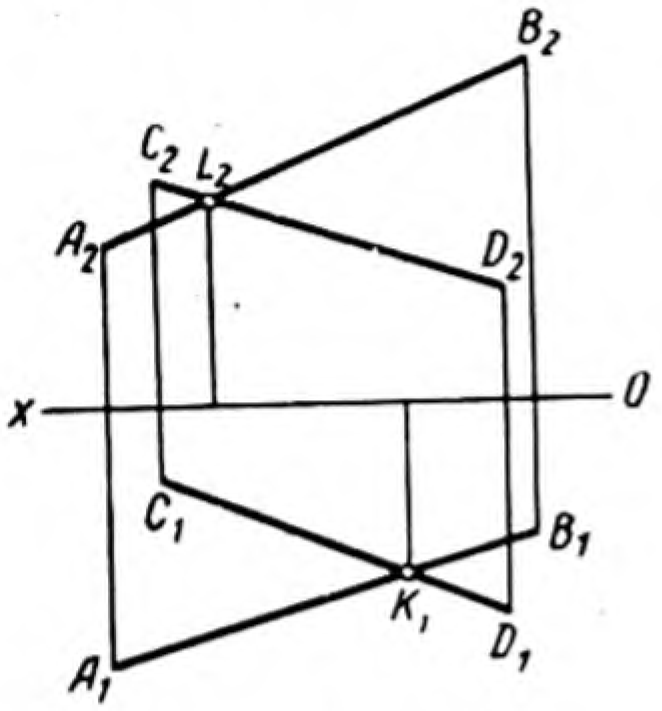


Fig. 65

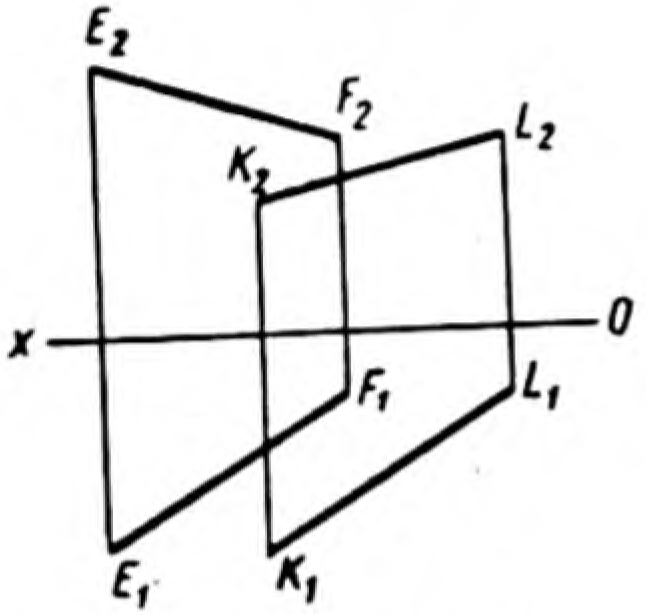


Fig. 66

3. If two lines are not parallel and do not intersect, they are called *non-intersecting* or *skew*.

In the multiview drawing (Fig. 65) the points of intersection (K_1 and L_2) of corresponding projections of non-intersecting lines AB and CD (A_1B_1 with C_1D_1 and A_2B_2 with C_2D_2) do not lie on one perpendicular to the ground line Ox.

Through point A in space a line parallel to CD is passed and through point C, a line parallel to AB. To avoid overburdening the drawing, Fig. 65, these lines are not shown. The planes formed by the two pairs of mutually parallel lines are parallel to each other. These planes are called the *planes of parallelism* of non-intersecting lines.

If the planes of parallelism of two non-intersecting lines are perpendicular to one of the planes of projection (Π_1, Π_2 or Π_3), then on that plane of projection the non-intersecting lines are projected as parallel lines (Fig. 66).

If the planes of parallelism of non-intersecting lines are parallel to one of the planes of projection (Π_1, Π_2 or Π_3), then the non-intersecting lines are projected onto the remaining two planes of projection as parallel lines perpendicular to the ground line (Fig. 67).

Example 1. At a distance of 14 mm from plane Π_2 draw a frontal to intersect two parallel lines AB and CD (Fig. 68).

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Draw a line at a distance of 14 mm below the ground line Ox and parallel to it. This line intersects the horizontal projections A_1B_1 and C_1D_1 of the given lines in points K_1 and L_1. The line K_1L_1 is the horizontal projection of the frontal. The frontal projections of points K and L are found in the usual manner. These are found at the intersection of projections A_2B_2 and C_2D_2 with verticals passing through points K_1 and L_1. Points K_2 and L_2 are then joined up. Line K_2L_2 is the frontal projection of the frontal KL.

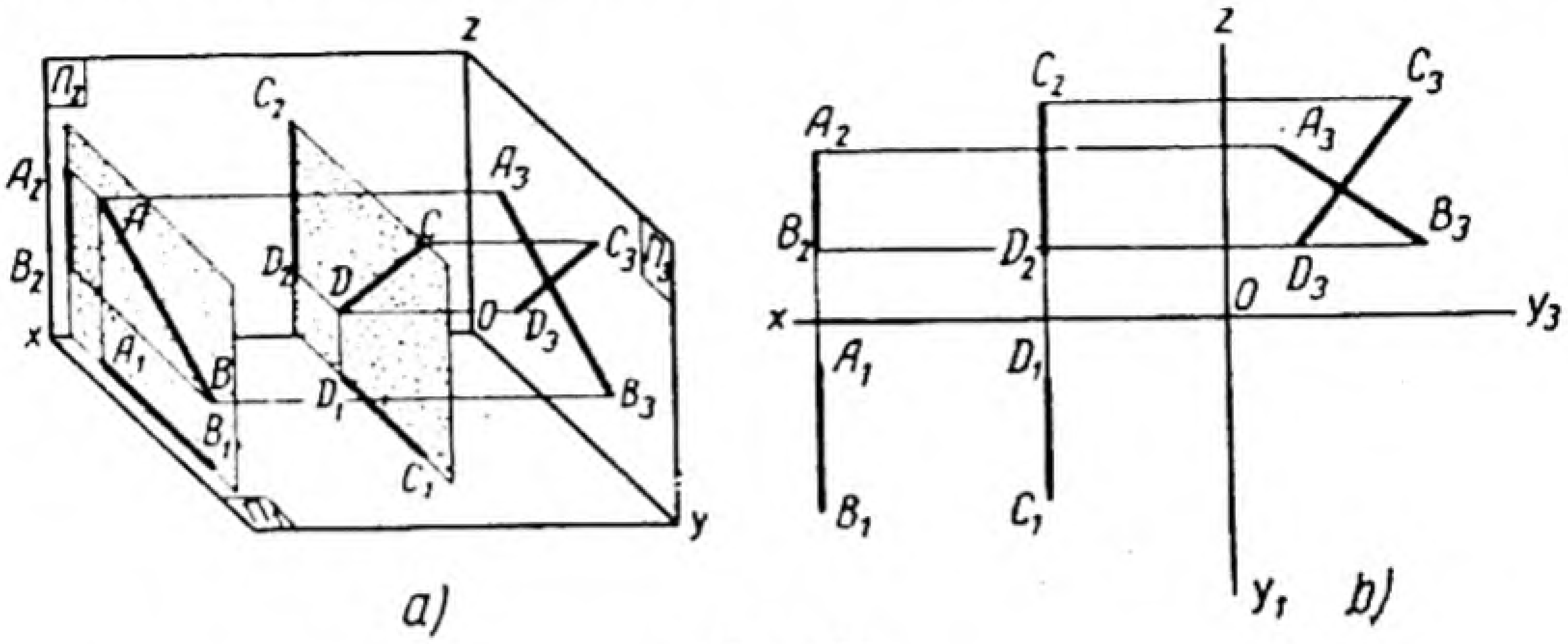

z
П2
C2
A2
A
A3
C
C3
П3
B2
D2
D
O
D3
x
A1
B
D1
B3
B1
C1
П1
y
a)
z
C2
C3
A2
A3
B2
D2
B3
O
D3
x
y3
A1
D1
B1
C1
y1
b)

Fig. 67

Example 2. Construct a parallelogram, one diagonal of which is line AC (Fig. 69).

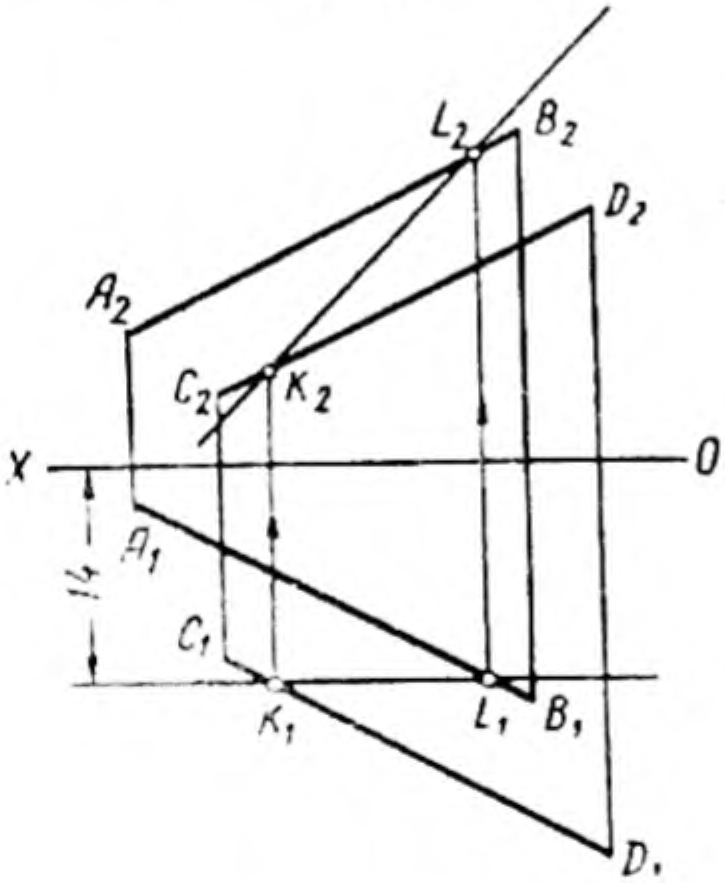

L2
B2
D2
A2
C2
K2
X
O
14
A1
C1
K1
L1
B1
D1

Fig. 68

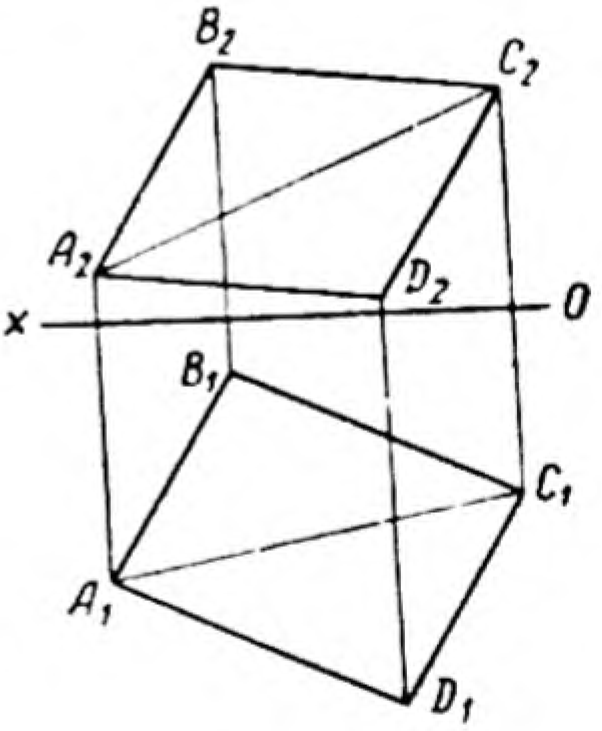

B2
C2
A2
D2
x
O
B1
C1
A1
D1

Fig. 69

Since in a parallelogram opposite sides are equal and parallel, we may take line AB of any length and direction as the side of the parallelogram. Then through point C pass line CD equal and parallel to line AB. Joining up corresponding projections of the pairs of points A and D and B and C, we obtain two projections of the parallelogram $ABCD$, which meet the stipulated condition.

§ 13. Projections of Plane Angles

1. A plane or linear angle formed by two straight lines, whether it is an acute, right or obtuse angle, is projected onto a plane in its true size, if both of its sides are parallel to that plane. This statement follows from the theorem of elementary geometry which states that angles formed by mutually parallel and similarly directed sides are equal.

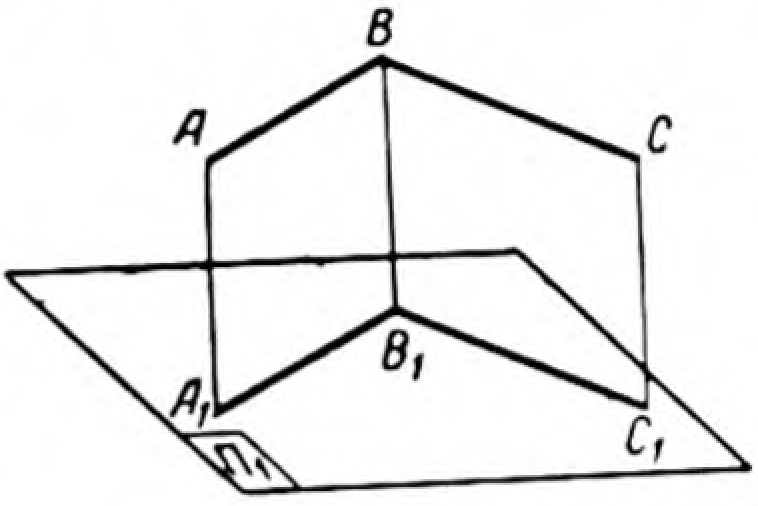
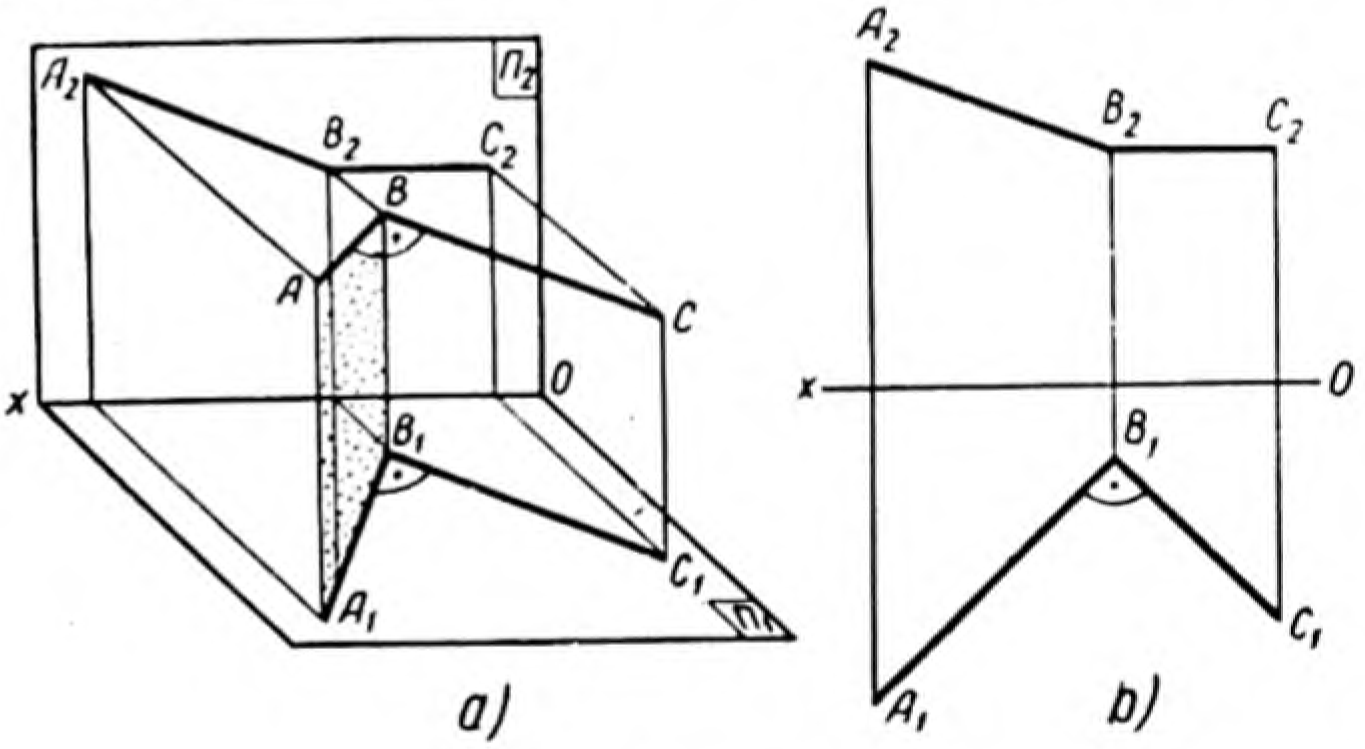


Fig. 70

In Fig. 70 the sides of the angle *ABC* are parallel to the plane of projection Π_1. Orthographic projections of lines which are parallel to the plane of projection Π_1 are parallel to the lines themselves, since they are the opposite sides of rectangles formed by these lines, their projections and the projectors, i. e., $A_1B_1 \parallel AB$ and $B_1C_1 \parallel BC$. So the angles $A_1B_1C_1$ and *ABC*, the corresponding sides of wich are parallel, are equal. .

Nevertheless it should be noted that to obtain the projection of an angle equal to the angle being projected, the parallel disposition of the sides of the latter relative to the plane of projection is a condition which, although sufficient, is not essential. It has been proved by Prof. N. Chetveroukhin in his theorem* that it is possible



a)

b)

Fig. 71

for linear angles in space to occupy such positions that, while neither of their sides is parallel to the plane of projection, nevertheless the angles are projected orthographically onto the given plane in their true size. In such a case the inclination of the sides of the given angle to the plane of projection will necessarily be different.

---

* "Any angle α $(0 < \alpha° < 180°)$ may be considered as being the orthographic projection of any other angle β $(0 < \beta° < 180°)$."

2. If one of the sides of a right angle is parallel to the plane of projection, the right angle is projected onto the plane in its true size.

A right angle ABC in space is shown in Fig. 71, *a*. Its side BC is parallel to the horizontal plane of projection Π_1 and its side AB is neither parallel nor perpendicular to that plane. In this case angle ABC is projected onto plane Π_1 in its true size.

This may be demonstrated as follows: the side BC is perpendicular to AB and to BB_1, since $BB_1 \perp \Pi_1$, and $BC \parallel \Pi_1$. Therefore BC is also perpendicular to the plane ABB_1A_1 which in Fig. 71, *a* is shaded. The projection B_1C_1 is parallel to BC itself. Therefore the projection B_1C_1 is also perpendicular to the plane ABB_1A_1, and so, perpendicular to the line A_1B_1 contained in that plane. In other words, angle $A_1B_1C_1$ is a right angle.

From Fig. 71, *b* it may be seen that the right angle ABC is projected onto the horizontal plane Π_1 without distortion, i. e., angle $A_1B_1C_1 = 90°$, precisely because the side BC of the original angle is parallel to plane Π_1.

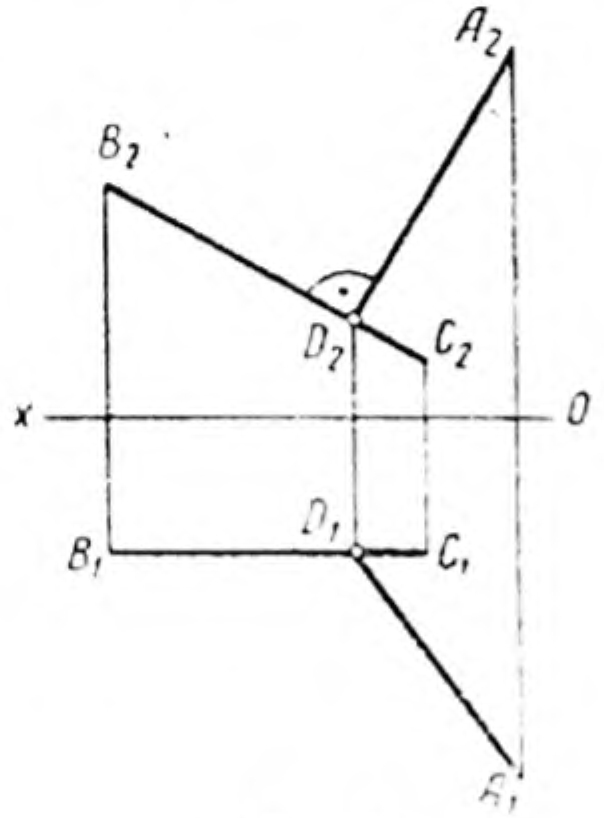


Fig. 72

On the frontal plane of projection Π_2 angle ABC is projected as angle $A_2B_2C_2$ which is not equal to 90°. This is because neither of the sides of angle ABC is parallel to plane Π_2. Angle $A_2B_2C_2$ may vary from 0 to 180°, depending on the position of its sides.

Example. From a point A draw a line perpendicular to the frontal BC (Fig. 72).

Since in this case the right angle will be projected onto plane Π_2 in its true size, we draw from point A_2 a perpendicular to B_2C_2 and mark the point D_2, where it intersects with B_2C_2. Having

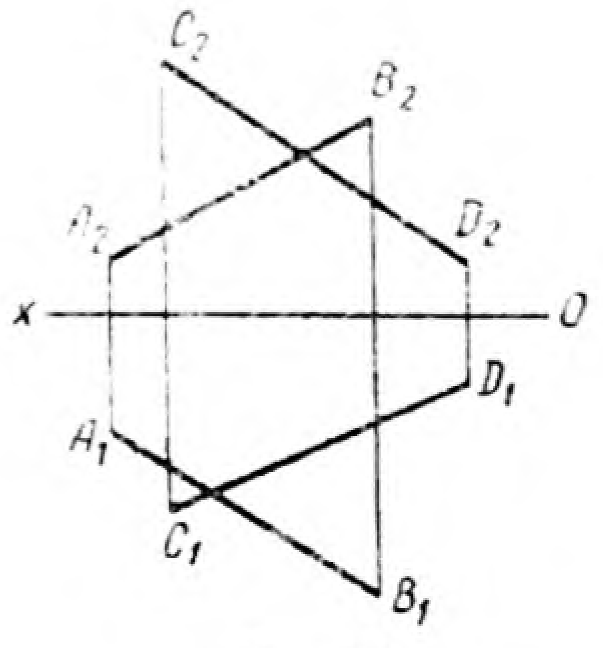


Fig. 73

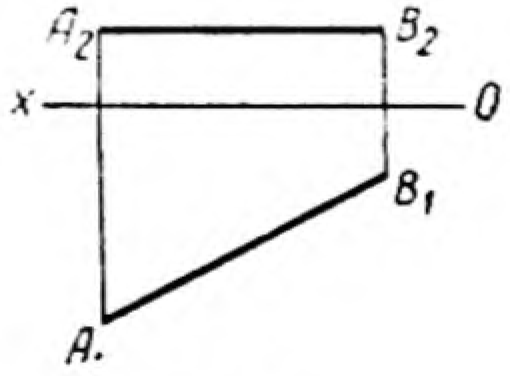


Fig. 74

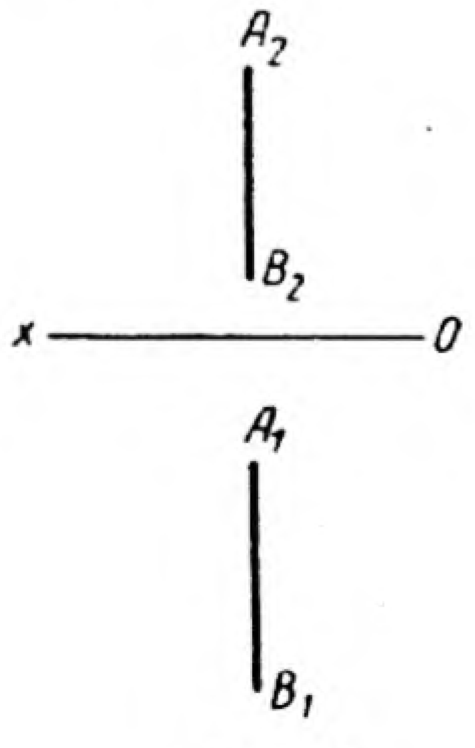


Fig. 75

found with the help of a projector point D_1, points A_1 and D_1 are then joined up. Then lines A_1D_1 and A_2D_2 are the projections of line AD which is perpendicular to BC.

Problems and Exercises

1. Prove that corresponding projections of parallel lines are parallel to each other.

2. How are two intersecting lines shown in the multiview drawing?

3. How are two non-intersecting lines shown in the multiview drawing?

4. When is a plane right angle projected in true size: a) onto one plane of projection, b) onto two planes of projection?

5. Draw a horizontal to cut the non-intersecting lines AB and CD and determine the distance between the intersection points obtained (Fig. 73).

6. Construct a triangle with sides of different lengths and its base $AB \parallel \Pi_1$ (Fig. 74).

7. Determine the distance of profile line AB from the ground line Ox (Fig. 75).

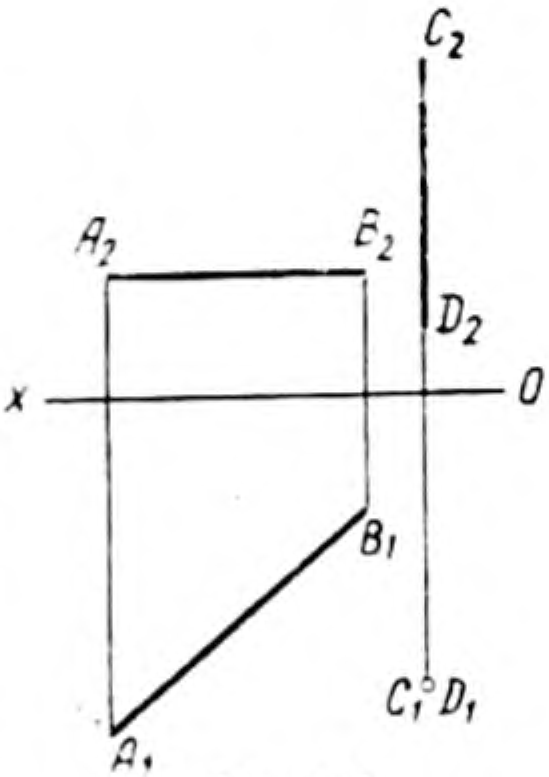


Fig. 76

8. Determine the distance between the non-intersecting lines AB and CD (Fig. 76).

9. Construct a rectangle $ABCD$ the side AB of which is a frontal line.

Chapter IV

PLANES

§ 14. Given Planes

1. In the multiview drawing a plane may be given by showing those geometrical elements by which the position of the plane in space is completely determined.

These elements are as follows:

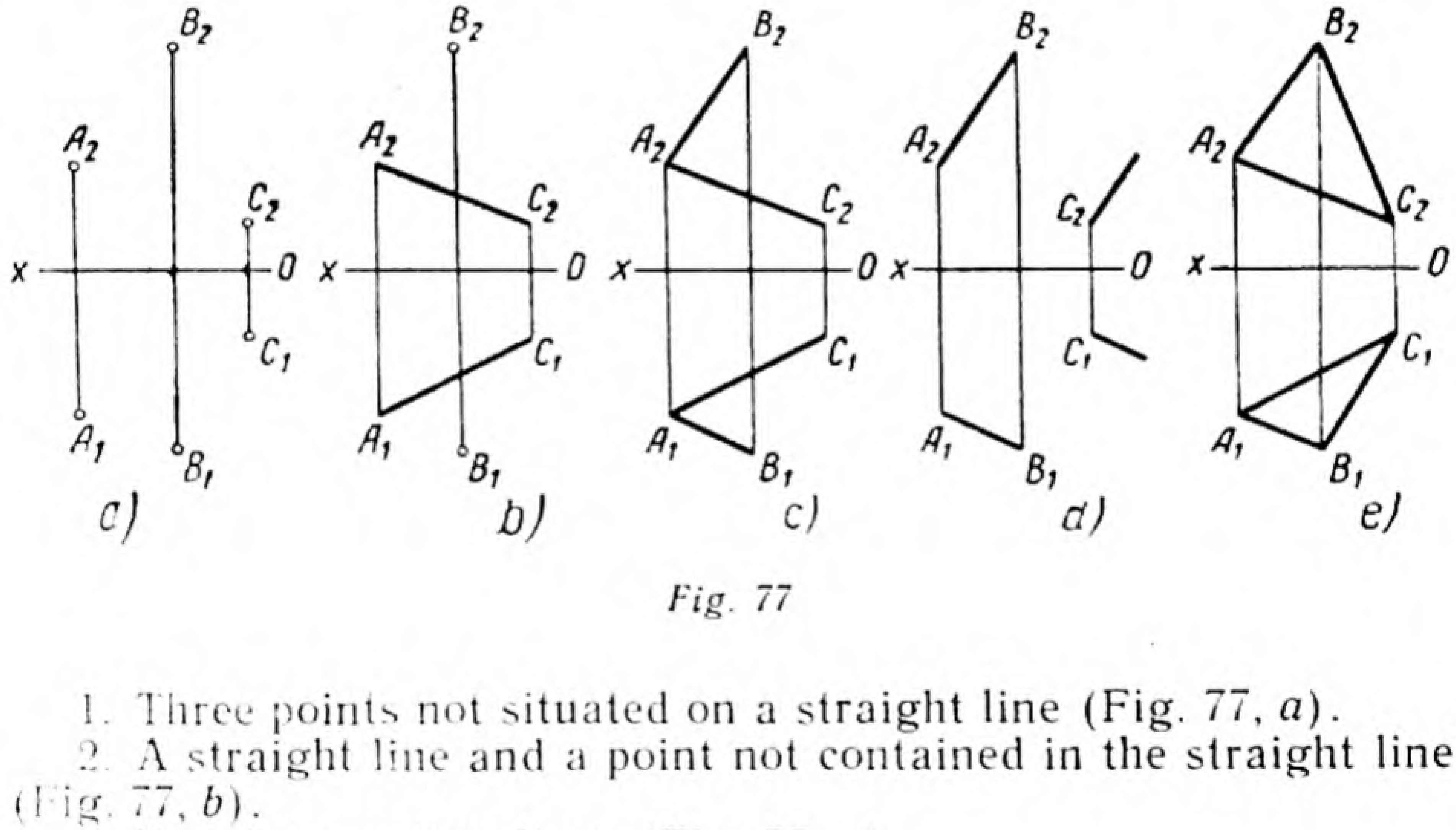


Fig. 77

1. Three points not situated on a straight line (Fig. 77, *a*).
2. A straight line and a point not contained in the straight line (Fig. 77, *b*).
3. Two intersecting lines (Fig. 77, *c*).
4. Two parallel lines (Fig. 77, *d*).
5. A triangle or any other plane figure (Fig. 77, *e*).

It is simple to change from one method of determining a plane to any other. It is sufficient, for instance, to join three given points by straight lines (Fig. 77, *a*) in order to obtain the representation of the planes of triangles (Fig. 77, *e*). Then again, for example, two points (of the three given) can be joined by a straight line and through the third point a line, parallel to the first, can be drawn. In this case instead of having a plane determined by three points, we have it determined by two parallel straight lines (Fig. 77, *d*). Other similar changes may be made.

2. The elements determining a plane (Fig. 77) are sufficient to allow straight lines and points contained in the plane to be constructed.

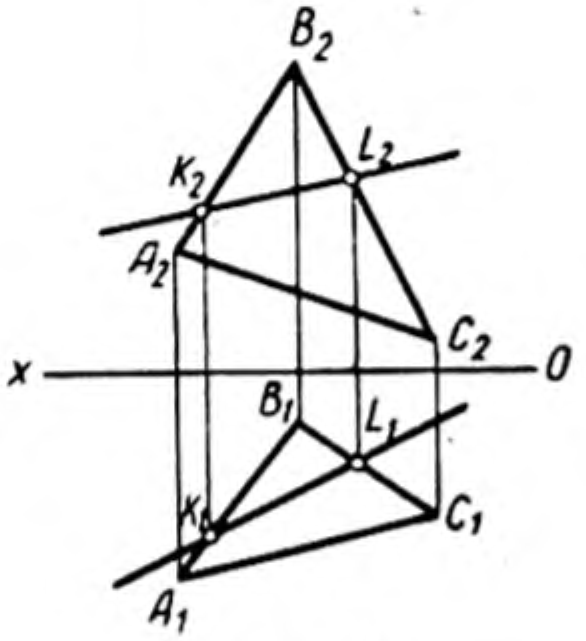


Fig. 78

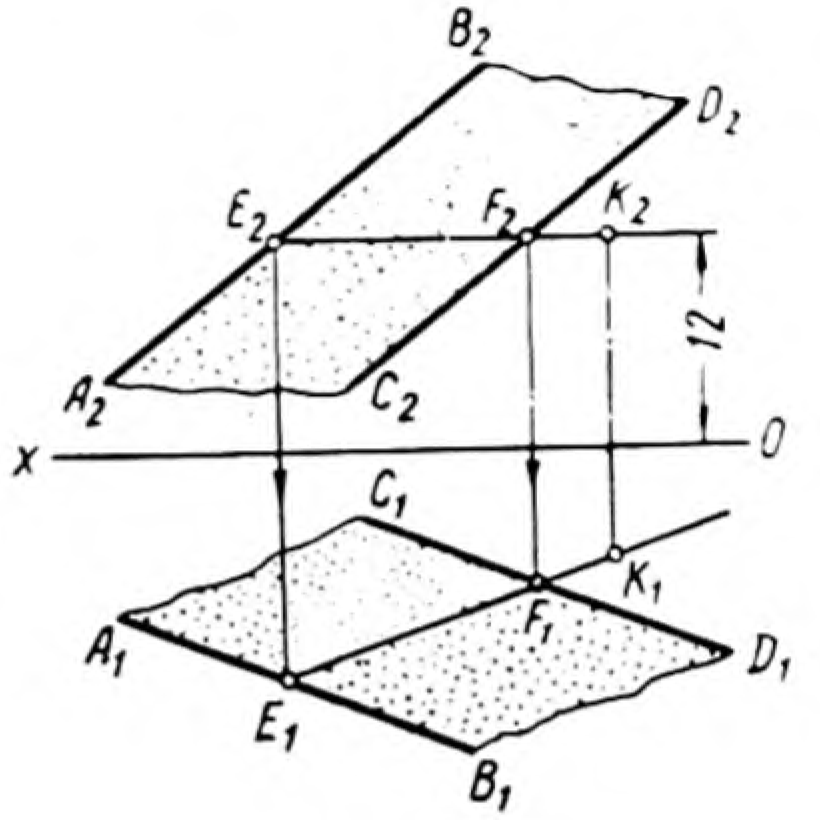


Fig. 79

For instance, if a plane is given in the multiview drawing by triangle *ABC* (Fig. 78), it is possible to construct the projections of any line, such as *KL*, lying in the given plane. A straight line is contained in a plane if two points of the line lie in that plane. Take two points *K* and *L* on the sides *AB* and *BC* of the triangle. In accordance with § 9 each projection of these points on corresponding projections of the sides *AB* and *BC* must lie on a line perpendicular to the ground line *Ox*, that is on a vertical projector. Then through the corresponding projections of points *K* and *L* the lines K_1L_1 and K_2L_2 are drawn. Line *KL* then lies in the plane of triangle *ABC*.

3. The construction of points contained in a given plane may be reduced in general to drawing an auxiliary line passing through a point and lying in a plane. For instance, if a plane is determined by the parallel lines *AB* and *CD* (Fig. 79), in order to construct a point *K* lying in the given plane and 12 mm from plane Π_1, first, at a distance of 12 mm from the ground line *Ox*, the frontal projection E_2F_2 of an auxiliary horizontal *EF* is drawn. Next, with the help of projecting lines passing through points E_2 and F_2, the points E_1 and F_1 are found, as shown by arrows. Thus the projection E_1F_1 is also found. Then on the projections E_1F_1 and E_2F_2 of line *EF*, the projections K_1 and K_2 of point *K* are marked, taking into account that line K_1K_2 must be perpendicular to the ground line *Ox*.

4. Traces of planes. When solving problems of descriptive geometry it is often convenient to determine a plane in the multiview drawing by giving its traces.

The *trace of a plane* is the straight line of intersection of that plane with the plane of projection.

A plane α and its traces — the horizontal trace k, the frontal trace l and the profile trace m — are shown in Fig. 80, *a*.

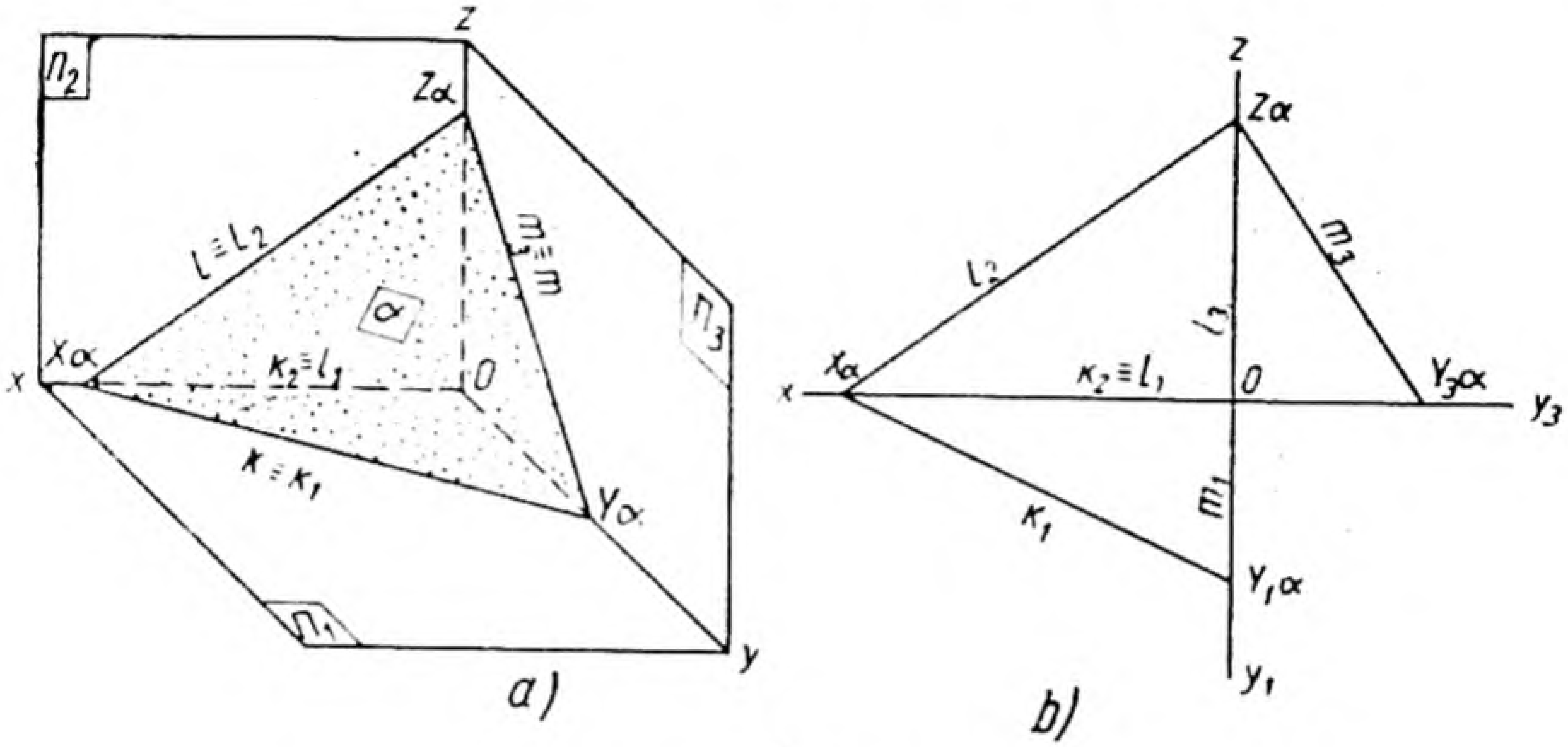


Fig. 80

The traces of planes coincide with the corresponding projections of those traces, i. e., trace $k \equiv k_1$ which is its projection on Π_1, trace $l \equiv l_2$ which is its projection on Π_2, and trace $m \equiv m_3$ which is its projection on Π_3.

The other two projections of each trace coincide with the corresponding ground lines and, as a rule, they are not shown in the multiview drawing of a plane.

If a plane is inclined to all three planes of projection (Π_1, Π_2, Π_3), as illustrated in Fig. 80, *a*, then each pair of its traces intersect in a point. These are called the *points of convergence of the traces.*

The points of convergence X_α, Y_α, Z_α of the traces are contained in the corresponding ground lines. This can be demonstrated as follows: trace k, being the line of intersection of planes α and Π_1, is common to both of those planes. The trace l, being the line of intersection of planes α and Π_2, is common to planes α and Π_2. The point of intersection X_α of traces k and l of plane α, in other words, the point of convergence of the traces, must therefore belong to the three planes Π_1, Π_2 and α. Moreover it follows that this point must lie on the ground line Ox common to planes Π_1 and Π_2. Similarly it can be proved that the points of convergence Y_α and Z_α of the traces lie on the ground lines Oy and Oz, respectively.

§ 15. Oblique Planes

1. *An oblique plane* is inclined to all three planes of projection Π_1, Π_2 and Π_3, such as plane α, shown in Fig. 80. Both in space and in the multiview drawing all the three traces *k*, *l* and *m* of plane α are inclined to ground lines Ox, Oy and Oz (Fig. 80, *a*, *b*). It should be noted that the angles formed by the traces of plane α with the ground lines are not the same as the angles of inclination of plane α itself to the planes of projection (see § 17).

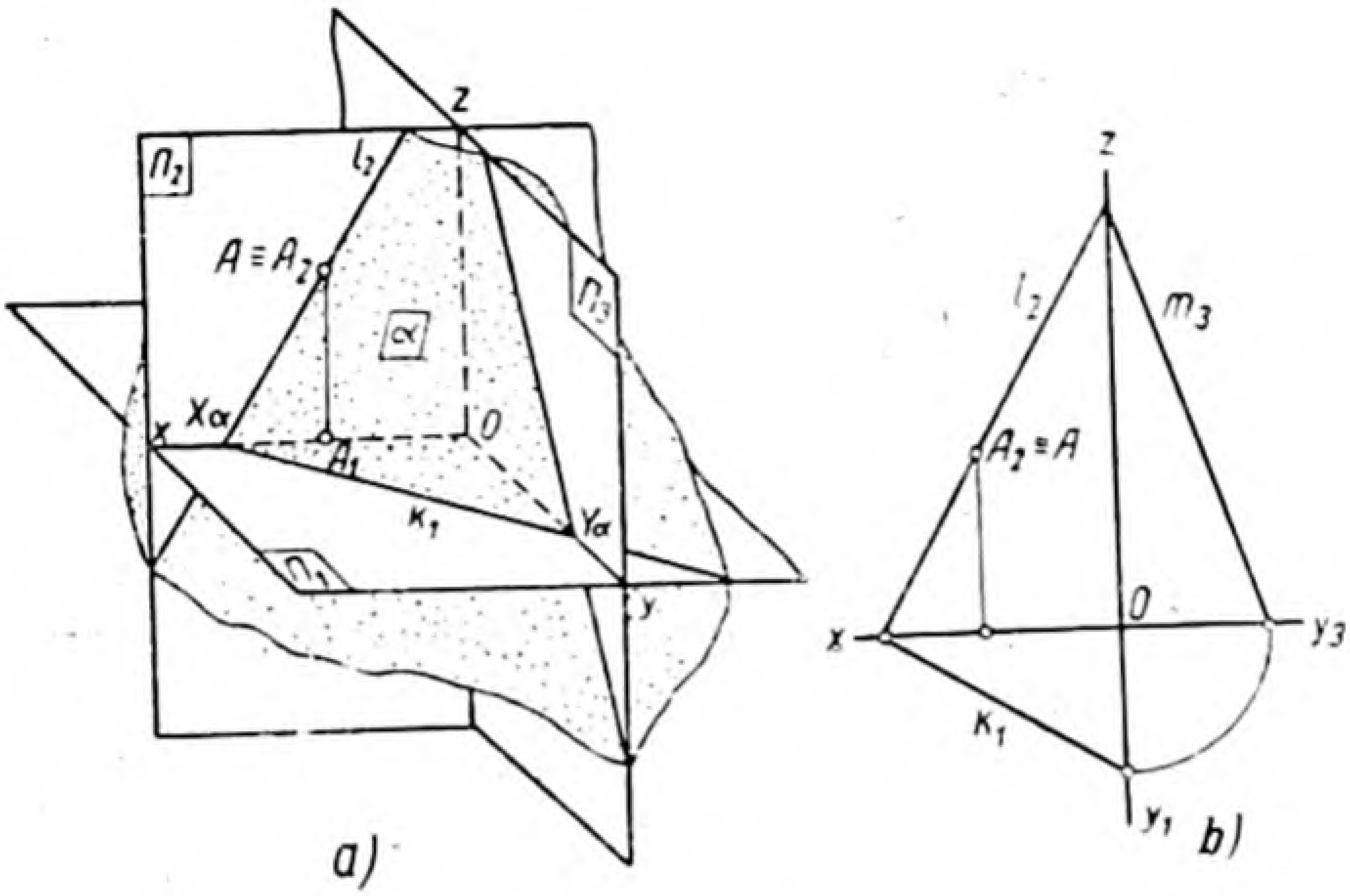


Fig. 81

It should also be noted that the traces of oblique planes contain the projections of only those of its points which actually lie on the traces, such as the point *A* which lies on the frontal trace *l* of plane α in Fig. 81. No other points belonging to oblique planes can have their projections lying on any of the traces of the given planes. Projections of such points in oblique planes can be found by means of auxiliary lines contained in the planes, as shown in Fig. 79.

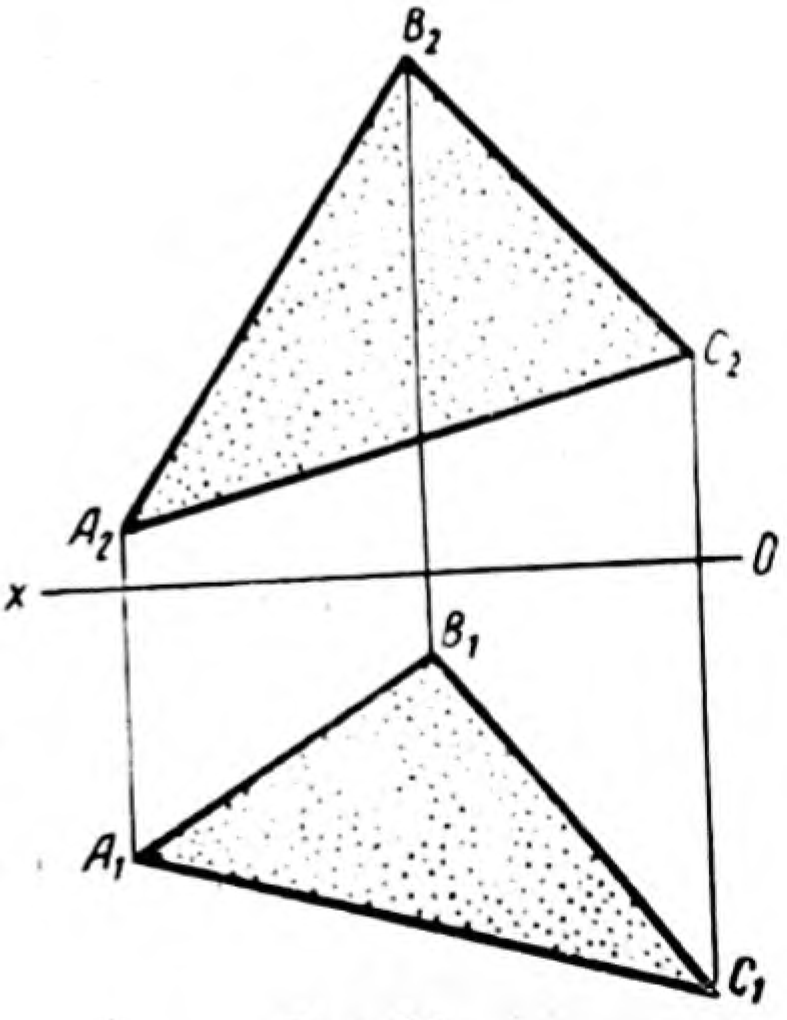


Fig. 82

2. The upper part of the oblique plane α, depicted in Fig. 81, *a*, slopes away from the reader. Each

of the three plane angles formed by the intersection of the traces of this plane in the first octant, is an acute angle. Even so, on the multiview drawing the angles between the traces are not necessarily acute angles. In Fig. 81, *b*, for example, the angle α formed by the traces l and k of plane α is actually an obtuse angle.

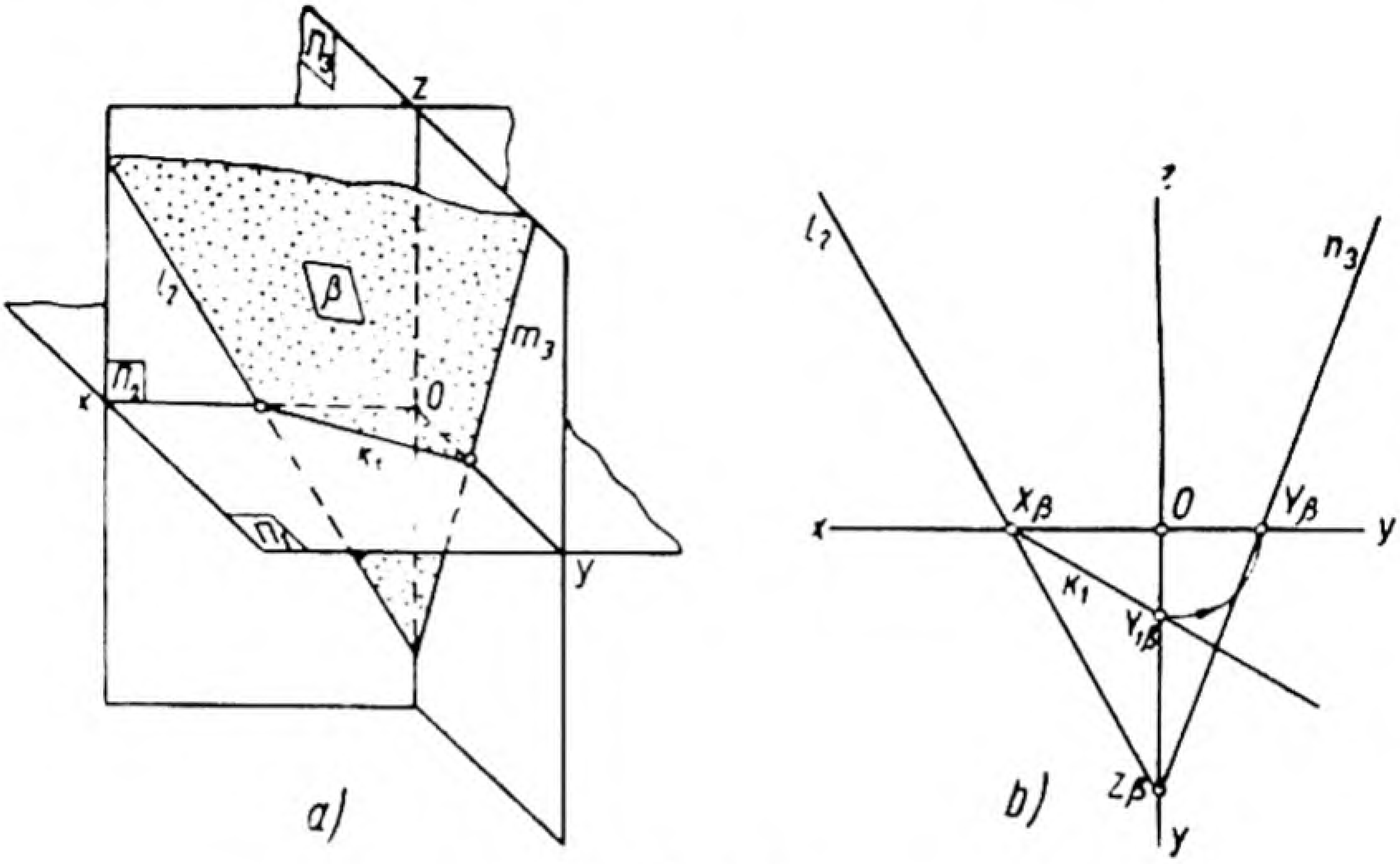


Fig. 83

3. The oblique plane in Fig. 82 is determined by the triangle ABC. The top part of this plane also slopes away from the reader. So, when the two projections on planes Π_1 and Π_2 are constructed the observer will see one and the same side of the plane. The projections of this side of the plane are indicated by dots in Fig. 82.

4. The top of plane β shown in Fig. 83, *a* slopes toward the observer. The angle between its horizontal and frontal traces, within the limits of the first octant, is an obtuse angle. The traces k and m of plane β, within the limits of the first octant, also form an obtuse angle.

The construction in the multiview drawing (Fig. 83, *b*) of profile trace m of this plane, when the horizontal k and frontal l traces are given, consists, as before, in finding the points of convergence Z_β and Y_β and drawing a straight line m_3 through them.

5. The top of the oblique plane, determined by the triangle ABC in Fig. 84, is also inclined toward the observer. Therefore the side, seen when projecting the plane onto Π_2 — the frontal projection of this side is indicated by hatching — is not seen when projecting the plane onto Π_1. The horizontal projection of the latter is indicated by dots.

6. The oblique plane γ, shown in Fig. 85, *a*, is characterized by the fact that in the multiview drawing (Fig. 85, *b*) its traces k_1 and l_2 merge as one straight line. In space these traces cannot constitute one straight line since they lie in different planes, Π_1 and Π_2, and neither of them coincides with the ground line Ox.

The fact that in the multiview drawing (Fig. 85, *b*) the angles formed by the traces k and l with the ground line Ox are the same is because these angles are also equal in space.

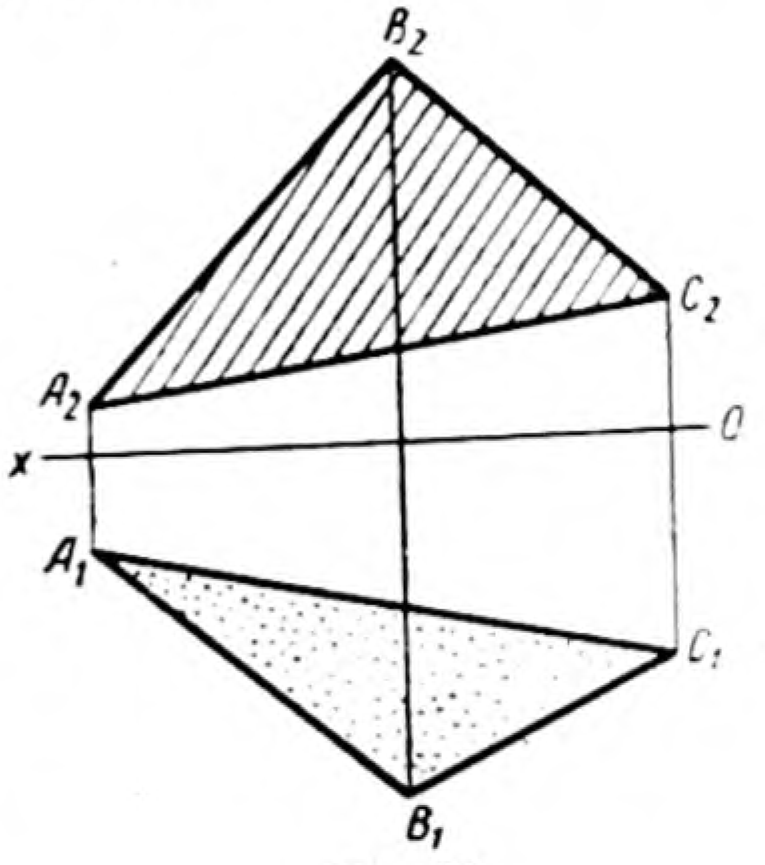


Fig. 84

7. The construction of traces of planes. In Fig. 86 the line AB shown is contained in plane α and pierces the planes of projection Π_1 and Π_2 in points M and N which are its traces. These traces lie on the corresponding traces k and l of plane α. Indeed, if it be assumed that the traces of the given line AB do not lie on the corresponding traces of plane α, then two points, M and N, of straight line AB, which does lie in plane α, do not lie in the plane; this is obviously impossible.

Therefore the traces of straight lines contained in a plane lie on the corresponding traces of that plane.

It follows that the construction of the traces of oblique planes, determined by a combination of points and lines not situated in the planes of projection, consists in the construction of the traces of two lines contained in the given plane, and then drawing through these points the corresponding traces of the planes.

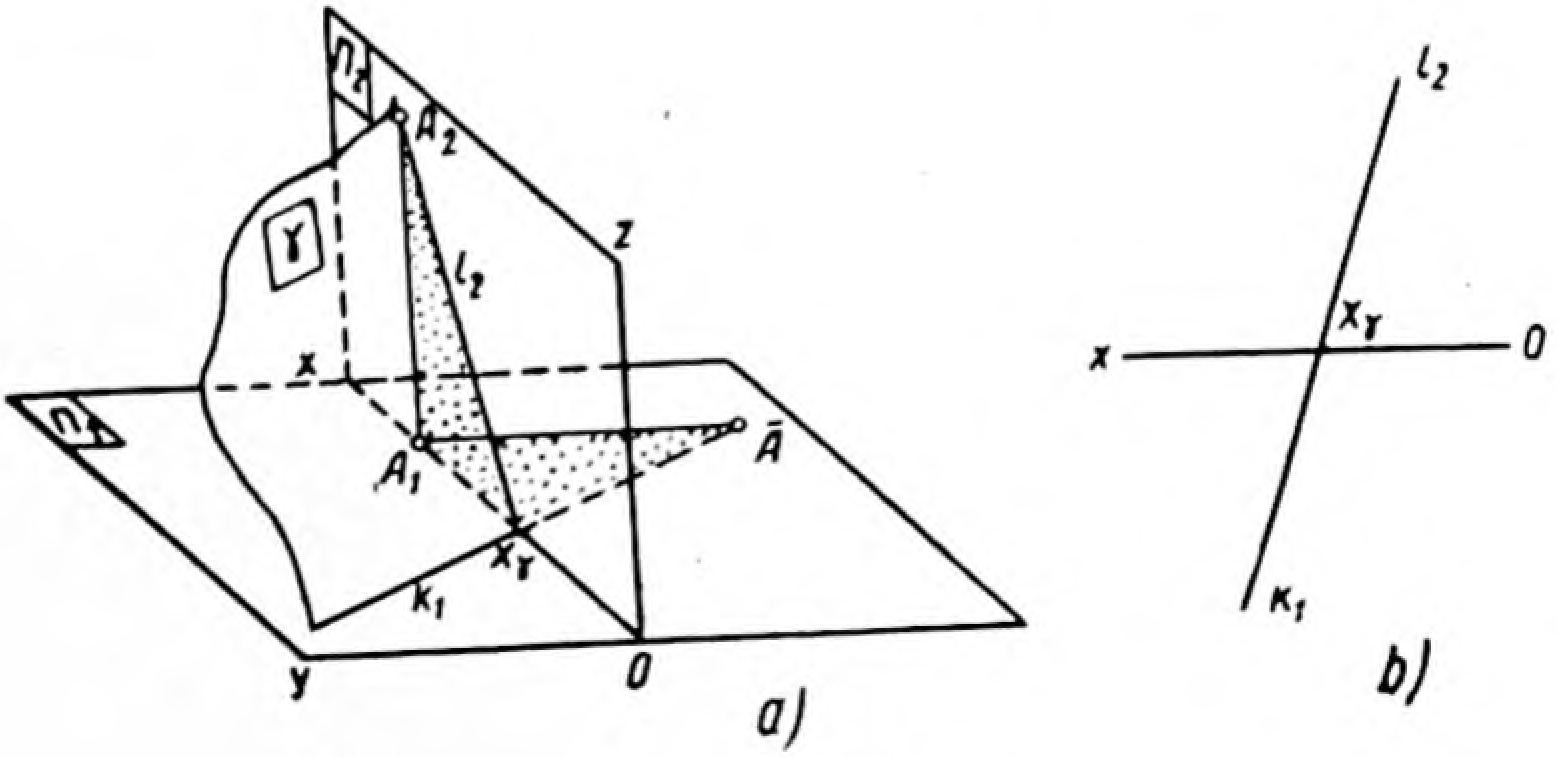


Fig. 85

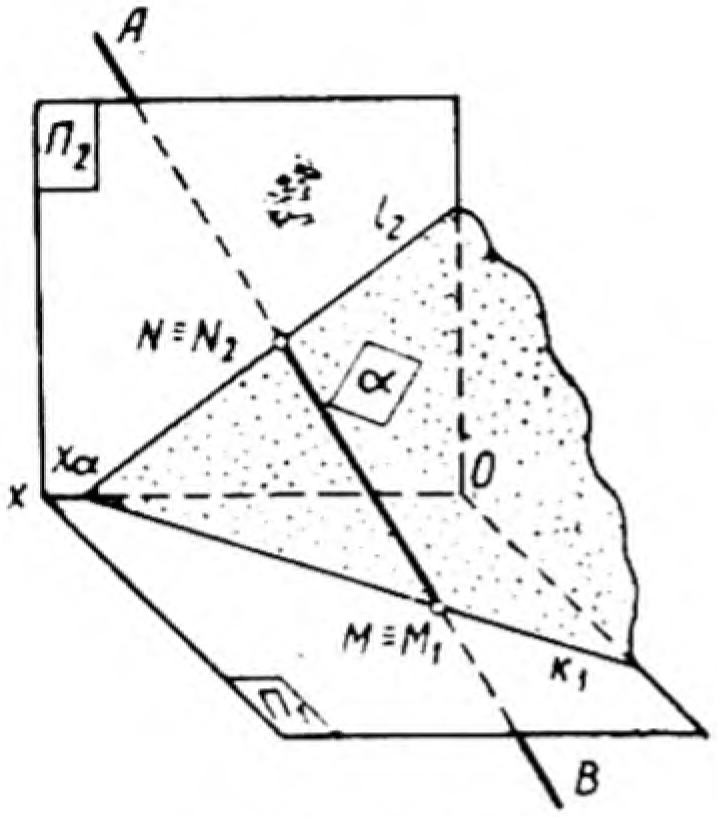
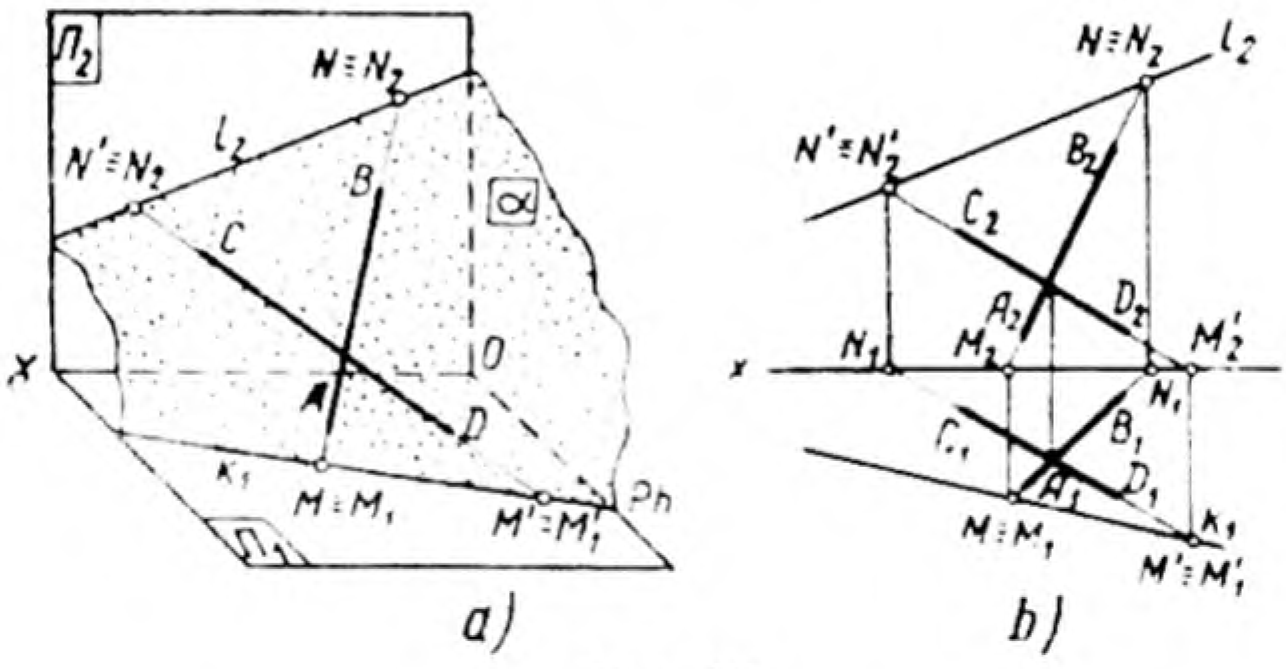

A
П2
l2
N ≡ N2
α
Xα
x
0
M ≡ M1
K1
П1
B

Fig. 86


П2
N ≡ N2
N′ ≡ N2
l2
B
C
α
x
A
0
D
K1
M ≡ M1
M′ ≡ M′1
Ph
П1
N ≡ N2
l2
N′ ≡ N′2
B2
C2
A2
D2
x
N1
M2
M′2
0
N1
C1
B1
A1
D1
M ≡ M1
K1
M′ ≡ M′1

a) b)

Fig. 87

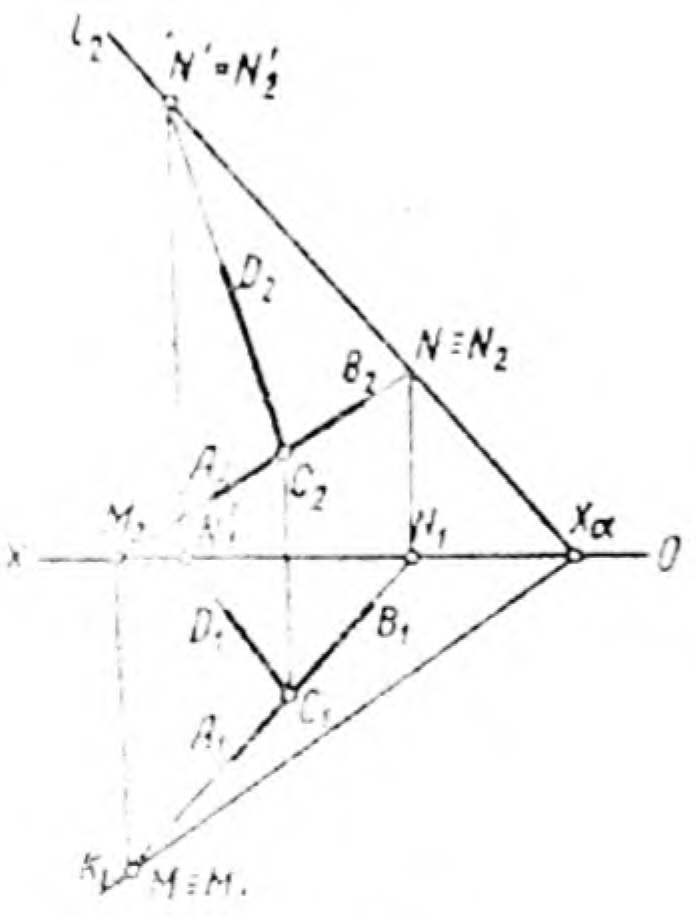

l2
N′ ≡ N′2
D2
N ≡ N2
B2
A2
C2
x
M2
N′1
N1
Xα
0
D1
B1
C1
A1
K1
M ≡ M1

Fig. 88

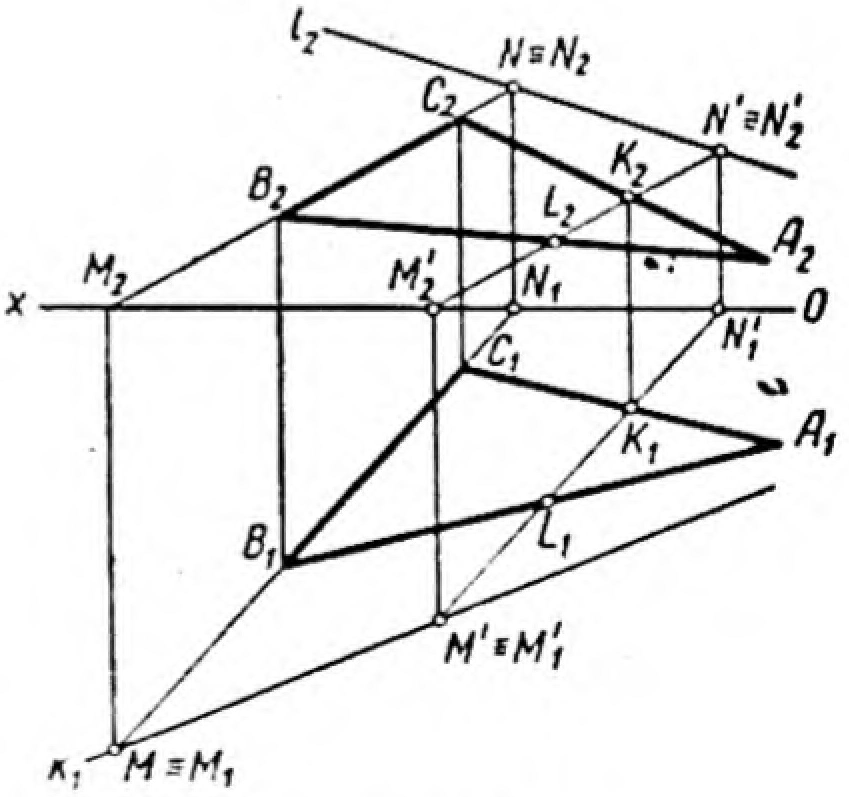

l2
N ≡ N2
C2
N′ ≡ N′2
K2
B2
L2
A2
x
M2
M′2
N1
0
N′1
C1
K1
A1
B1
L1
M′ ≡ M′1
K1
M ≡ M1

Fig. 89

In Fig. 87 the plane α is given by two intersecting lines AB and CD. To construct its traces k and l on the multiview drawing (Fig. 87, *b*) the horizontal traces M and M' of the given lines are found (§ 11). Next, through M and M' a line k_1 is passed. Then the frontal traces N and N' of the given lines are found, and the line l_2 is drawn through them. The straight lines k_1 and l_2 are the horizontal and frontal traces of plane α.

When the point X_α of convergence of the traces of a plane is situated within the limits of the drawing, as in Fig. 88, to construct the traces of the plane given by intersecting lines, such as AB and CD, it is sufficient to find two traces of one line and only one trace of the other line. For instance, it is enough to find the traces M and N of the line AB and frontal trace N' of the line CD. The horizontal trace k_1 of plane α will pass through trace M and the point of convergence of the traces X_α.

Example. To construct traces of the plane given by triangle ABC (Fig. 89).

In this particular case only one side of the triangle, namely BC, has traces within the limits of the drawing, whereas the sides AB and AC pierce the planes of projection Π_1 and Π_2 outside the drawing.

To solve the problem in the plane of triangle ABC an auxiliary line KL parallel to BC is drawn through any point K on the side AC. Once the traces M and N of line BC and traces M' and N' of line KL have been found, a horizontal trace k_1 is drawn through points M and M'. Through points N and N' a frontal trace l_2 is drawn. These are traces of plane α given by the triangle ABC.

§ 16. Projecting Planes

1. A *horizontal-projecting plane* is a plane (α), perpendicular to the horizontal plane of projection Π_1 (Fig. 90).

Since the plane α is perpendicular to Π_1, the projector $AA_1 \perp \Pi_1$, drawn through point A of plane α, also lies in plane α. This projector will pierce plane Π_1 in point A_1 on the line of intersection of planes α and Π_1, i. e., on the horizontal trace (k_1) of plane α. This trace coincides with line α_1 in which plane α projects on Π_1 ($k_1 \equiv \alpha_1$).

All the projectors drawn perpendicular to plane Π_1 through any points of plane α pierce plane Π_1 on that same trace $k_1 \equiv \alpha_1$.

Therefore the horizontal projections of all points contained in the horizontal-projecting plane (α) lie on the horizontal trace (k_1) of that plane. This property of the horizontal trace of plane α will be called the *collecting* property, and traces possessing this property will be called the *trace-projections of planes.*

Since planes α and Π_2 are perpendicular to one and the same plane Π_1, the line of their intersection (trace l) is perpendicular to Π_1. This means that the trace l is also perpendicular to every other line on Π_1 which passes through it, including the ground line Ox.

It follows that the frontal trace of the horizontal-projecting plane is perpendicular to the ground line Ox.

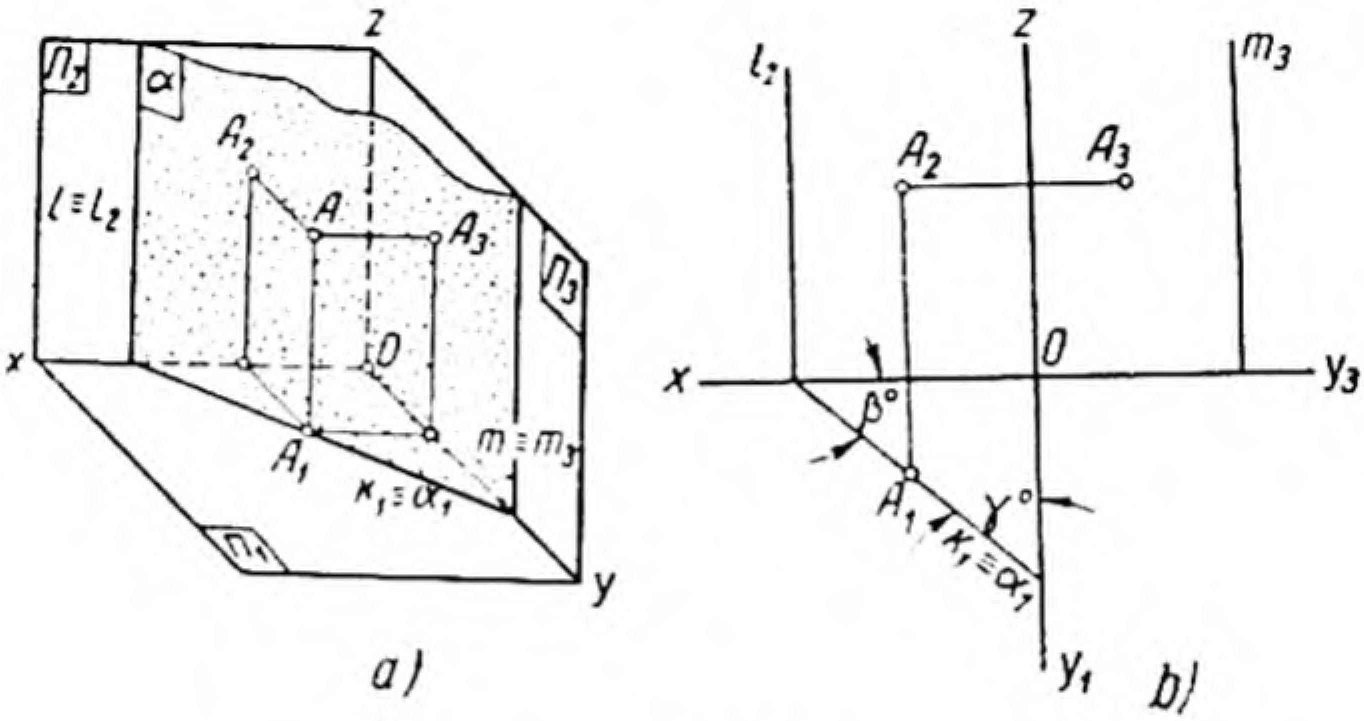


Fig. 90

In the same way the profile trace m_3 of this plane both in space and in the multiview drawing (Fig. 90, *b*) is perpendicular to the ground line Oy_3.

Since the plane of projection Π_1 is perpendicular to the frontal trace l, i. e., to the edge of the dihedral between planes α and Π_2, the angle β formed by the horizontal trace-projection $k_1 \equiv \alpha_1$ and the ground line Ox, is the true angle of inclination of plane α to the frontal plane of projection Π_2 (Fig. 90, *b*).

Since plane Π_1 is also perpendicular to trace m, the angle γ between the trace-projection $k_1 \equiv \alpha_1$ and ground line Oy_1 gives the true angle of inclination of plane α to the profile plane of projection Π_3 (Fig. 90, *b*).

Example. It is required to pass a horizontal-projecting plane through line AB (Fig. 91).

Since the horizontal trace-projection α_1 of this plane possesses collecting properties, it should be drawn as a continuation of projection A_1B_1. The frontal trace l is perpendicular to ground line Ox.

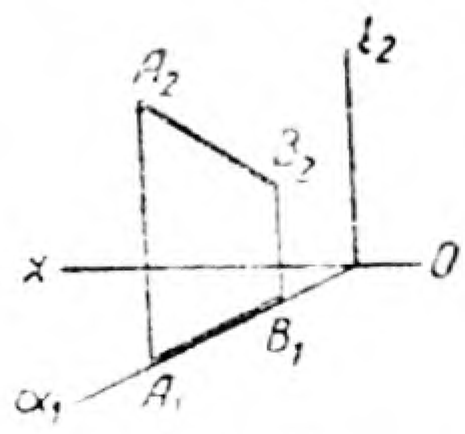


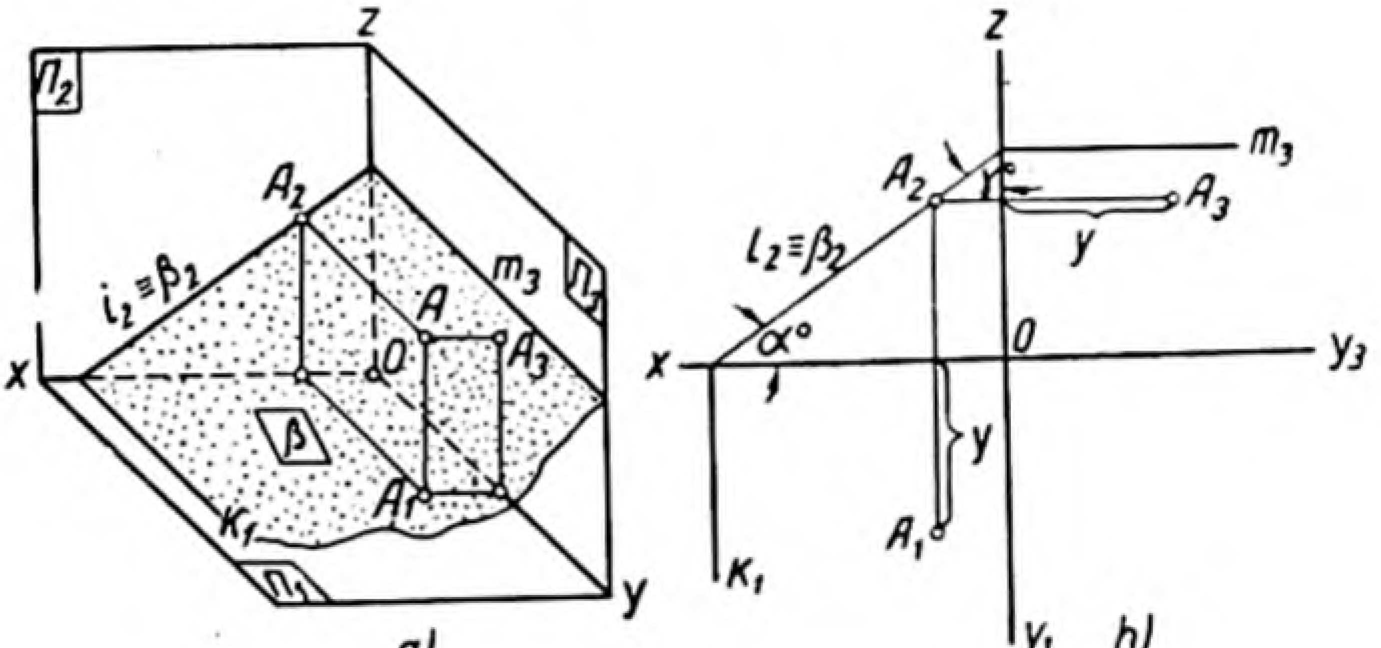
Fig. 91

2. The *frontal-projecting plane* is a plane perpendicular to the frontal plane of projection. In Fig. 92, *a* the frontal-projecting plane β is shown.

Its frontal trace-projection ($l_2 \equiv \beta_2$) possesses collecting properties, i. e., the frontal projection of any point of the plane lies on this trace-projection. An example of this is the frontal projection A_2 of point A.

The other two traces, k_1 and m_3, of the frontal-projecting plane are respectively perpendicular to ground lines Ox and Oz (Fig. 92 *a, b*).

Angles α and γ are the true angles of inclinations of plane β to planes of projection Π_1 and Π_3 respectively.



a) *b)*

Fig. 92

Example. In Fig. 93 the frontal projection $A_2B_2C_2$ of triangle ABC coincides with the frontal trace-projection β_2 of plane β. This shows that triangle ABC is contained in plane $\beta \perp \Pi_2$.

3. The *profile-projecting plane* is a plane perpendicular to the profile plane of projection. In Fig. 94 the profile-projecting plane γ is shown.

Its profile trace-projection γ_3 possesses collecting properties, i. e., the profile projection of any point belonging to plane γ lies on this trace-projection as does, for instance, the profile projection A_3 of point A.

The remaining two traces, k_1 and l_2, of the profile-projecting plane are respectively perpendicular to ground lines Oy_1 and Oz (Fig. 94, *b*).

The angles α and β give the true values, of the angles of inclination of the profile-projecting plane to the planes of projection Π_1 and Π_2, respectively.

4. The *horizontal plane* α (Fig. 95) is a special type of projecting plane. Since this plane, being parallel to plane Π_1, is at the same time perpendicular to the planes of projection Π_2 and Π_3, its traces on Π_2 and Π_3 are perpendicular to ground line Oz and possess collecting properties.

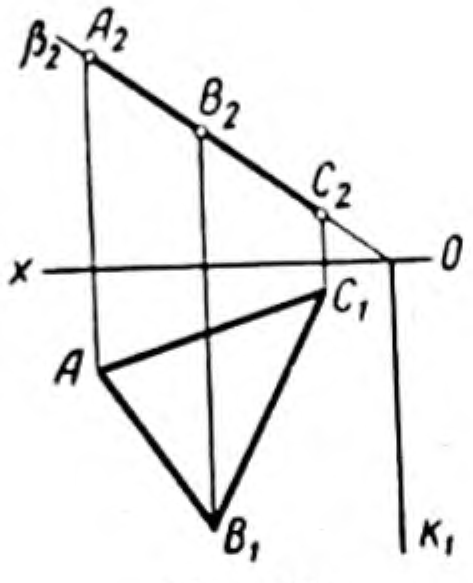
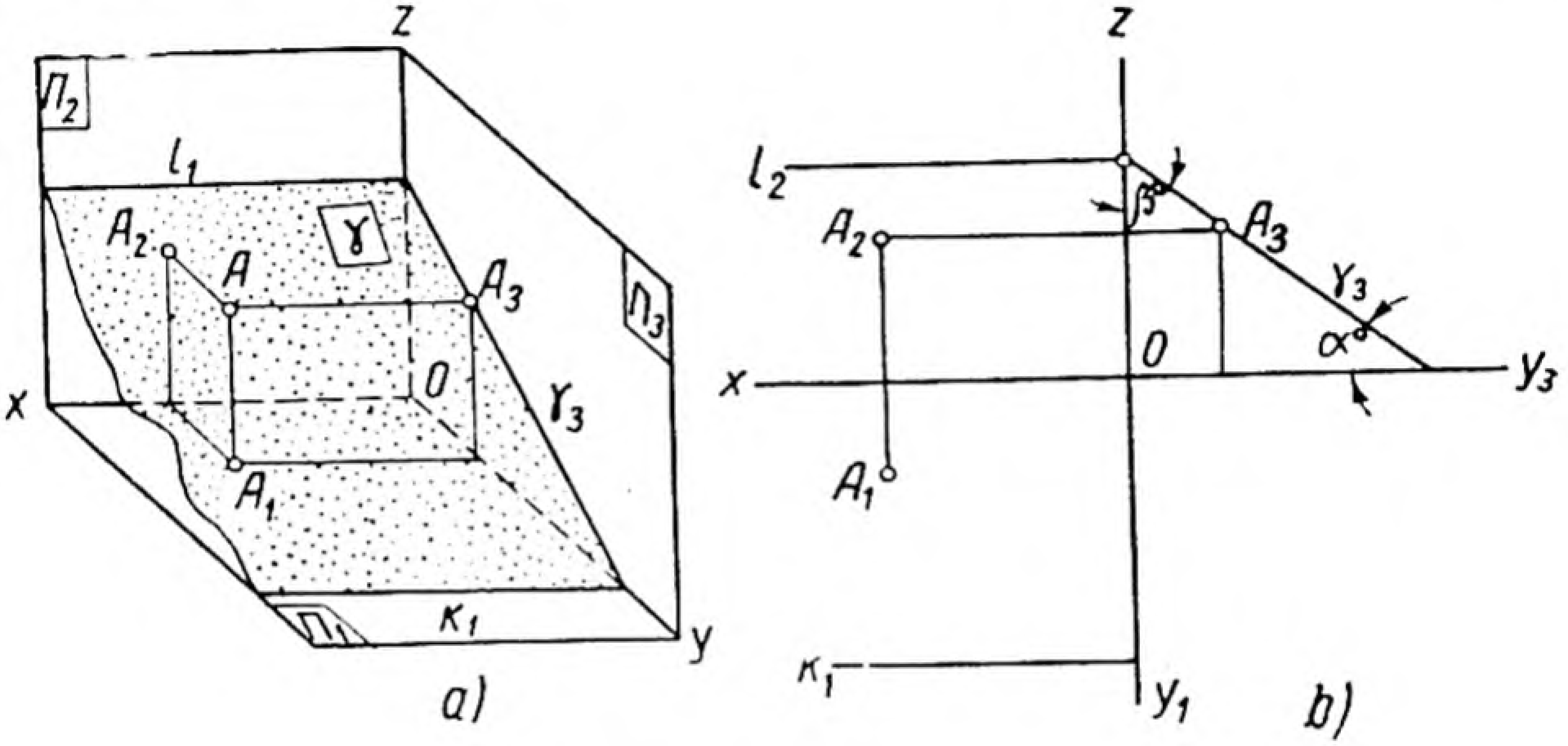


Fig. 93

So, on the trace-projections α_2 and α_3 the corresponding projections A_2 and A_3 of point A contained in the plane α are to be found. Any plane figure contained in the horizontal plane is projected onto the plane of projection Π_1 in its true size. For example, the regular pentagon $ABCDE$ (Fig. 96), which lies in the horizontal plane, is projected onto plane Π_1 without distortion. The frontal projection of this pentagon coincides with the trace-projection α_2 of plane α.



Z
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Π3
x
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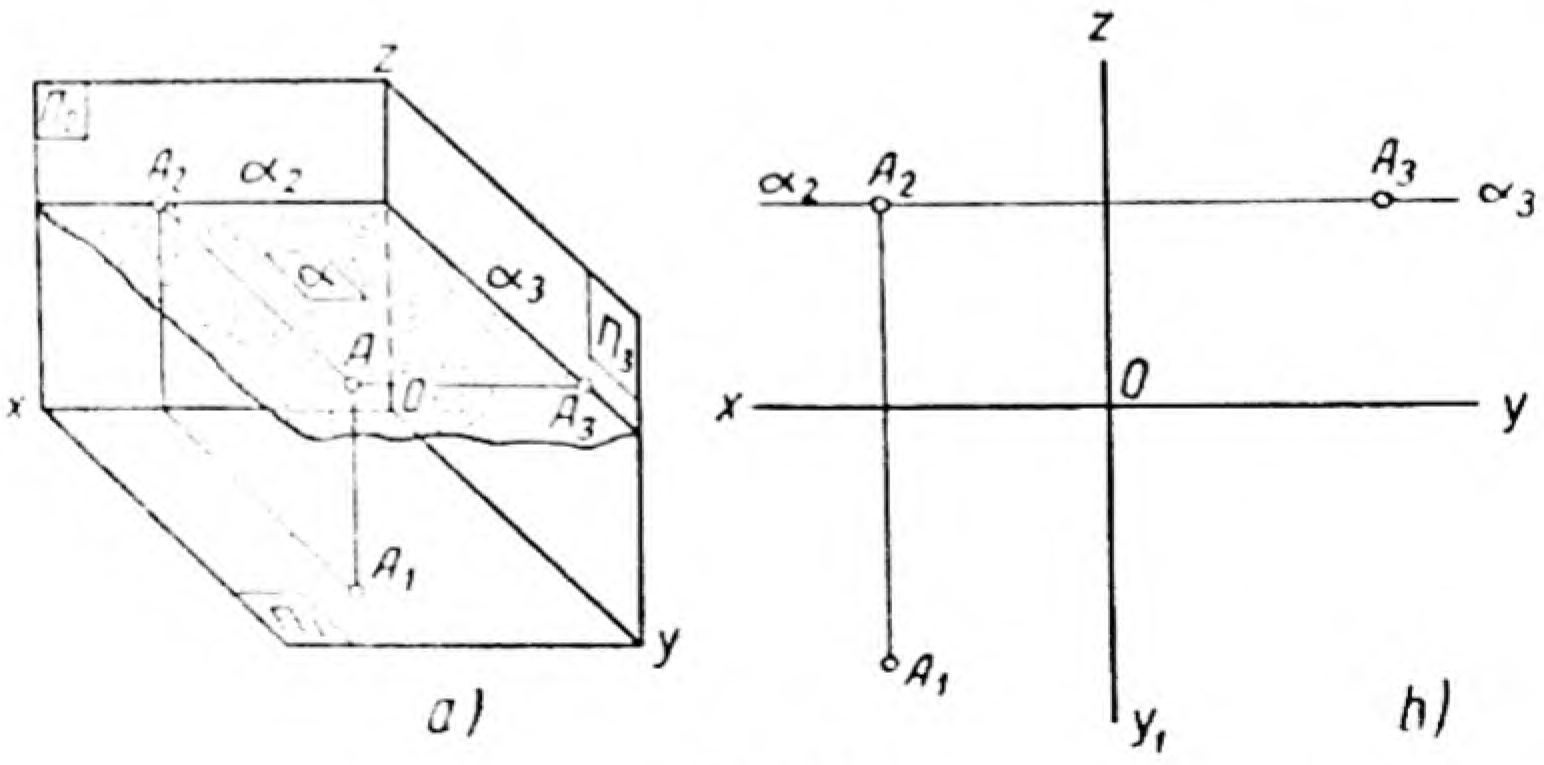
γ3
A1
K1
Π1
y
a)
Z
l2
β°
A2
A3
γ3
0
α°
x
y3
A1
K1
y1
b)

Fig. 94

5. The *frontal plane* $\beta \parallel \Pi_2$ (Fig. 97) and the *profile plane* $\gamma \parallel \Pi_3$ (Fig. 98) are also special cases of projecting planes. The trace-projections β_1 and β_3 of the frontal plane β are parallel to the respective ground lines Ox and Oz, and the trace-projections γ_1 and γ_2 of the profile plane γ on the multiview drawing merge into one



z
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α3
Π3
A
x
0
A3
A1
Π1
y
a)
z
α2
A2
A3
α3
x
0
y
A1
y1
b)

Fig. 95

line, which is perpendicular to the ground line Ox. Each of the traces indicated possesses collecting properties.

6. Let us consider one of the possible profile-projecting planes, namely plane α passing through the ground line Ox (Fig. 99). This plane is called an *axial plane*. It intersects planes Π_1 and Π_2 along the ground line Ox, therefore the traces k and l of plane α coincide with this ground line. The profile trace-projection α_3 passes through point O. The angles α and β give the true inclination of the axial plane to planes Π_1 ($\alpha°$) and Π_2 ($\beta°$).

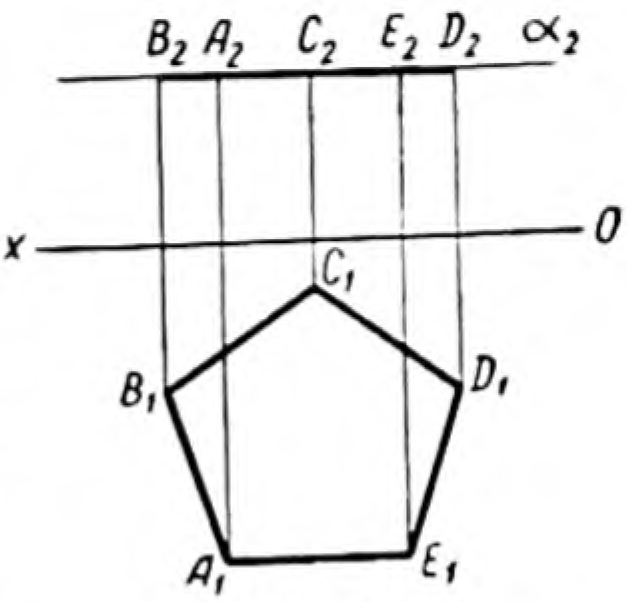
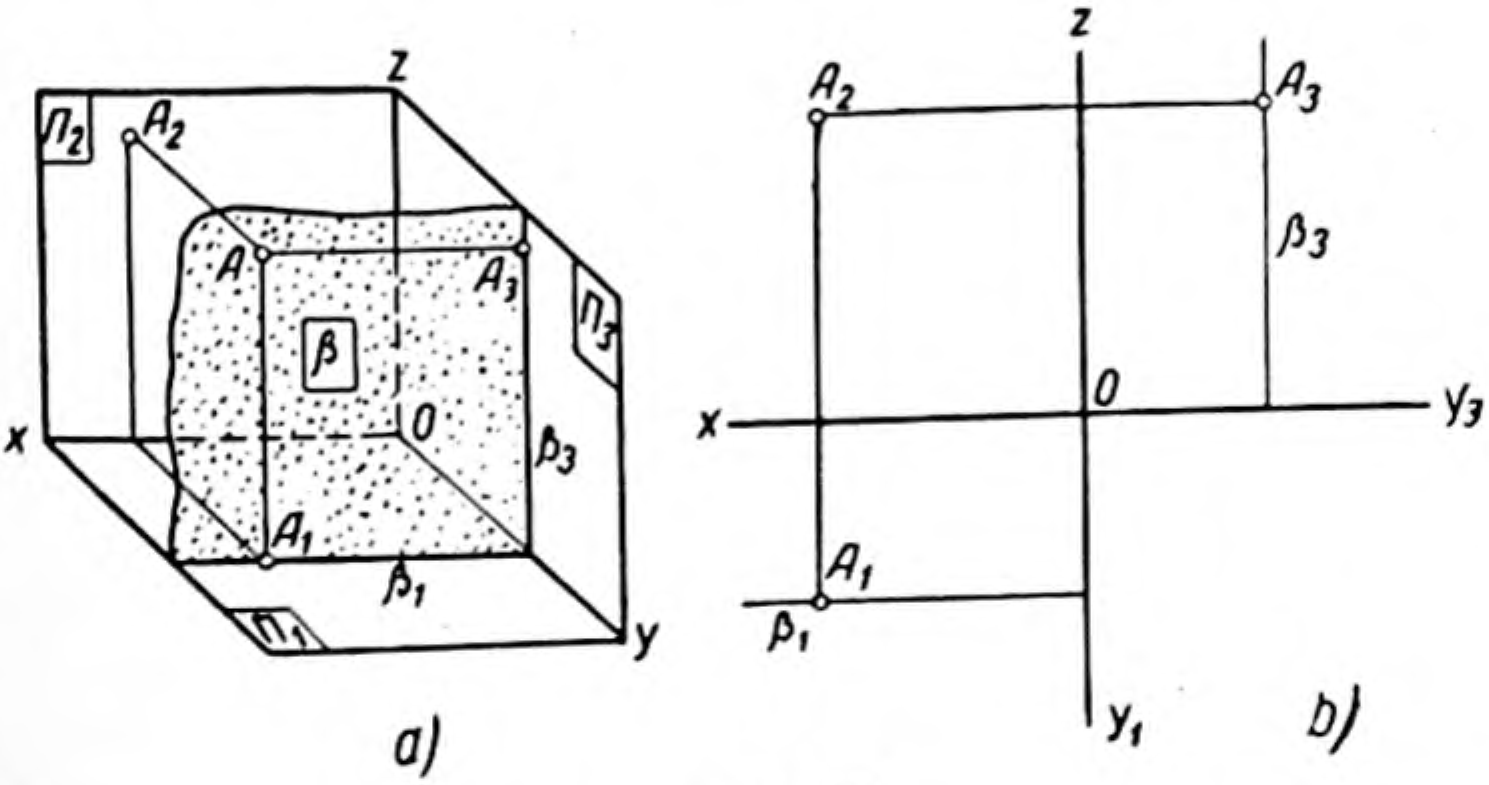


Fig. 96

In the special case when the axial plane α is represented in a system consisting of only two planes of projection Π_1 and Π_2 (Fig. 100), in order to determine the position of the plane in space some point, such as A contained in the plane, is given. The ground line Ox and point A make it possible to establish the position of the axial plane in space and to construct in it points, lines and figures.

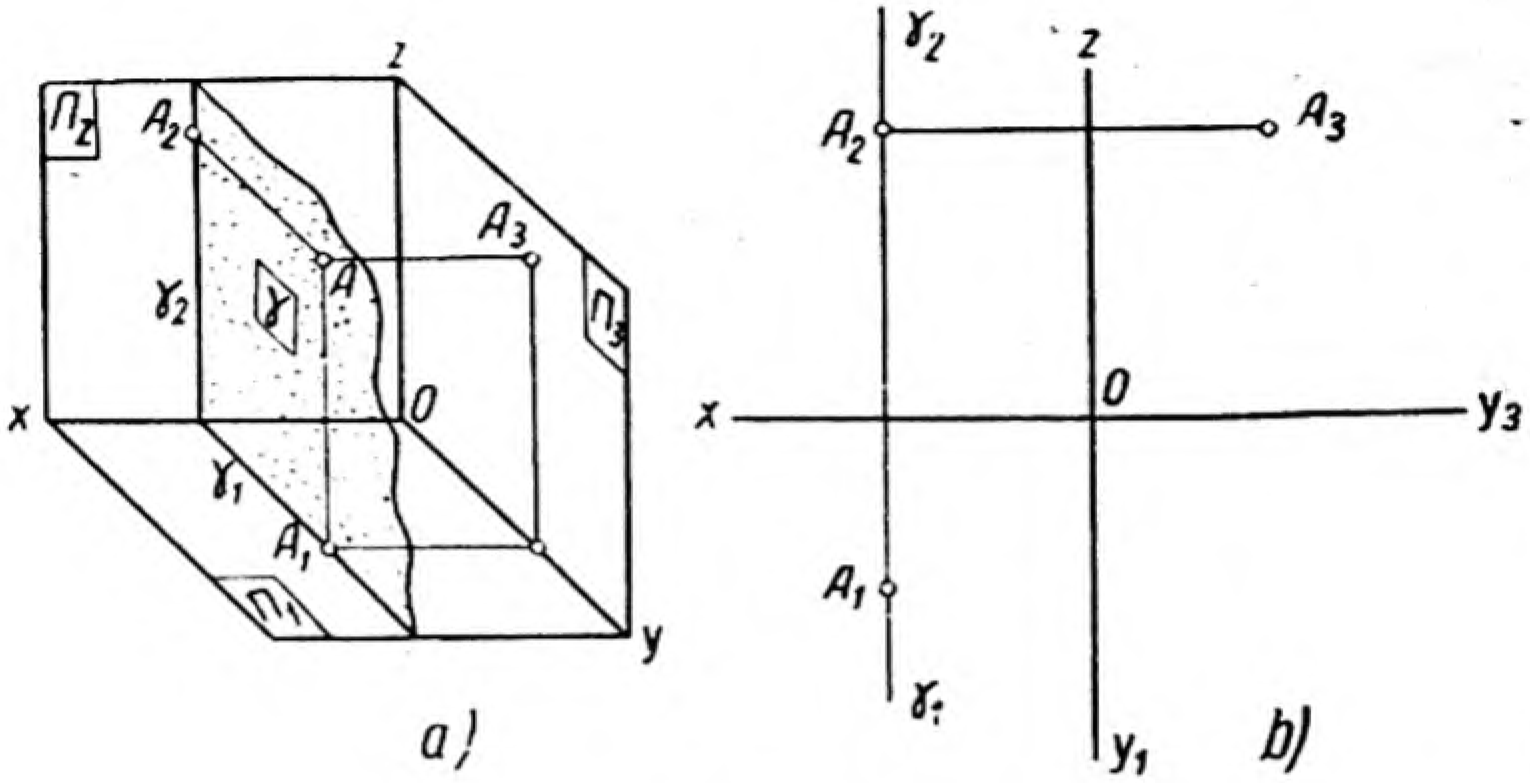
Example. The plane α is given by its ground line Ox and point A (Fig. 101). Construct the horizontal projection K_1 of point K contained in this plane, the frontal projection K_2 of point K being unknown.

Draw a line through points A_2 and K_2. Find the point ($B_1 \equiv B_2$) of intersection of this line with the ground line Ox. Join this point with point A_1. The line AB is then contained in the given plane α and passes through point K. It follows that the horizontal projection K_1 of this point must be contained in A_1B_1 and, at the same time, it must lie on the vertical line K_2K_1 which joins the projections.



a) b)

Fig. 97


z
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γ2
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x
O
γ1
A1
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y
a)
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x
O
y3
A1
γ1
y1
b)

Fig. 98

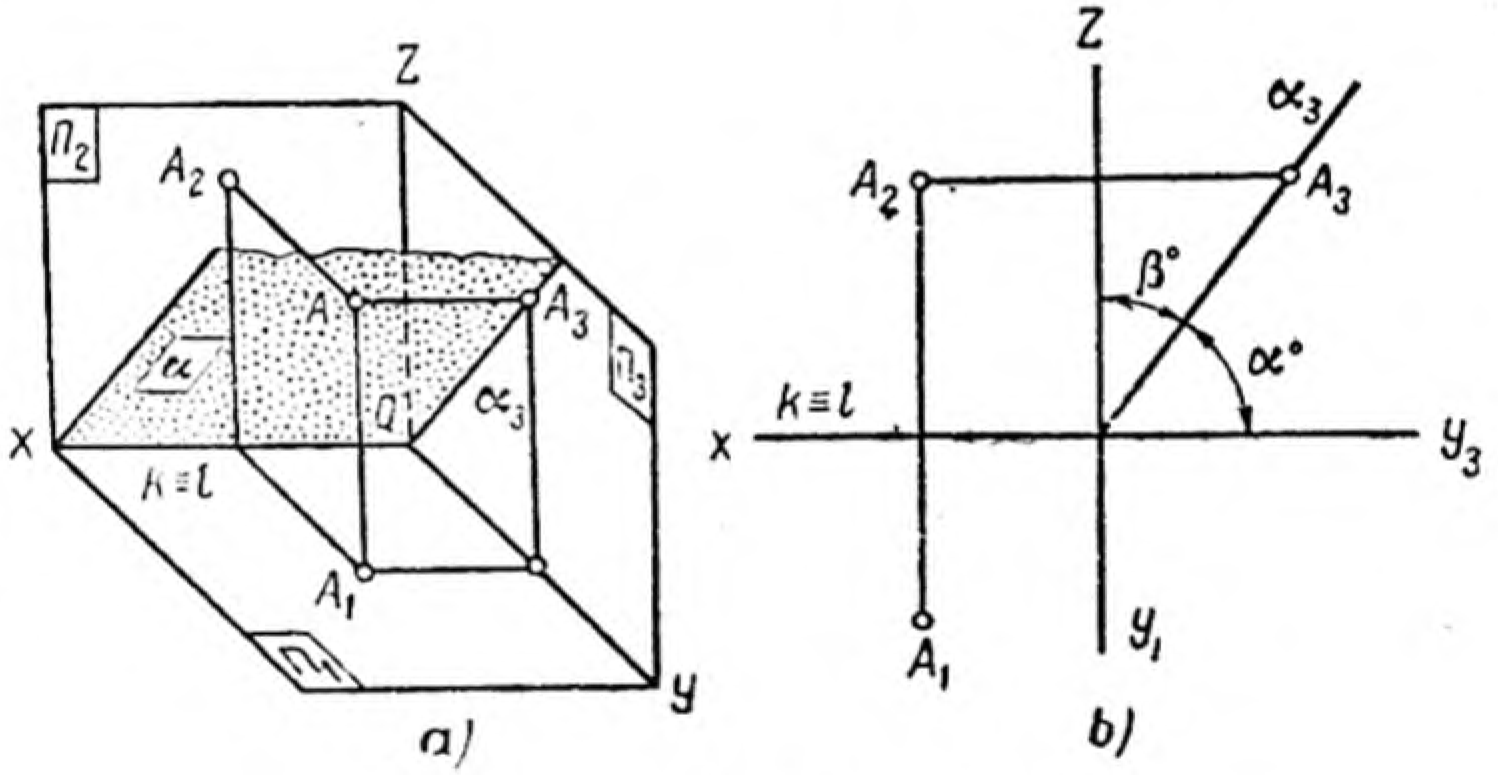

Z
П2
A2
A
A3
α
П3
X
O
α3
k≡l
A1
П1
y
a)
Z
α3
A2
A3
β°
α°
x
k≡l
y3
A1
y1
b)

Fig. 99

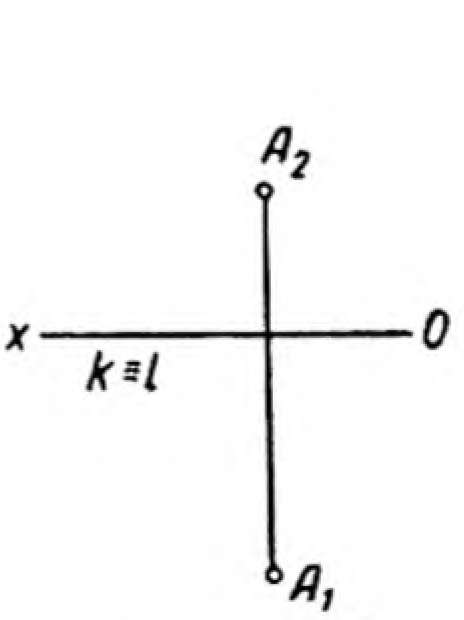


Fig. 100

If an axial plane has the same angle of inclination to two planes of projection, it is called a *bisecting plane*. In Fig. 102 a bisecting plane is depicted. It belongs to the first and third quadrants formed by the planes of projection Π_2 and Π_1. This bisecting plane is the geometrical locus of points equidistant from the planes of projection Π_1 and Π_2. The coordinates y and z of any point A assumed in this plane are equal. Since the coordinates y and z are equal when the plane bisects the second and fourth space quadrants, it follows that points belonging to the bisecting plane, when projected onto the planes Π_1 and Π_2, merge in the multiview drawing as a single point.

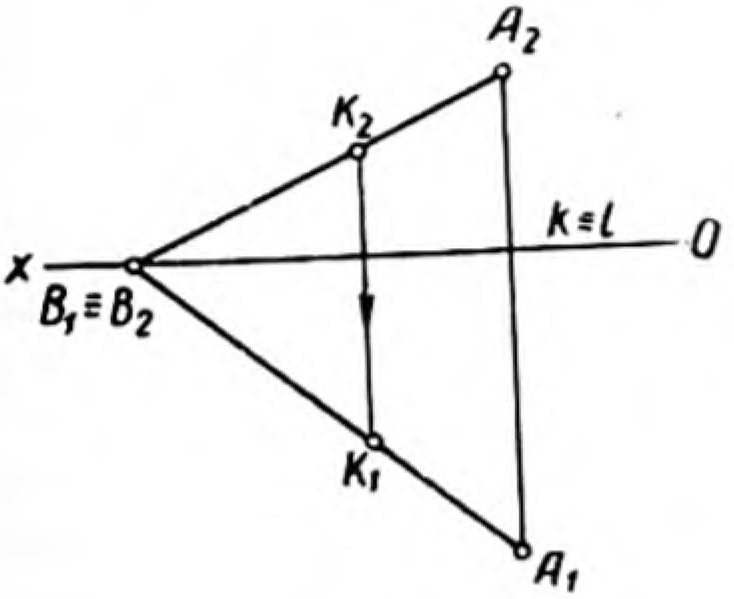


Fig. 101

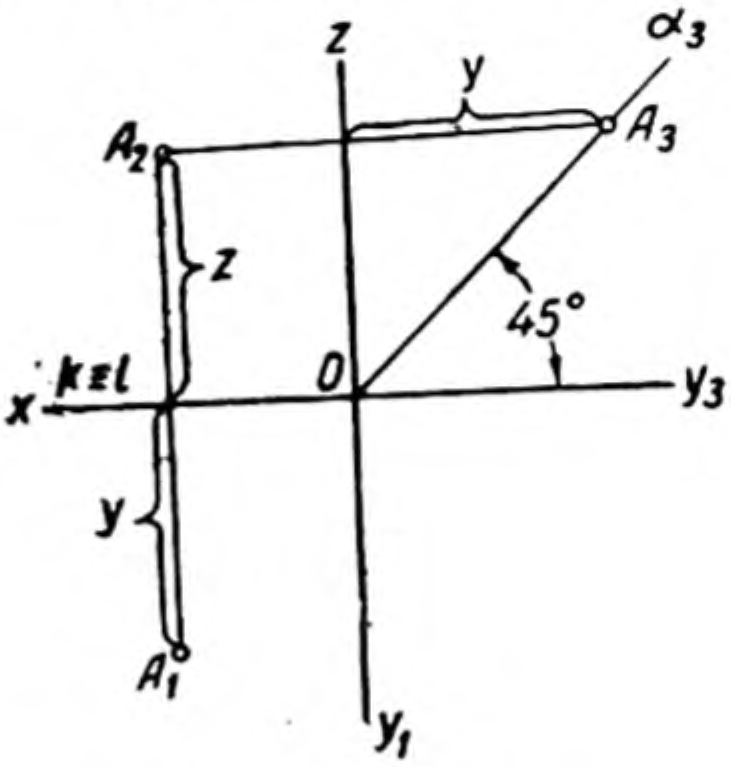


Fig. 102

§ 17. Level Lines and Lines of Maximum Inclination in a Plane

1. The horizontal of a plane. A straight line h contained in a given plane and parallel to the horizontal plane of projection Π_1 is called a *horizontal of that plane*.

For instance, a horizontal cornice may serve as an example of the horizontal of the sloping plane of the roof.

A horizontal of a plane, as any horizontal in space, is projected onto the frontal plane of projection Π_2 as a line parallel to the ground line Ox. Every point of a horizontal lies at the same altitude, for this reason they are also known as level lines.

In Fig. 103 two projections of triangle ABC are given. To construct the projections of a horizontal contained in the plane of this triangle, through the frontal projection of vertex C pass a frontal projection C_2D_2 of the horizontal. This line is parallel to ground line Ox. Point D_1 is found by passing a projector through point D_2 as shown by an arrow. Then through C_1 and D_1 draw the horizontal projection C_1D_1 of the horizontal h contained in the plane triangle ABC.

If the plane is given by its traces (Fig. 104), the construction of a horizontal lying in it may also be begun by drawing the horizontal projection of the horizontal since the horizontal projection of the horizontal of a plane is parallel to the horizontal trace of that plane. Two straight lines, trace k and horizontal h, may be considered as the intersection of plane α by two horizontal planes, Π_1 and the plane at the level of the projecting horizontal itself.

When constructing the horizontal h in the multiview drawing (Fig. 104, b) it should be remembered that the traces of lines, lying in the plane, lie on the similar traces of the plane. Therefore first mark the projections (N_1 and $N_2 \equiv N$) of the frontal trace of the horizontal h. Then through point N_1 draw the horizontal projection h_1 of the horizontal. This line is parallel to the horizontal trace of plane α. Next, through point $N_2 \equiv N$ of the frontal projection, draw projection h_2 of the horizontal parallel to ground line Ox.

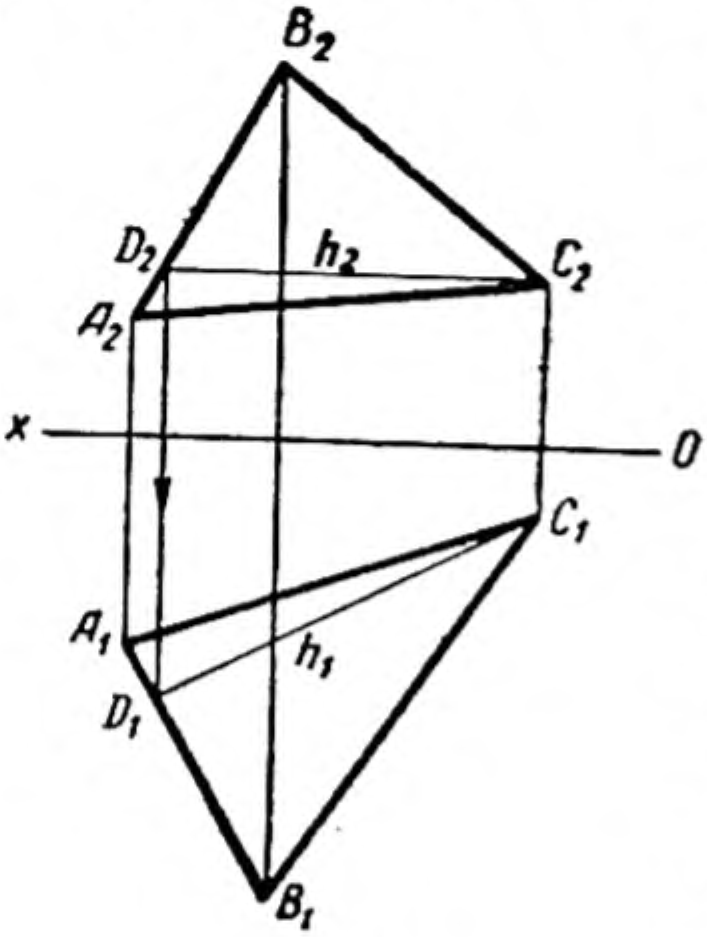


Fig. 103

2. The frontal of a plane. A straight line f contained in a given plane and parallel to the frontal plane of projection Π_2 is called the *frontal of the plane*.

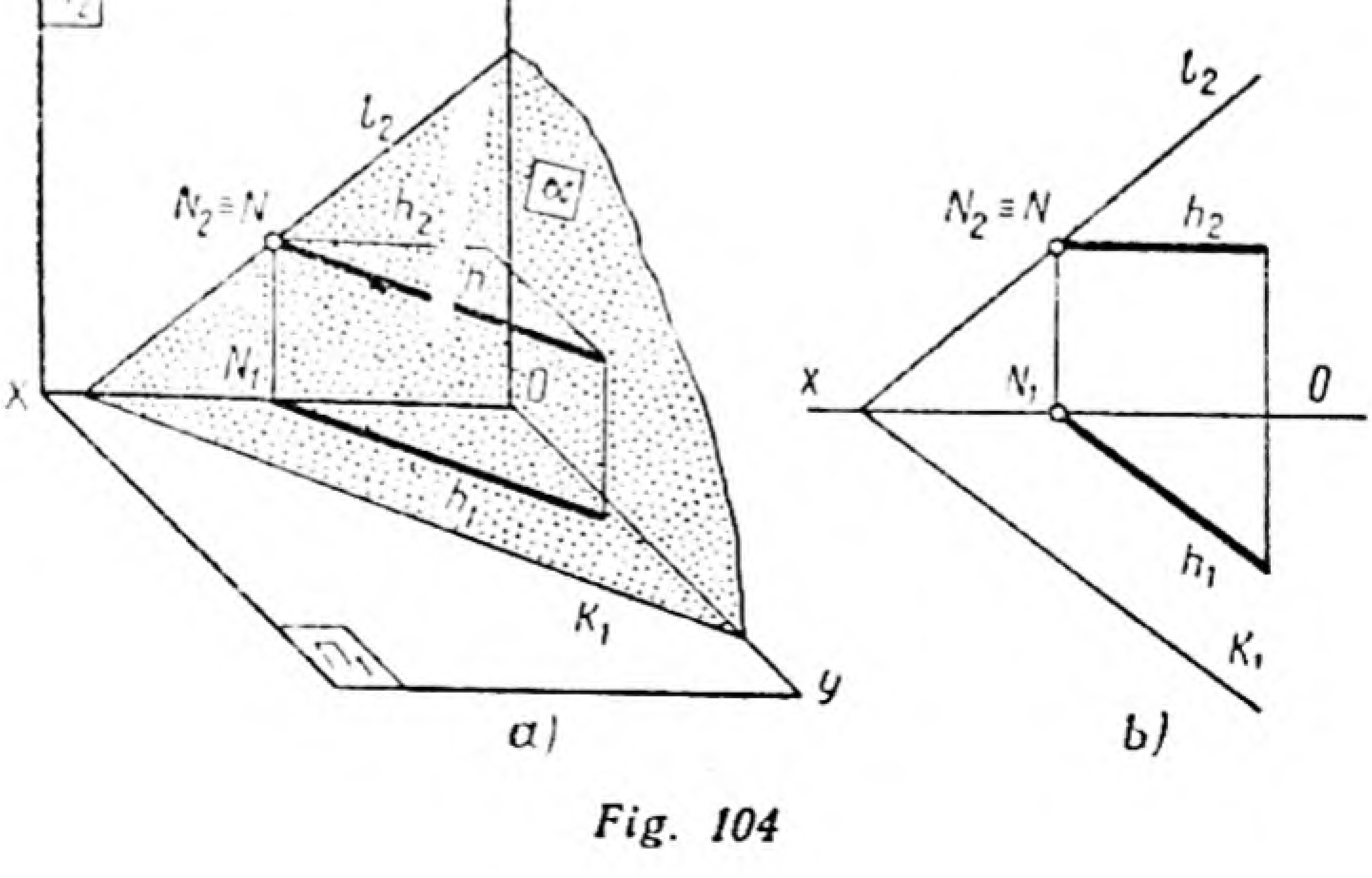


Fig. 104

From the drawing in Fig. 105 it may be seen that the horizontal projection f_1 of the frontal is parallel to the ground line Ox. The frontal projection f_2 of the frontal of the plane is parallel to the frontal trace l_2 of plane α, and the horizontal trace, point M, of the frontal of the plane lies on the horizontal trace of plane α.

The construction of the projections of the frontal f in the plane

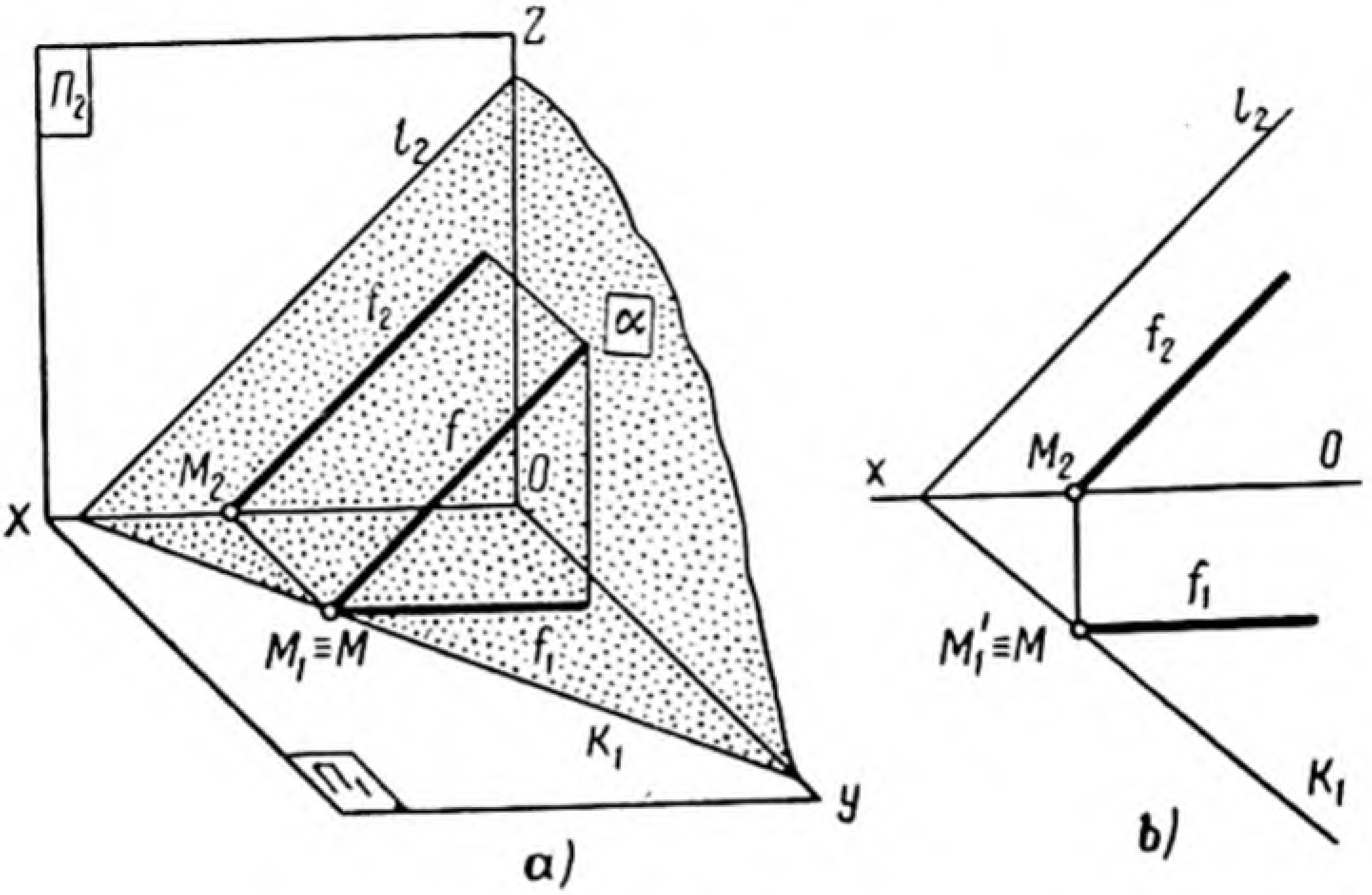


Fig. 105

given by the triangle ABC, is shown in Fig. 106. The construction is begun by drawing the horizontal projection f_1 of the frontal through point B_1. This line is parallel to ground line Ox. The intersection of f_1 with the extended projection A_1C_1 of side AC of the triangle determines point D_1. Since point D_1 has been found, the point D_2 on the extension of projection A_2C_2 can be determined. Join points B_2 and D_2. The straight lines f_1 and f_2 thus obtained are the projections of frontal f of the plane of triangle ABC.

The projections of the frontals of the plane of triangle ABC, passing respectively through the vertex A and the vertex C, can likewise be constructed.

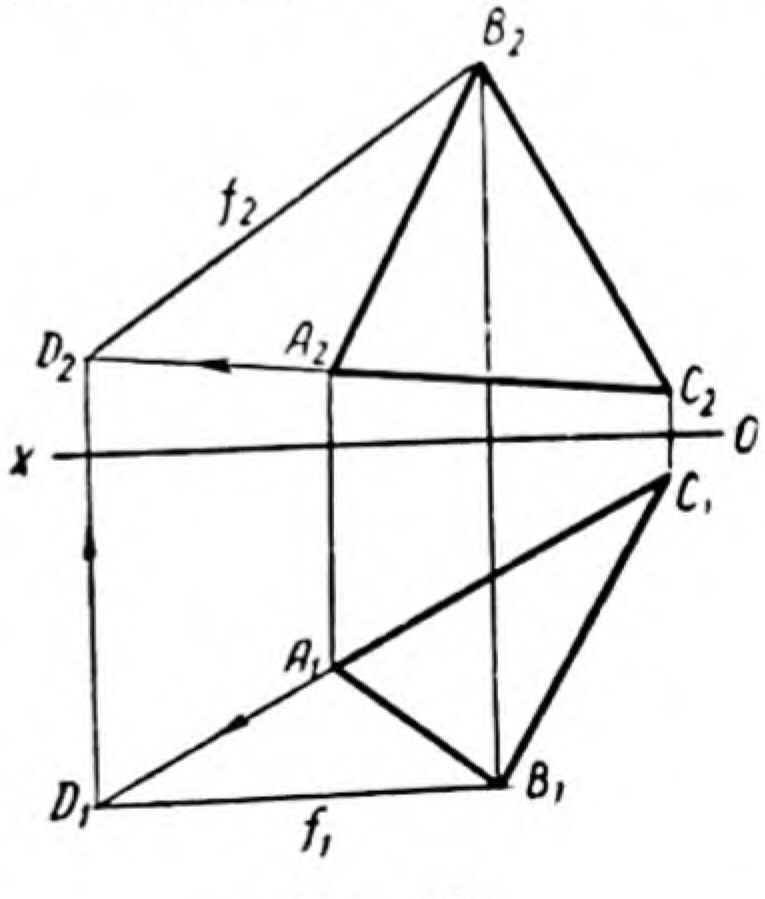


Fig. 106

3. A *profile line of a plane* is a line contained in a given plane and parallel to the plane of projection Π_3.

In the multiview drawing of a plane, which is given by its traces, the construction of the projections of a profile line in the plane is best begun by drawing its profile projection. This line is parallel to the profile trace of the plane. When a plane is given, say, by two parallel lines (Fig. 107), the construction should be begun by drawing the frontal projection of the profile line parallel to ground line Oz, or by drawing its horizontal projection parallel to ground line Oy_1.

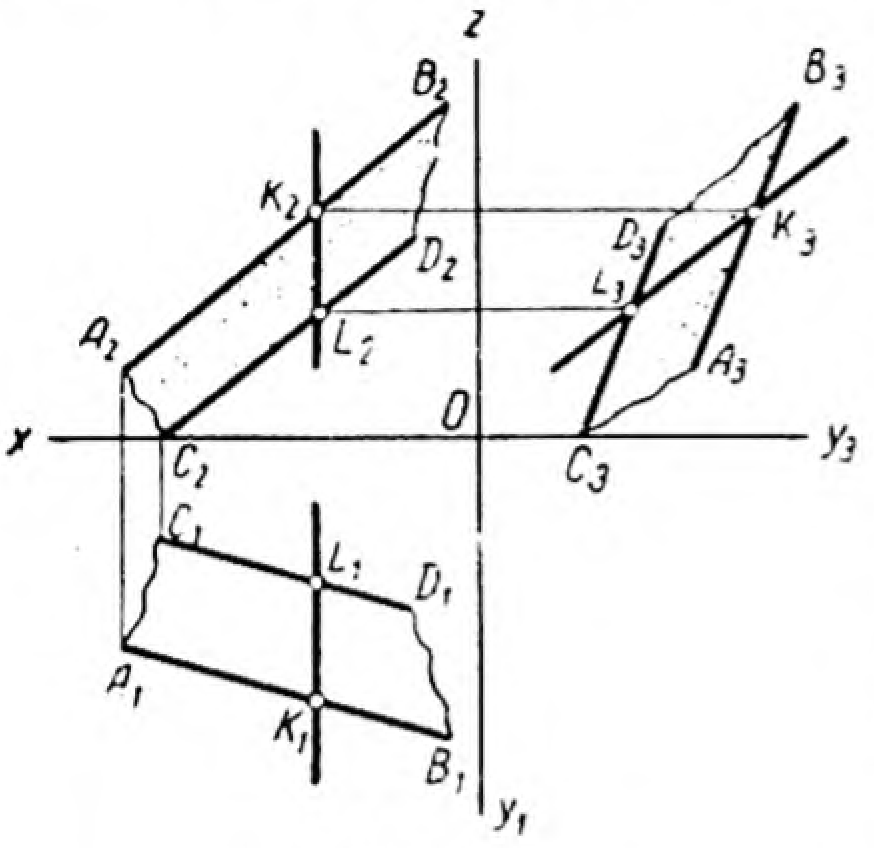


Fig. 107

Note. By analogy with the horizontals of a plane, the frontal and profile lines are also referred to as level lines.

4. In Figs. 108 to 111 the projections of horizontals and frontals contained in projecting planes are given. These are: the horizontal AB lying in a horizontal-projecting plane, the frontal CD in a horizontal-projecting plane, the horizontal KL in a frontal-projecting plane, and the frontal EF lying in a frontal-projecting plane.

Example 1. Construct the horizontal projection of line AB, contained in plane α and given by the frontal projection A_2B_2 (Fig. 112).

Through points A_2 and B_2 we draw the frontal projections, A_2N_2 and B_2N_2', of the auxiliary horizontals of plane α. Using points N_2 and N_2', determine the position on the ground line Ox of points N_1 and N_1' and through them, parallel to the trace k_1, draw the horizontal projections A_1N_1 and B_1N_1' of the auxiliary horizontals. By drawing projectors through points A_2 and B_2 points A_1 and B_1 are found. They lie at the intersections of the projectors with the horizontal projections of the corresponding auxiliary horizontals. The straight

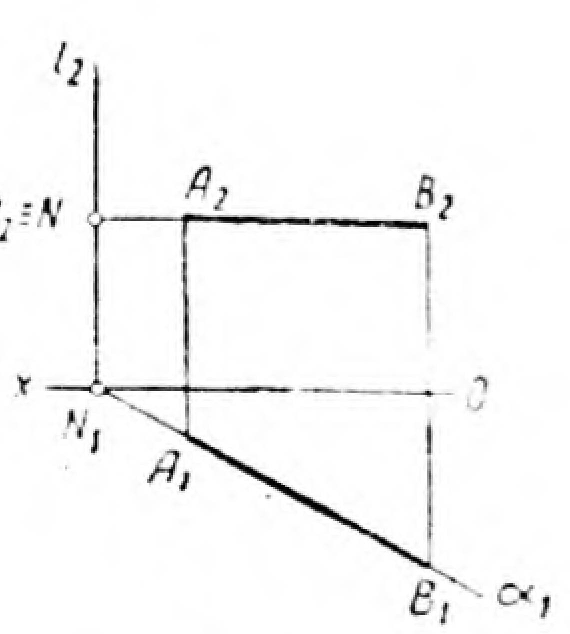


Fig. 108

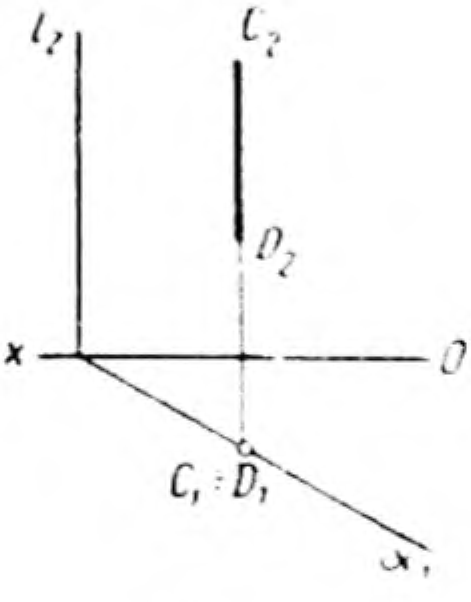


Fig. 109

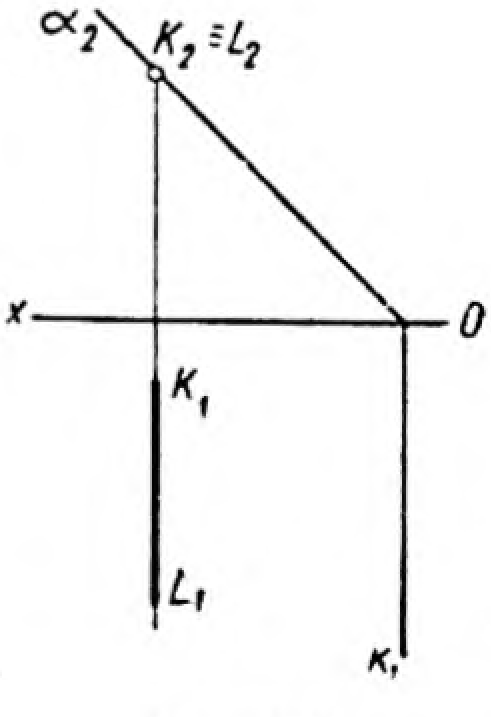
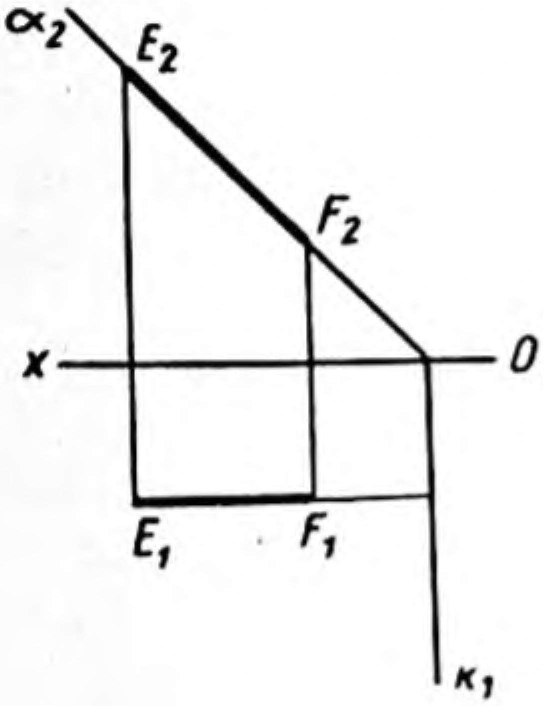


Fig. 110

line A_1B_1 drawn through points A_1 and B_1 is then the horizontal projection of line AB.

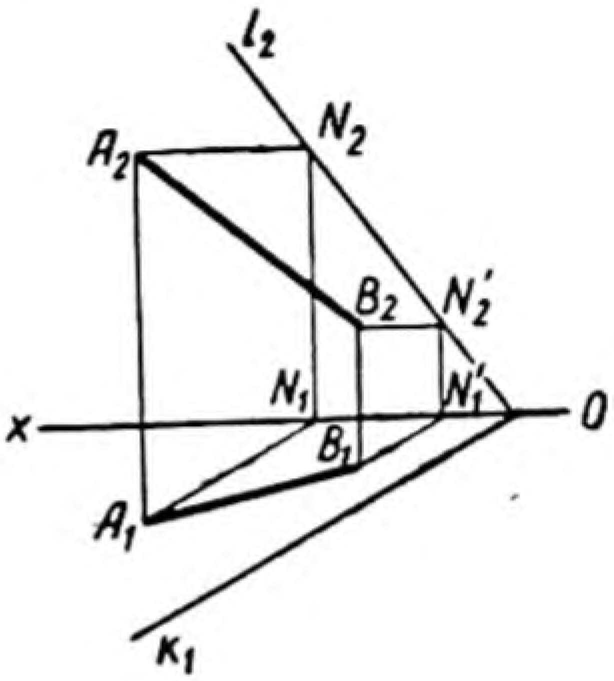
Example 2. Construct a plane α passing through point A and the point of convergence of traces X_α (Fig. 113).

An infinite number of planes may be drawn through the given two points. The construction of one of those planes is as follows:



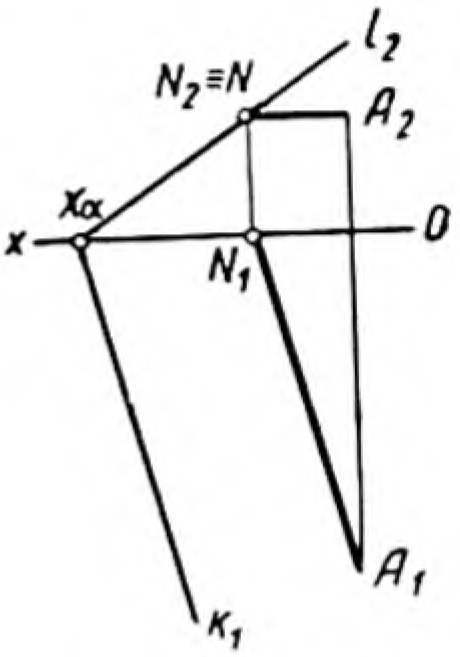
α_2
E_2
F_2
x
0
E_1
F_1
K_1

Fig. 111



l_2
N_2
A_2
B_2
N_2'
N_1
N_1'
x
0
B_1
A_1
K_1

Fig. 112



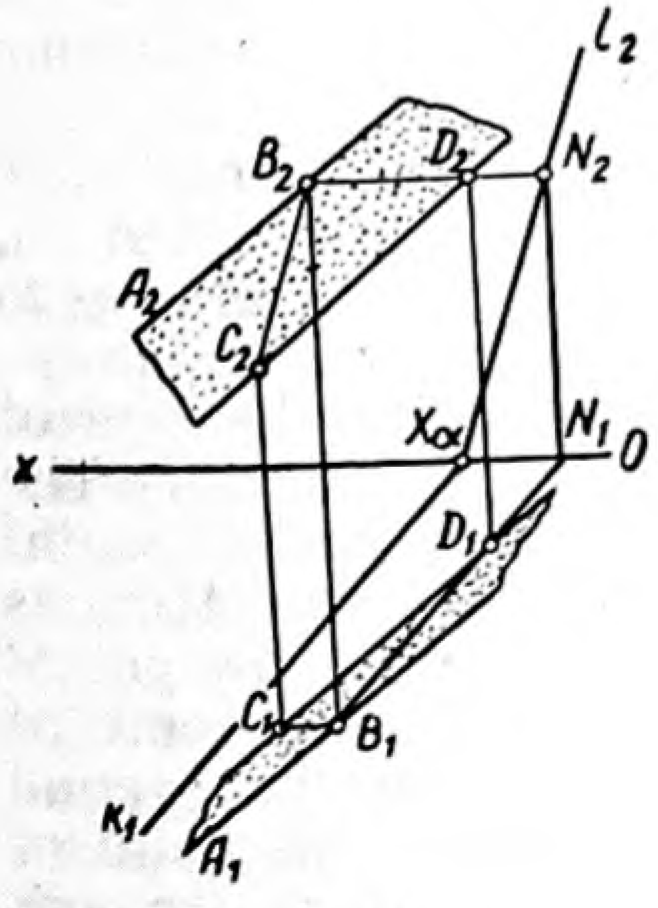
l_2
$N_2 \equiv N$
A_2
$X\alpha$
x
0
N_1
A_1
K_1

Fig. 113

Draw a horizontal through point A. Having found the trace N of this horizontal through point $N_2 \equiv N$ and X_α draw the frontal trace l_2. Draw the horizontal trace k_1 of the required plane α through the point X_α parallel to A_1N_1.

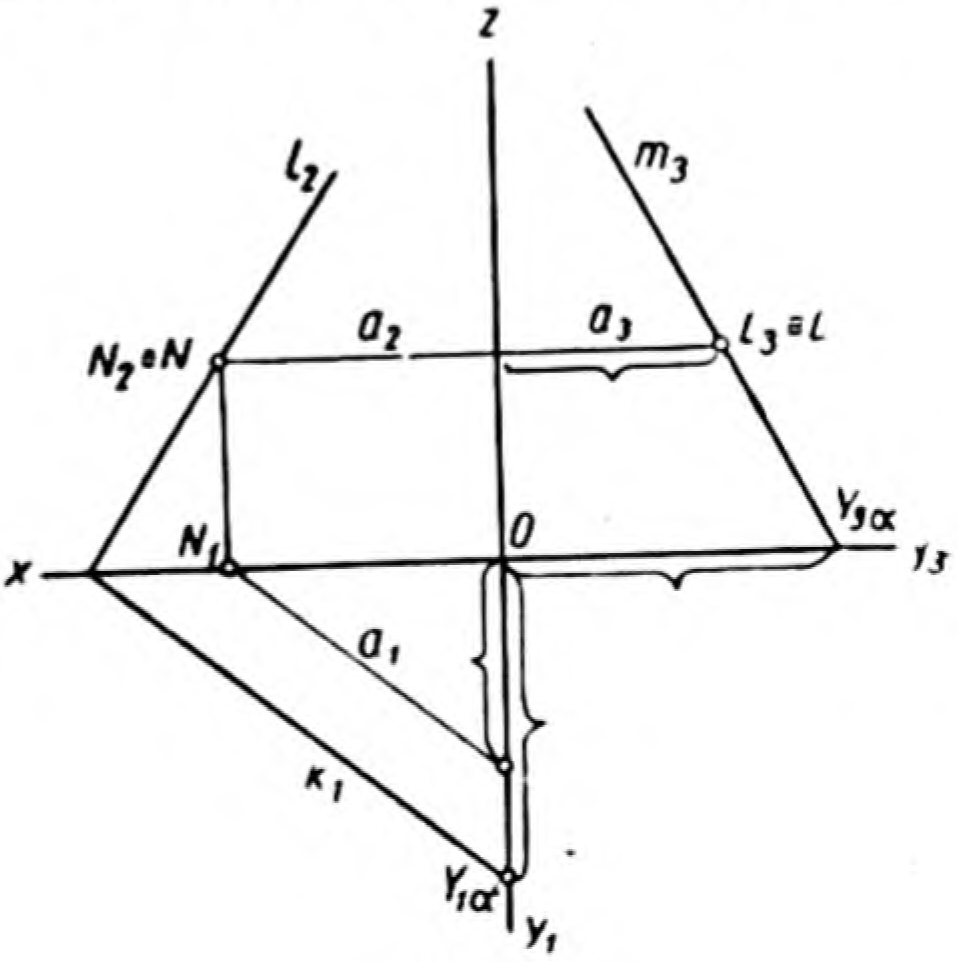
Example 3. Construct the traces k_1 and l_2 of plane α which is given by parallel lines AB and CD (Fig. 114).

First in the given plane draw the horizontal BD and frontal BC. Find trace N of the horizontal BD. Through point N_2, parallel to the projection of the frontal B_2C_2, draw the frontal trace l_2 of plane α



l_2
B_2
D_2
N_2
A_2
C_2
$X\alpha$
N_1
x
0
D_1
C_1
B_1
K_1
A_1

Fig. 114



z
l_2
m_3
$N_2 \equiv N$
a_2
a_3
$L_3 \equiv L$
N_1
0
$Y_{3\alpha}$
y_3
x
a_1
K_1
$Y_{1\alpha}$
y_1

Fig. 115

and extend it to intersect the ground line Ox in point X_α. Then, through point X_α parallel to B_1D_1 draw the horizontal trace k_1 of the plane.

Example 4. Construct the profile trace m_3 of plane α which is given by traces k_1 and l_2, the point of convergence Z_α of traces being located beyond the limits of the drawing (Fig. 115).

Draw any horizontal in the plane α and find its profile trace L. The point of convergence $Y_{1\alpha}$ is carried over to ground line Oy_3 as point $Y_{3\alpha}$. Join up the points $L_3 \equiv L$ and $Y_{3\alpha}$. This line m_3 is the profile trace of plane α.

5. Lines of maximum inclination of a plane. *A line of maximum inclination of a plane* is a line contained in the given plane and perpendicular to its trace.

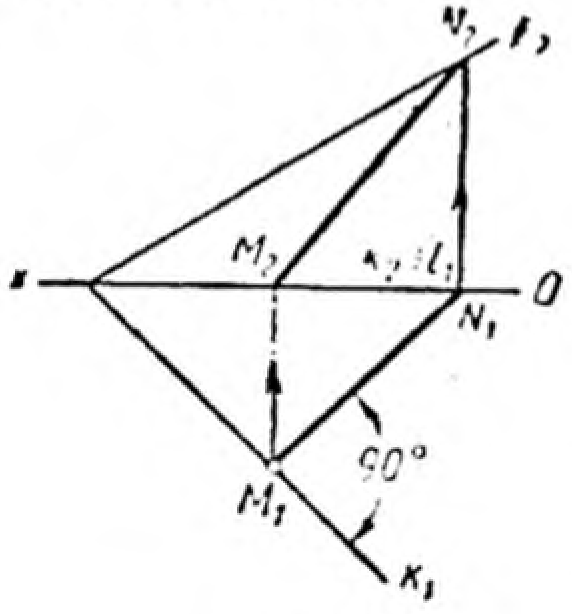


Fig. 116

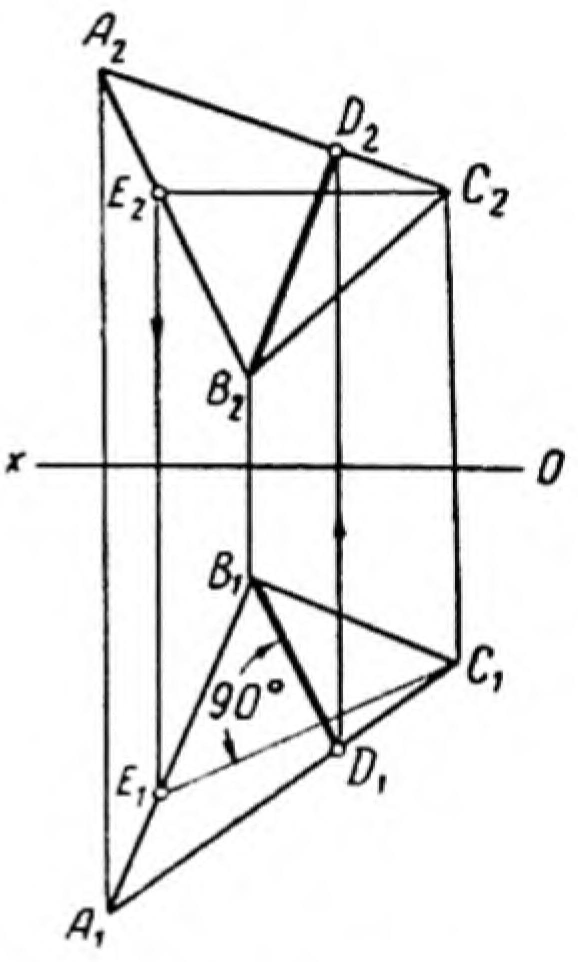


Fig. 117

Hint. The seam of metal sheet roofing perpendicular to the cornice may serve to illustrate what is meant by a line of maximum inclination of a plane.

In a given plane different lines of maximum inclination may be found, i. e., a) relative to the horizontal plane of projection Π_1, b) relative to the frontal plane of projection Π_2, and c) relative to the profile plane of projection Π_3.

In Fig. 116 the plane α is shown together with the line of maximum inclination MN of the plane relative to the plane Π_1. The construction of the line $MN \perp k$ is begun by drawing the horizontal projection $M_1N_1 \perp k_1$ (see § 13). The frontal projection M_2N_2 is drawn through points M_2 and N_2, which are determined by points M_1 and N_1. Point M_2, being the frontal projection of point M contained in the horizontal trace k_1, is located on the ground line Ox. The frontal projection N_2 of point N lies at the point of intersection of trace l_2 and the vertical projector drawn through point N_1.

The line of maximum inclination or slope of plane α to plane Π_1, being perpendicular to the horizontal trace of plane α, is also obviously perpendicular to any horizontal in the plane α. Therefore the line of maximum inclination of plane α, when the plane is given by triangle ABC (Fig. 117), is constructed with the help of an auxiliary horizontal drawn through the vertex C of the triangle. The horizontal projection B_1D_1 of the line of maximum inclination is then passed through B_1 perpendicular to the horizontal projection C_1E_1 of the horizontal CE. The frontal projection is drawn through point B_2 and point D_2, the latter being determined by a vertical projector passing through D_1.

The line of maximum inclination MN of plane α relative to the frontal plane of projection Π_2 is shown in Fig. 118. The frontal projection M_2N_2 of this line MN is perpendicular to the trace l_2.

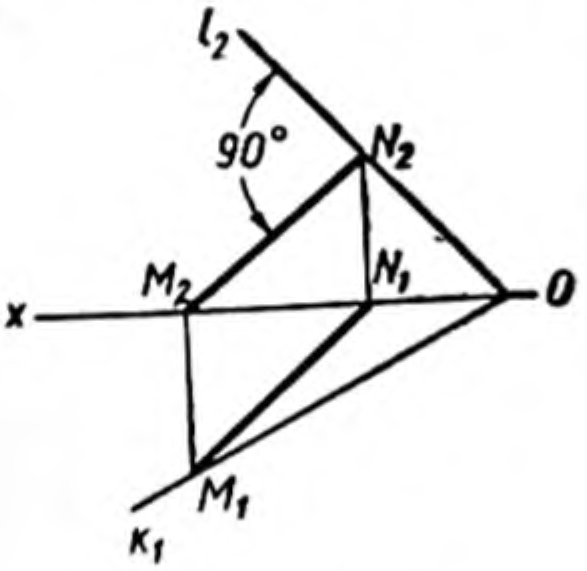


Fig. 118

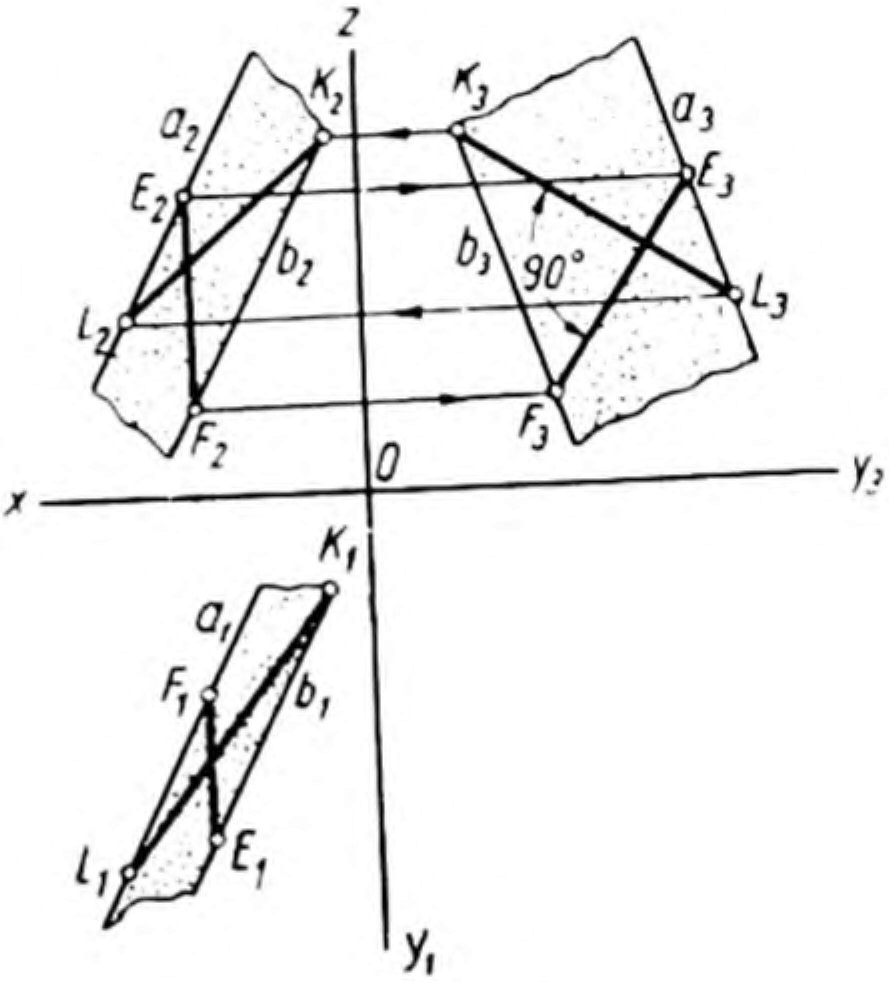


Fig. 119

The points M_1 and N_1 are constructed with the help of points M_2 and N_2, as these are the projections of points lying on the traces k_1 (point M) and l_2 (point N). M_1N_1 is the horizontal projection of the line of maximum inclination of plane α to plane Π_2.

Line KL of maximum inclination of plane γ relative to plane Π_3 (Fig. 119) is constructed in a similar manner. The profile projection K_3L_3 will be perpendicular to the profile projection E_3F_3 of the profile line EF. Then, by using points K_3 and L_3, points K_2 and L_2, K_1 and L_1 are found. Through them draw the frontal projection K_2L_2 and the horizontal projection K_1L_1 of the line of maximum inclination of plane γ to plane Π_3.

6. With the assistance of lines of maximum inclination the angles of inclination of a given plane to the planes of projection may be determined.

These angles are given by the angles formed by the corresponding lines of maximum inclination to planes of projection Π_1, Π_2 and

Π_3. The true value of these angles may be determined by the right triangle method (see § 8).

For example, determine the true magnitude of the angle of inclination of plane α to the frontal plane of projection Π_2 (Fig. 120).

In the plane α construct a line MN perpendicular to the frontal trace l. This will be the line of maximum inclination of plane α to plane Π_2. The frontal projection M_2N_2 of this line is drawn perpendicular to trace l_2. Once the frontal projection is found, the horizontal projection M_1N_1 is easily determined. It should be borne in mind that point N lies on the frontal trace and M on the horizontal trace. The projection M_1 of the latter therefore lies on k_1.

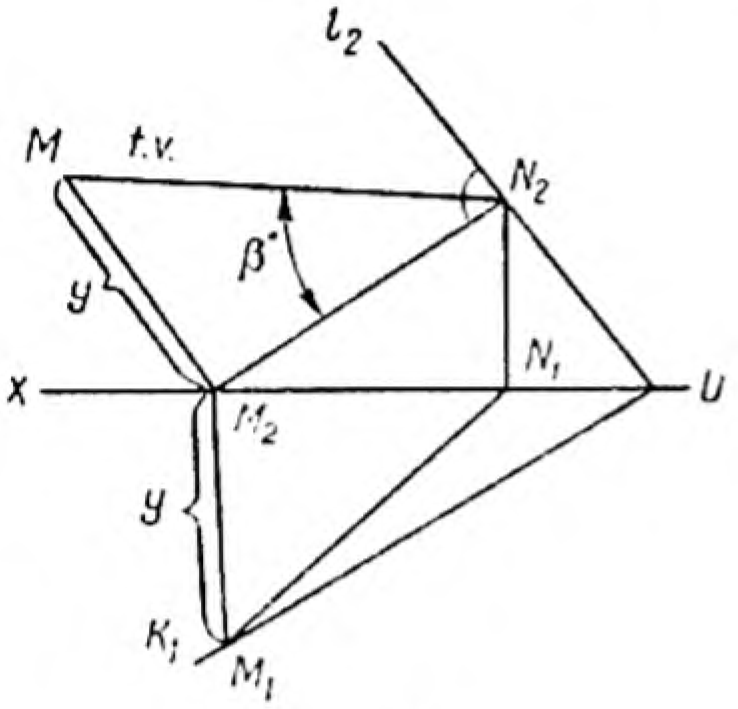


Fig. 120

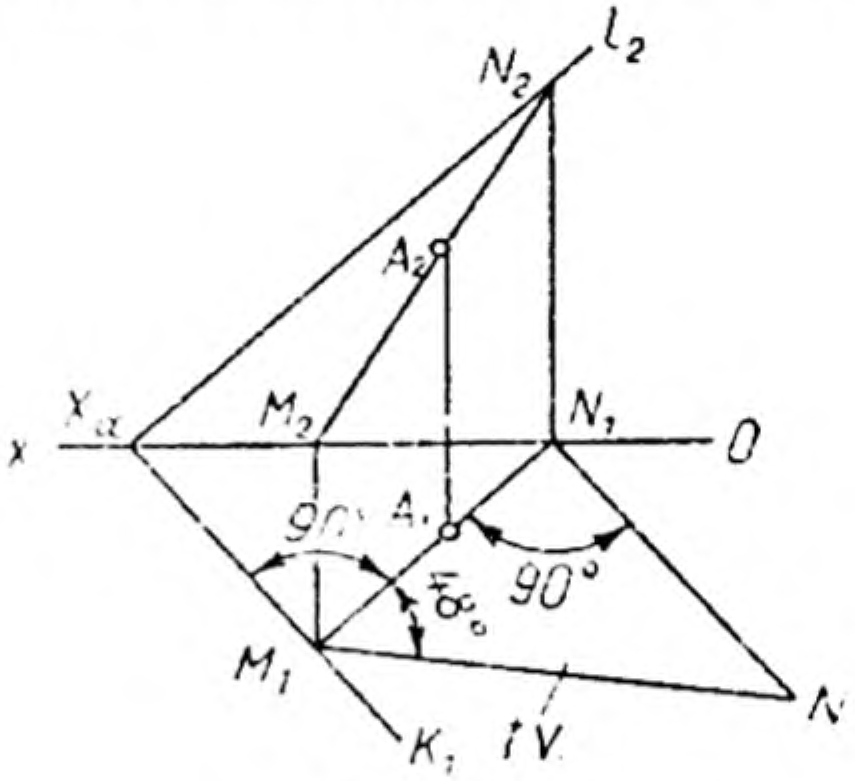


Fig. 121

The angle of inclination of line MN to plane Π_2 is determined by constructing the right triangle $M_2N_2\overline{M}$. One side of this triangle is the projection M_2N_2. The second side is equal to the difference between the distances of points M and N from plane Π_2, i. e., to the value of y for point M_1. The angle β between the hypotenuse $N_2\overline{M}$ of the resulting right triangle and projection M_2N_2 is the angle of inclination of the line MN to plane Π_2. At the same time $\beta°$ is the true inclination of plane α to that same plane Π_2.

7. By means of lines of maximum inclination a plane may be constructed so that it forms a given angle to the plane of projection.

The method of constructing a plane passing through a point A and inclined at 48° to the plane Π_1 (Fig. 121) is given below.

First the line MN, passing through point A and making an angle of 48° to plane Π_1, should be constructed. In order to do this draw line M_1N_1 at any angle to the ground line Ox. This line will serve as the side of the right triangle. Lay off from this side two angles — an angle of 48° from point M_1 and an angle of 90° from point N_1.

A distance equal to the length of the side $N_1\overline{N}$ is then laid off from point N_1 on the perpendicular above ground line Ox. The point N_2 thus found is connected to point M_2 which is obtained by drawing through point M_1 a perpendicular to ground line Ox to their intersection.

The lines M_1N_1 and M_2N_2 are the projections on Π_1 and Π_2 of line MN drawn through point A at an angle of 48° to plane Π_1.

Next, through this line MN a plane α should be drawn so that MN serves as its line of maximum inclination relative to Π_1. In order to do this draw the trace k_1 at an angle 90° to the projection M_1N_1 and the trace l_2 passing through the point X_α of convergence of the traces and point N_2.

Problems and Exercises

1. How may a plane be determined?
2. What is a trace of a plane?
3. What is meant by the term an "oblique plane"?
4. What is meant by the term a "projecting plane"?

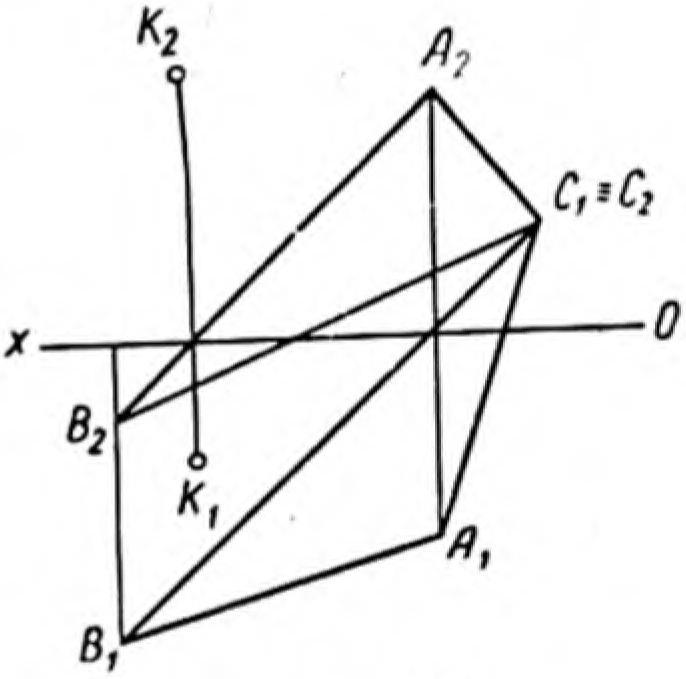


Fig. 122

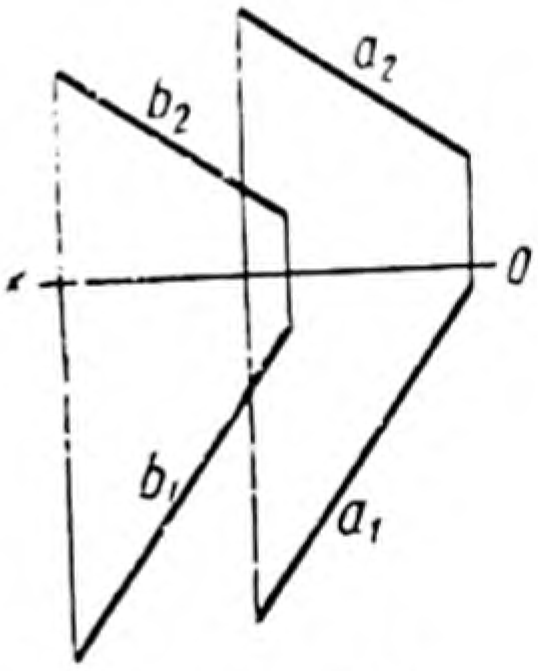


Fig. 123

5. Describe the positions of the traces of a plane lying parallel to the ground line Ox.

6. When do the traces of planes on the multiview drawing merge into one line cutting through ground line Ox?

7. How are the traces of a plane constructed when it is given by two intersecting lines?

8. How is a point lying in an oblique plane constructed?

9. How can it be checked whether a point is contained in a plane?

10. What is meant by a "horizontal of a plane"?

11. How is a plane constructed to pass through a given point?

12. What is meant by a "line of maximum inclination of a plane"?

13. How is the angle of inclination of a plane to the horizontal plane of projection determined?

14. Is the point *K* contained in the plane of the triangle *ABC* (Fig. 122)?

Hint. Through projections A_1 and K_1 draw the projection of an auxiliary straight line contained in the plane of the triangle *ABC*.

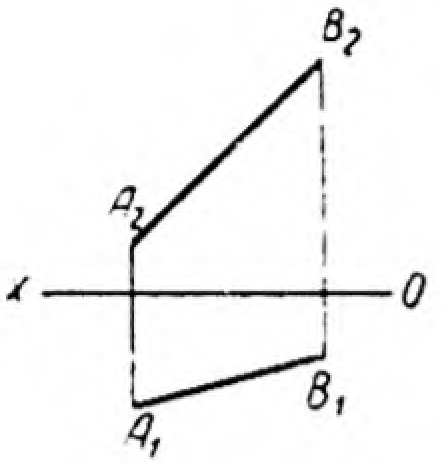


Fig. 124

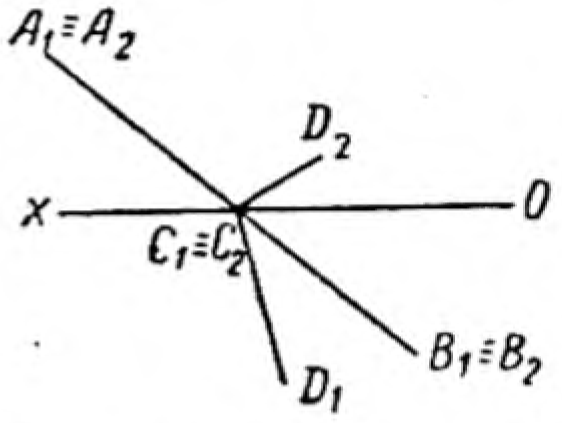


Fig. 125

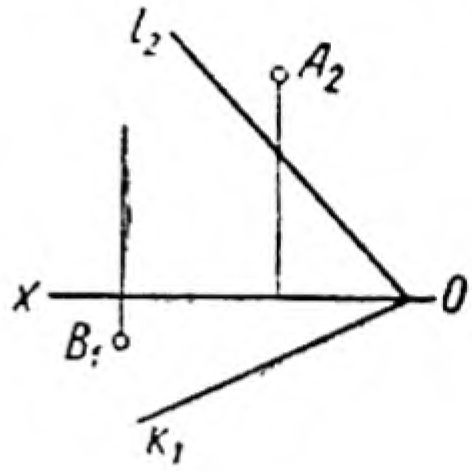


Fig. 126

When the frontal projection of this line is found, it may be seen that the line constructed in the plane *ABC* does not pass through point *K* in space.

15. Construct the traces of a plane given by two parallel lines (Fig. 123).

Hint. It is convenient to draw a horizontal and a frontal in the given plane and find the trace of one of them. This is of course not the only method. Any line lying in the given plane may be used, including lines *a* and *b*, but the method recommended is the most simple.

16. Through a given point *A* draw a plane α, the traces of which merge into one straight line.

17. Through line *AB* draw an oblique plane and a horizontal-projecting plane (Fig. 124).

18. Construct the traces of the plane given by the intersecting lines *AB* and *CD* (Fig. 125). (See exercise 15.)

19. Construct the projection of triangle *ABC* contained in the given plane (Fig. 126). Use lines of altitude.

20. Plane α is given by point *A* and a frontal trace. Construct the horizontal trace of the plane (Fig. 127).

Hint. Through point *A* in the given plane draw a horizontal.

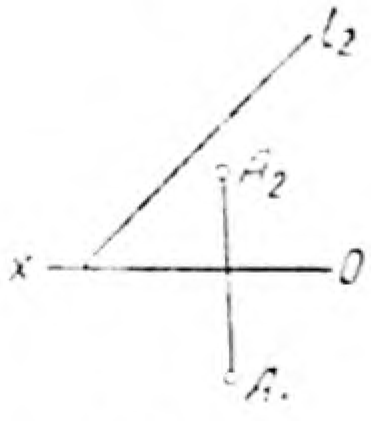


Fig. 127

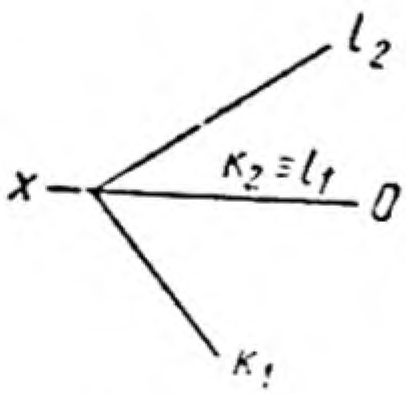


Fig. 128

Remember that the traces of lines lie on the corresponding traces of its planes.

21. Determine the angles of inclination of the plane of triangle ABC to the planes of projection Π_2 and Π_3 (Fig. 117).

Hint. To solve the problem it is necessary to draw in the plane of triangle ABC the frontal and profile lines and, perpendicular to them, lines of maximum inclination of plane ABC relative to the planes of projection Π_2 and Π_3 (see text pertaining to Fig. 117).

22. Determine the angles of inclination of plane α to planes of projection Π_1 and Π_3 (Fig. 128).

23. A plane α passing through ground line Ox and through a point A is given, as is the frontal projection B_2C_2 of line BC contained in this plane. Construct the horizontal projection B_1C_1 without using the profile plane of projection Π_3.

Hint. To solve this problem it is necessary to draw an auxiliary line intersecting BC through point A of plane α.

Chapter V

LINES AND PLANES

§ 18. Straight Lines Parallel to Planes. Parallel Planes

1. In elementary geometry it is demonstrated that a straight line is parallel to a given plane if the line is parallel to at least one line contained in the plane, or if it lies in a plane parallel to the given plane.

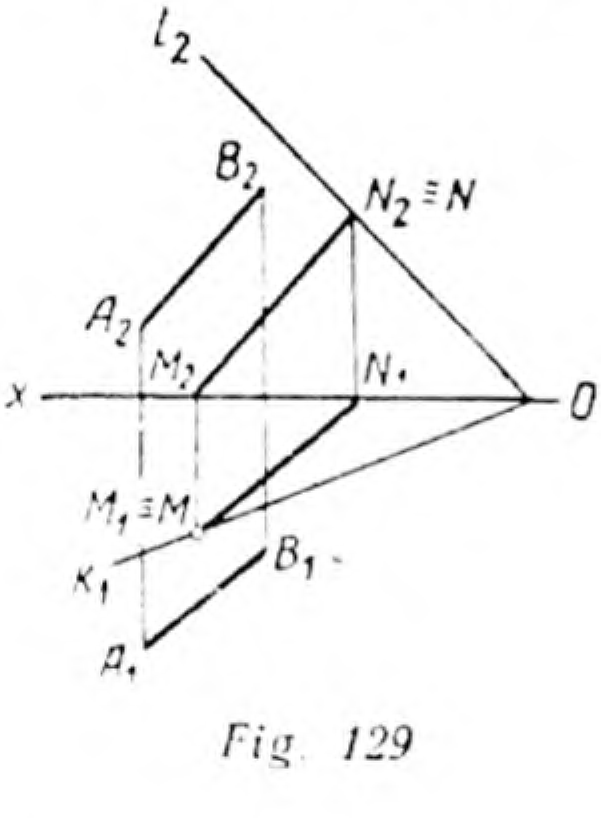


Fig. 129

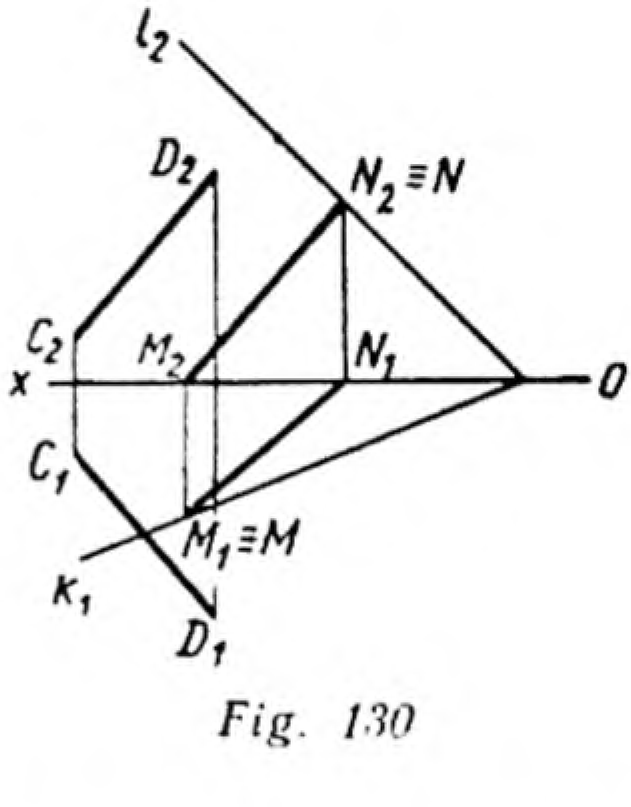


Fig. 130

The corresponding projections of two parallel lines are also parallel. It follows that the line AB in Fig. 129 is parallel to the plane α, since the projections A_1B_1 and A_2B_2 of this line are parallel to the corresponding projections M_1N_1 and M_2N_2 of line MN contained in plane.

The line CD, shown in Fig. 130, is not parallel to plane β because it is not possible to draw in the given plane β a line parallel to the given line CD. This can be demonstrated as follows: parallel to projections C_2D_2 draw the projection M_2N_2 of line MN contained in plane β and construct the projection M_1N_1 from projection M_2N_2. It is seen that M_1N_1 is not parallel to the projection C_1D_1. This means that line CD is not parallel to the plane β.

2. If it is required to draw through point K (Fig. 131) a straight line AB parallel to the plane of triangle CDE, first in the plane of the triangle draw an auxiliary line EF. Then through point K draw a line AB parallel to EF, i. e., so that $A_1B_1 \parallel E_1F_1$ and $A_2B_2 \parallel E_2F_2$. The constructed line AB is parallel to the plane of the triangle CDE. Another solution would be to draw through point K a line parallel to one of the sides of the triangle. In this case it is not necessary to construct an auxiliary line in the plane of the triangle.

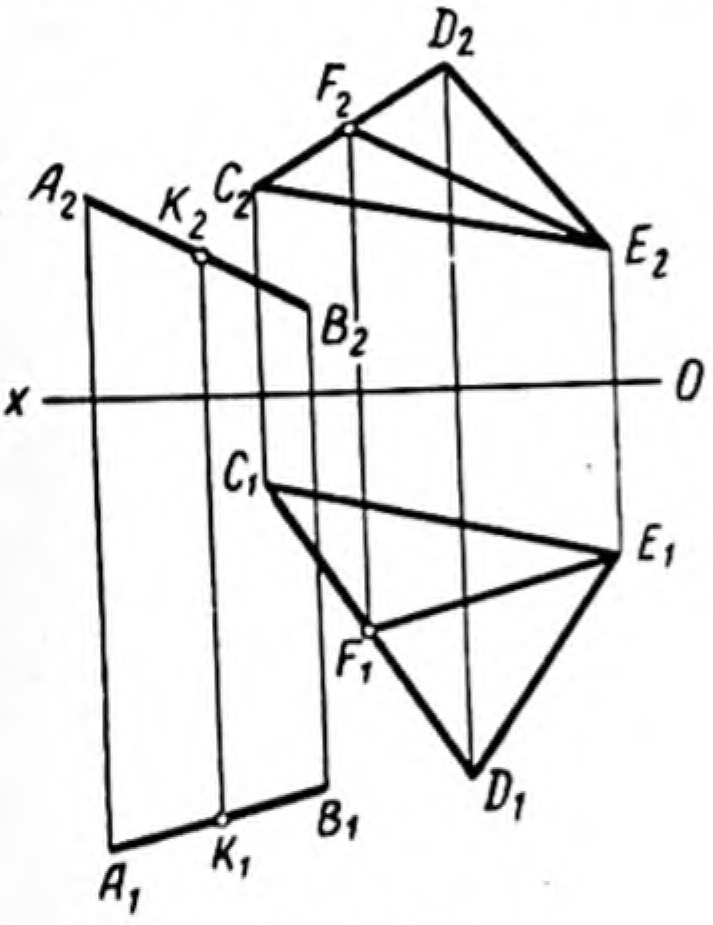


Fig. 131

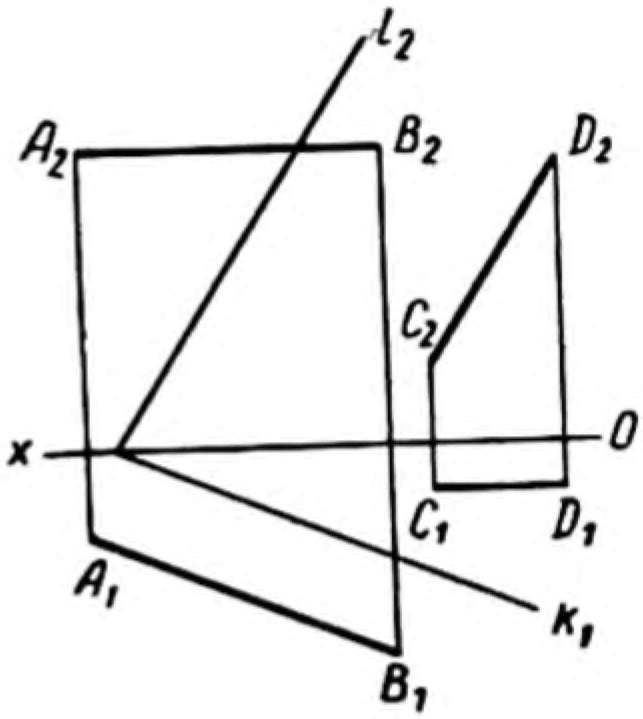


Fig. 132

The construction of horizontal and frontal lines parallel to a given plane, when it is defined by its traces, is also carried out directly, without using auxiliary lines, since the lines required will be parallel to the respective traces. The construction of the horizontal AB parallel to the horizontal trace k and frontal CD parallel to the frontal trace l of plane α is shown in Fig. 132. Lines AB and CD are parallel to α.

3. The construction of a plane parallel to a given line is carried out with the help of an auxiliary line. A plane passing through an auxiliary line drawn parallel to the given line will be parallel to the latter.

In order to construct a plane α parallel to the given line AB and passing through point C (Fig. 133) draw a straight line CD of any suitable length parallel to AB and join the points C and D with any point E. The plane defined by triangle CDE is one of the possible planes parallel to line AB passing through point C.

To construct the traces of a plane which is parallel to a given line AB and passes through a given point K (Fig. 134), through point K draw an auxiliary line KL parallel to AB. Then construct the traces M and N of the line KL. Through these traces and a point taken at random on the ground line Ox, which is assumed as X_α, the point of convergence of the traces, draw the traces k_1 and l_2 of the required plane α, which is one of the planes satisfying the given conditions. If the traces of the plane were to be drawn through points M and N parallel to ground line Ox, then the plane constructed would be parallel to line KL and to ground line Ox.

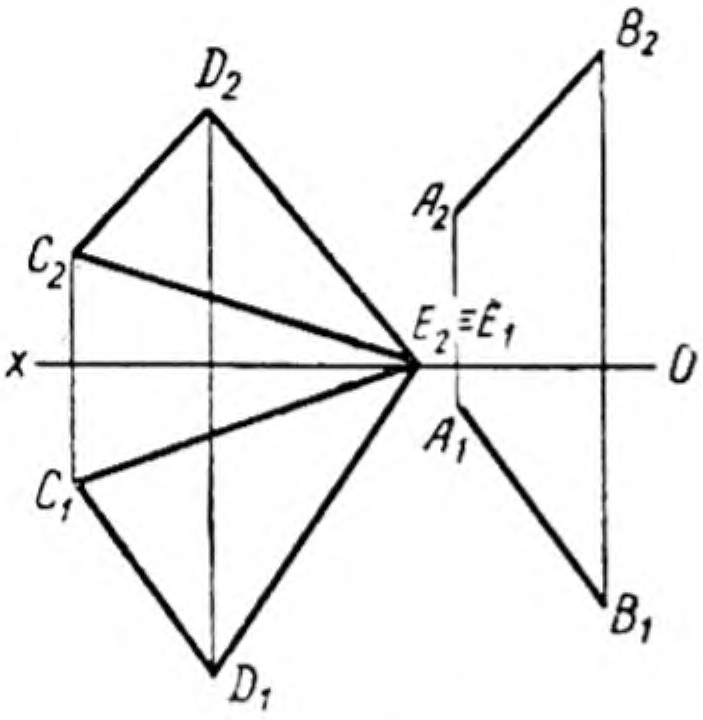


Fig. 133

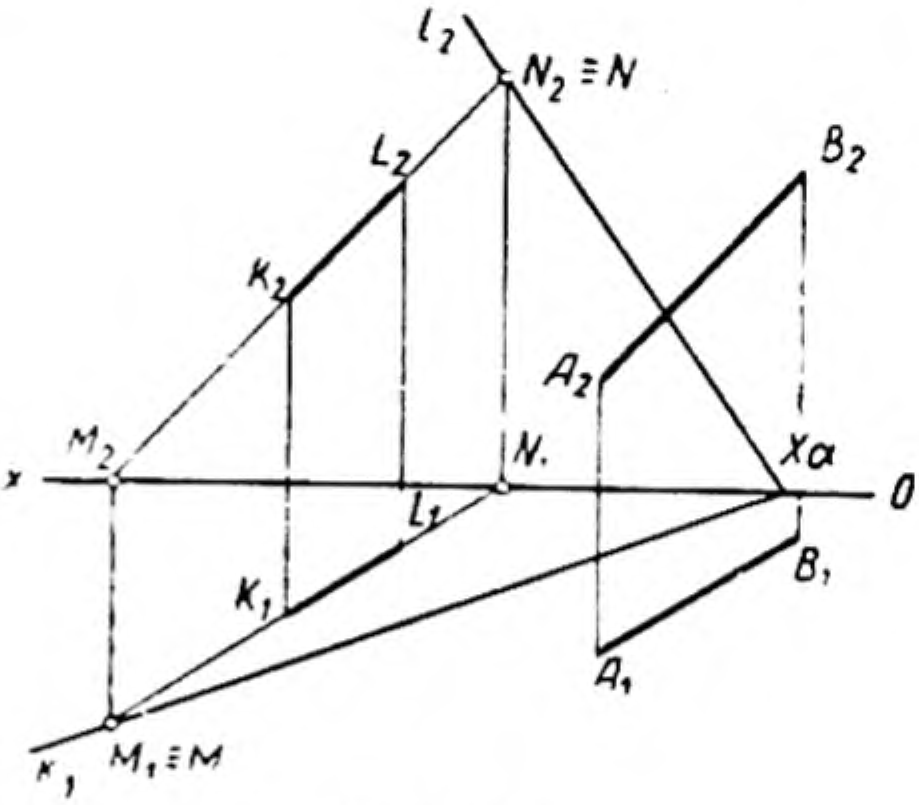


Fig. 134

4. The lines of intersection of two mutually parallel planes with any third plane are parallel lines (Fig. 135, where $AB \| CD$). Since the traces of planes are straight lines along which the given planes intersect the planes of projection Π_1, Π_2 and Π_3, corresponding traces of parallel planes are parallel to each other (Fig. 136, where $k_1 \| k_1'$ and $l_2 \| l_2'$).

When the traces of two planes are not given, these planes are recognized as parallel, if two intersecting lines contained in the first plane are respectively parallel to two intersecting lines contained in the second plane. Horizontal and frontal lines may serve as the two intersecting lines in each of the planes. The two planes given by triangles ABC and DEF (Fig. 137) are parallel to each other since the corresponding projections of their horizontals

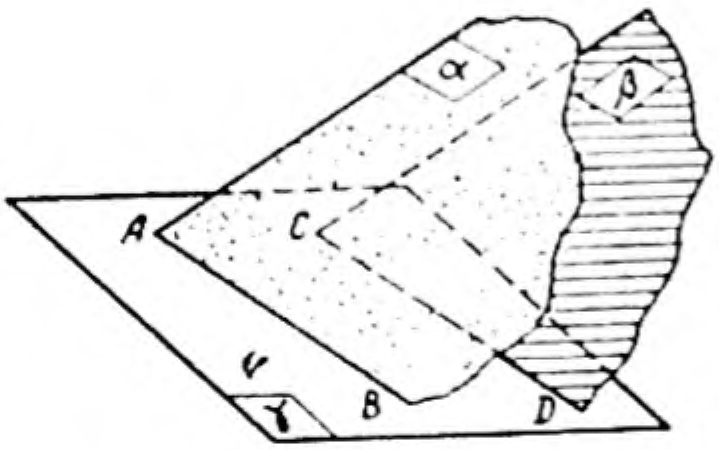


Fig. 135

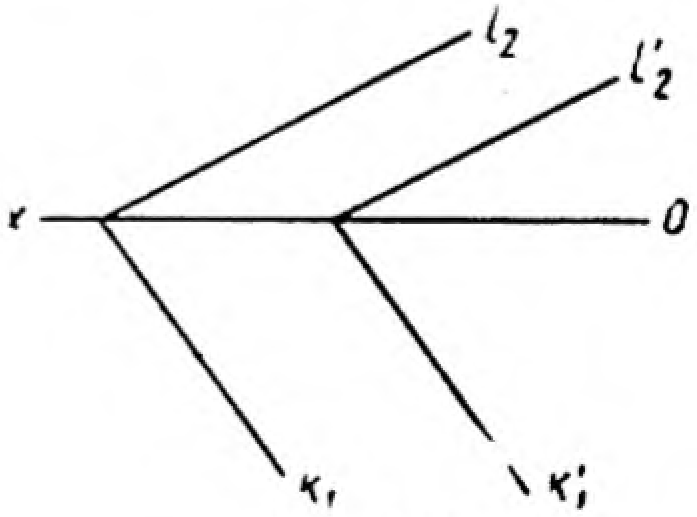


Fig. 136

CM and *DK* and of the frontals *AN* and *EL* are parallel ($CM \| D_1K_1$ and $A_2N_2 \| E_2L_2$).

5. Even if two planes are parallel to ground line *Ox* and their horizontal and frontal traces are parallel to that ground line, it may well happen that such planes are not parallel to each other. To determine whether such planes are really parallel it is necessary to construct their profile traces or to check whether the ratios of the distances of corresponding traces from the ground line *Ox* are equal.

The planes α and β, shown in Fig. 138, are not parallel to each other, although their corresponding traces in Π_1 and Π_2 are parallel ($k_1 \| k_1'$ and $l_2 \| l_2'$). As can be seen the profile traces α_3 and β_3 intersect.

The planes α and β, shown in Fig. 139, are parallel to each other, since the corresponding pairs of traces on the planes of projection Π_1, Π_2 and Π_3 are parallel. Due to this the ratios mentioned above are equal, e. g., $m : n = q : s$.

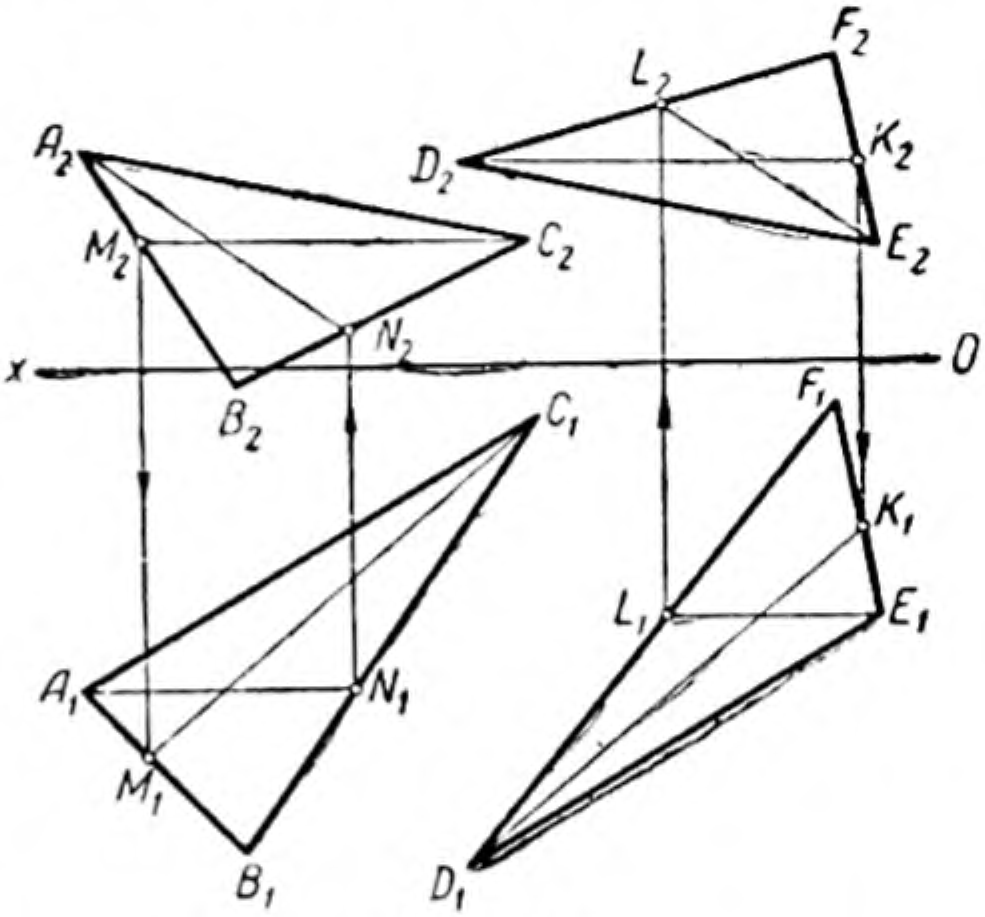
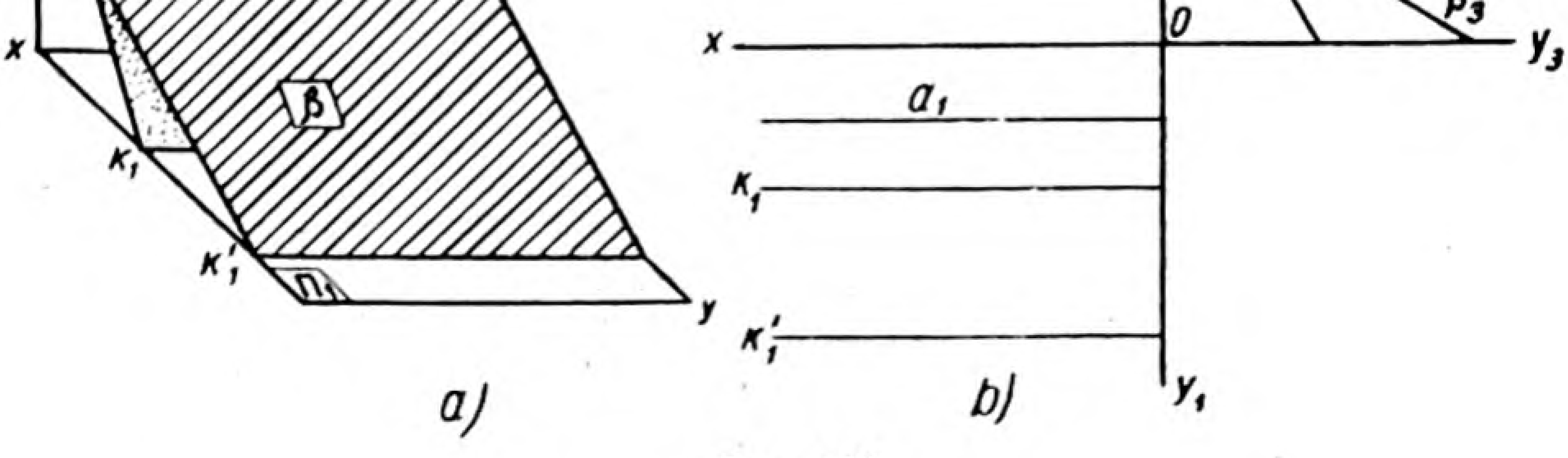

A_2
D_2
L_2
F_2
K_2
M_2
C_2
E_2
N_2
x
0
B_2
C_1
F_1
K_1
L_1
E_1
A_1
N_1
M_1
B_1
D_1

Fig. 137

6. The method of drawing a plane parallel to a given plane is based on the fact that the plane to be construct-

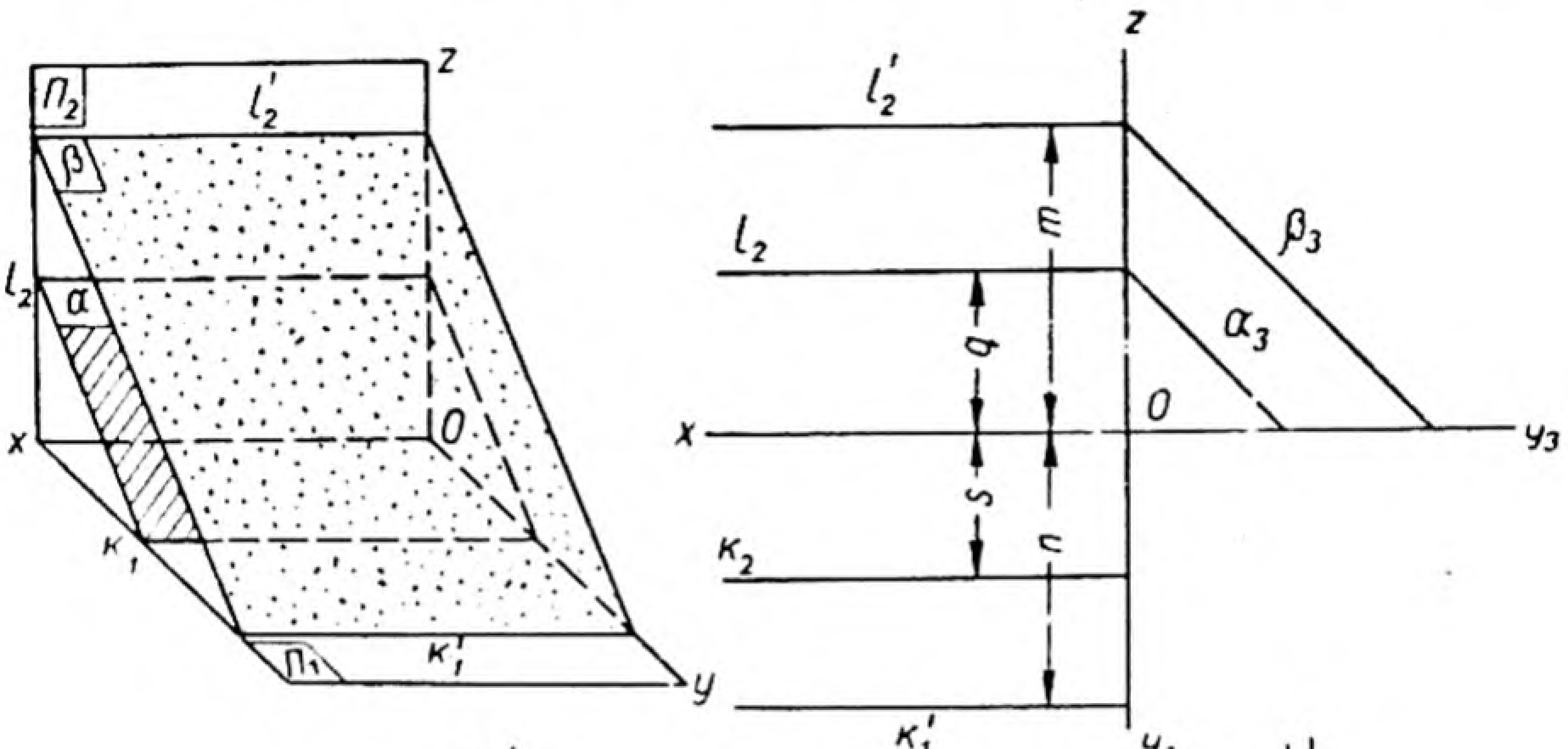

z
Π_2
l_2
l_2'
α
a
x
β
K_1
K_1'
Π_1
y
z
l_2
l_2'
a_2
α_3
a_3
β_3
0
x
y_3
a_1
K_1
K_1'
y_1

a) b)

Fig. 138

ed should contain two intersecting lines respectively parallel to two intersecting lines contained in the given plane.

Let plane α be given by a triangle *ABC*. It is required to draw a plane β parallel to plane α and passing through point *K* (Fig. 140). The construction of the required plane may be carried out with the help of two intersecting lines parallel to two corresponding lines contained in the plane of triangle *ABC*. Two sides of the triangle may serve as such lines. Draw through point *K* line $a \| AC$ and line $b \| BC$. The plane β, passing through intersecting lines *a* and *b*, will be the required plane.



a) b)

Fig. 139

When plane α is given by its traces k_1 and l_2 and it is required to draw through point *A* plane β parallel to plane α the construction is carried out as follows (Fig. 141):

1. Through point *A* draw a horizontal line *AN* parallel to plane α ($A_1N_1 \| k_1$ and $A_2N_2 \| Ox$).
2. Find the frontal trace *N* of this line.
3. Through point $N_2 \equiv N$ draw the trace n_2 parallel to l_2.
4. Through point X_β draw the trace m_1 parallel to trace k_1. The plane β is then parallel to the given plane α.

Instead of using a line

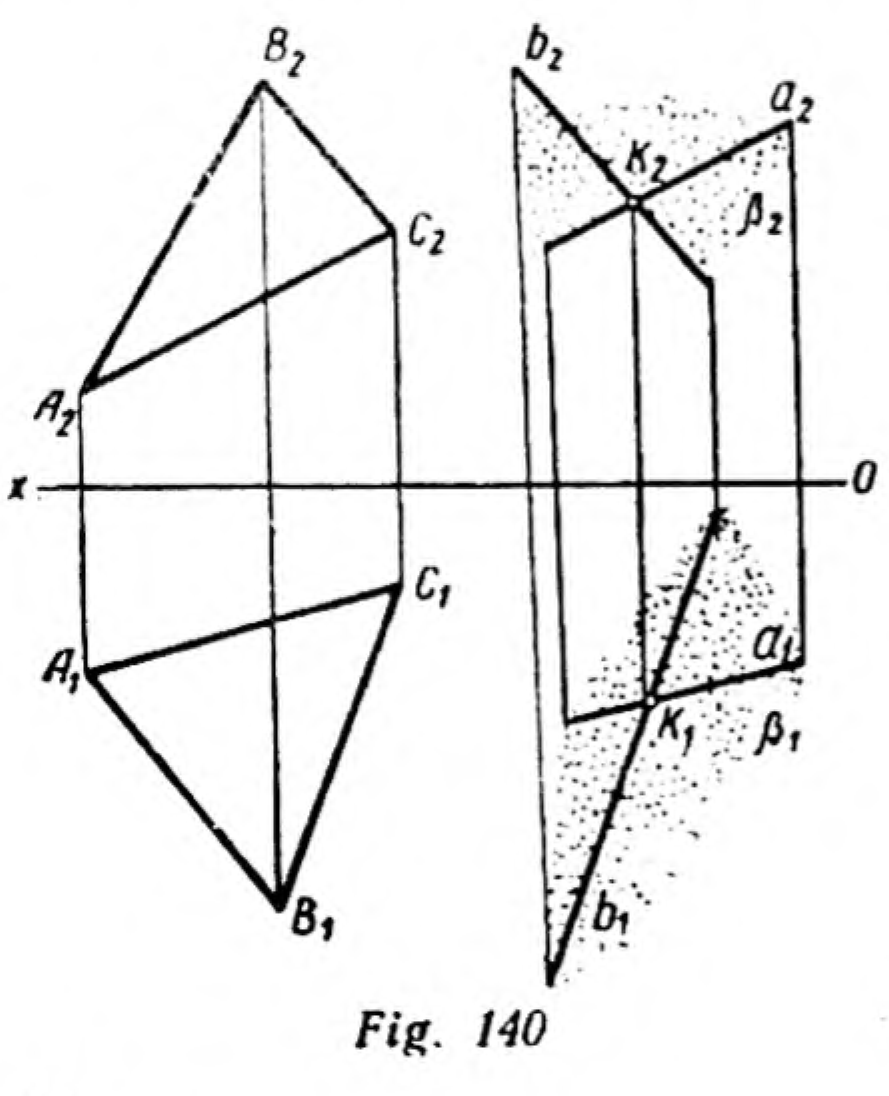


Fig. 140

parallel to the plane of projection when solving this problem (Fig. 141) it would, of course, also be possible to use oblique lines.

7. A plane parallel to a given plane can be made to pass through a given line only if the given plane and line are mutually parallel.

The plane α and the line a parallel to plane α ($a \| b$) are given. It is required to draw through line a a plane β parallel to plane α (Fig. 142). Construct the traces M and N of line a and through them draw the corresponding traces m_1 and n_2 of plane β parallel to the traces k_1 and l_2 respectively. The plane β constructed in this way is the required plane.

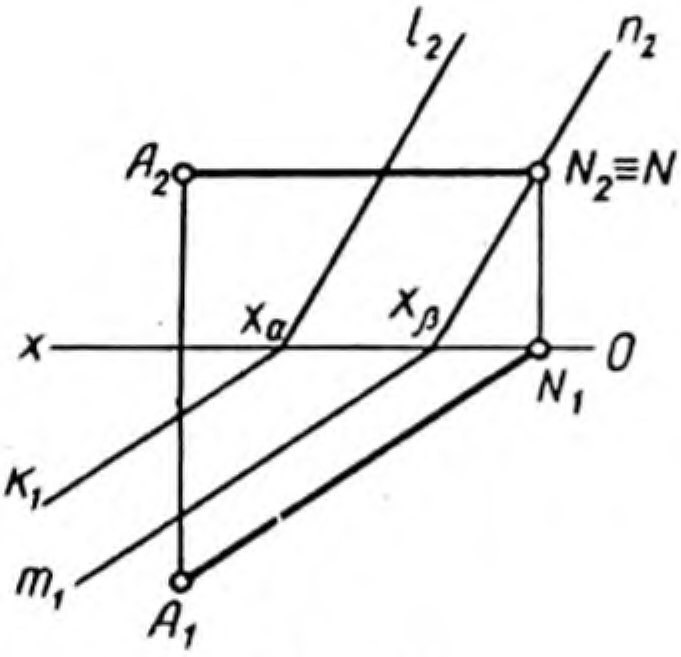


Fig. 141

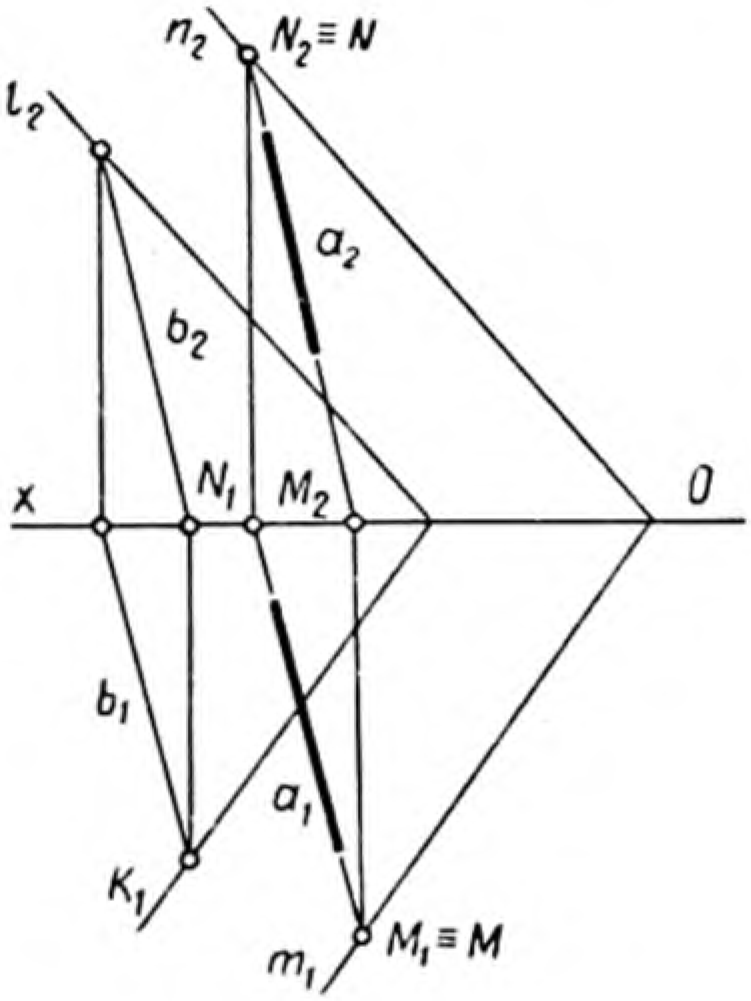


Fig. 142

Problems and Exercises

1. How is it possible to determine whether a given line and plane are parallel?
2. How may a plane be drawn parallel to a given plane?
3. How may a plane be drawn parallel to a given line?
4. Is the line a parallel to plane α (Fig. 143)?
5. Through point A draw a line parallel to the horizontal-projecting plane (Fig. 144).
6. Through point A draw a plane parallel to plane α which is parallel to the ground line Ox (Fig. 145).

Hint. First, in the given plane draw an auxiliary straight line passing through two points taken on the traces k and l. Then, having drawn through point A a second line, parallel to the auxiliary line, find the traces of this second line through which the traces of the required plane will pass (see Fig. 142 and text).

7. Construct the horizontal projection of triangle ABC the plane of which is parallel to the horizontal-projecting plane α (Fig. 146).

8. Through point D draw a plane α parallel to the plane given by the intersecting lines AB and BC (Fig. 147). The plane should be represented as a triangle.

9. Construct the traces of three mutually parallel planes α, β, and γ passing respectively through the given points A, B and C (Fig. 148).

Hint. Three mutually parallel horizontal lines, drawn through points A, B and C and their traces, should be used.

10. Through a given point A draw a plane parallel to a plane passing through ground line Ox and a point B. (See exercise 6).

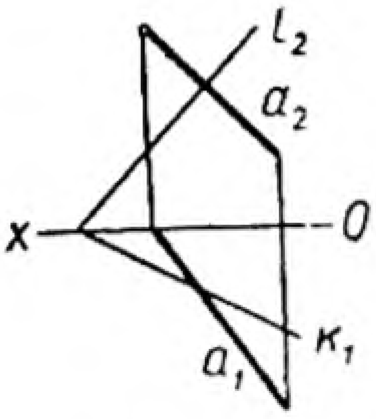


Fig. 143

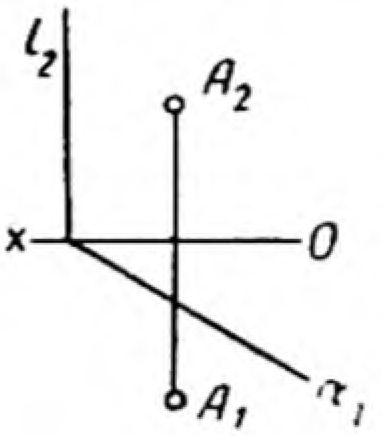


Fig. 144

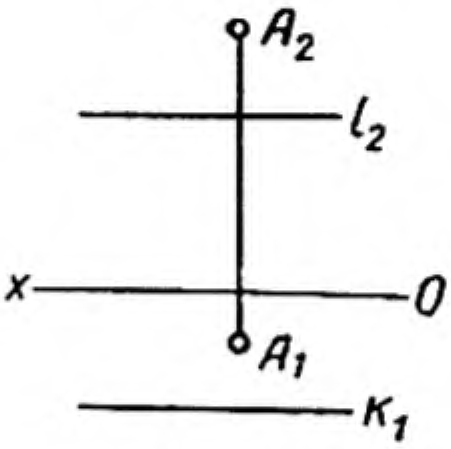


Fig. 145

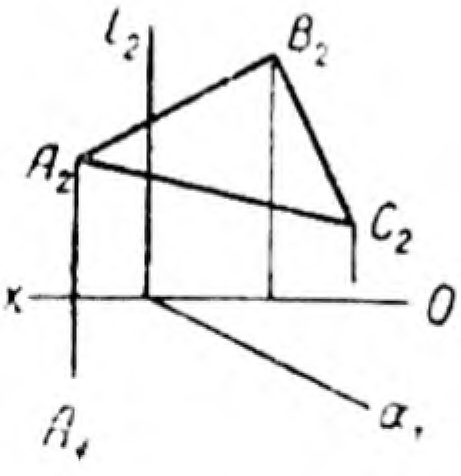


Fig. 146

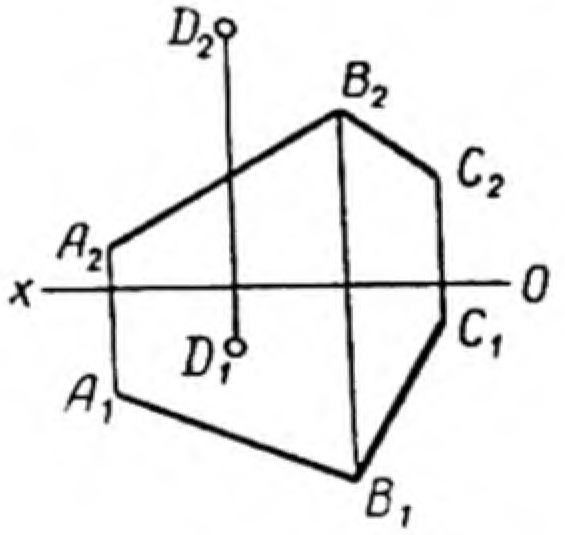


Fig. 147

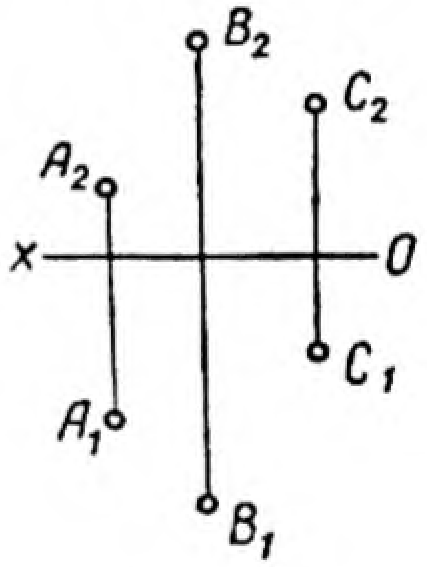


Fig. 148

§ 19. The Intersection of Planes

1. The intersection of two planes is a straight line defined by two points which are common to both planes. So the construction of the projections of the line of intersection of two planes, as a rule, can be reduced to the determination of the projections of two points, lying in both intersecting planes.

The method of determining points which lie on the line of intersection is called the *method of auxiliary sections*. It is widely used in all courses of descriptive geometry.

The application of the method of auxiliary sections is illustrated in Fig. 149, which shows the plane α, given by the parallel lines

AB and *CD* and the plane β, given by the triangle *EFM*. Points *K* and *L*, common to both planes α and β, are determined by using two auxiliary planes, γ and φ. The plane γ intersects planes α and β in lines *1-2* and *3-4*, respectively, and plane φ in lines *5-6* and *7-8*. The lines *1-2* and *3-4* intersect in point *K* common to planes α, β and γ. Therefore point *K* lies on the line of intersection of planes α and β. Point *L* is common to planes α, β and γ and lies on the line of intersection of planes α and β which is found by connecting up points *K* and *L*.

This method of auxiliary sections is used for solving problems in the multiview drawing. It is required to find the line of intersection of two planes, α and β, given by the pairs of parallel lines *AB* and *CD* and *EF* and *MN*, respectively (Fig. 150).

Begin the construction by drawing an auxiliary horizontal plane γ to intersect both planes.

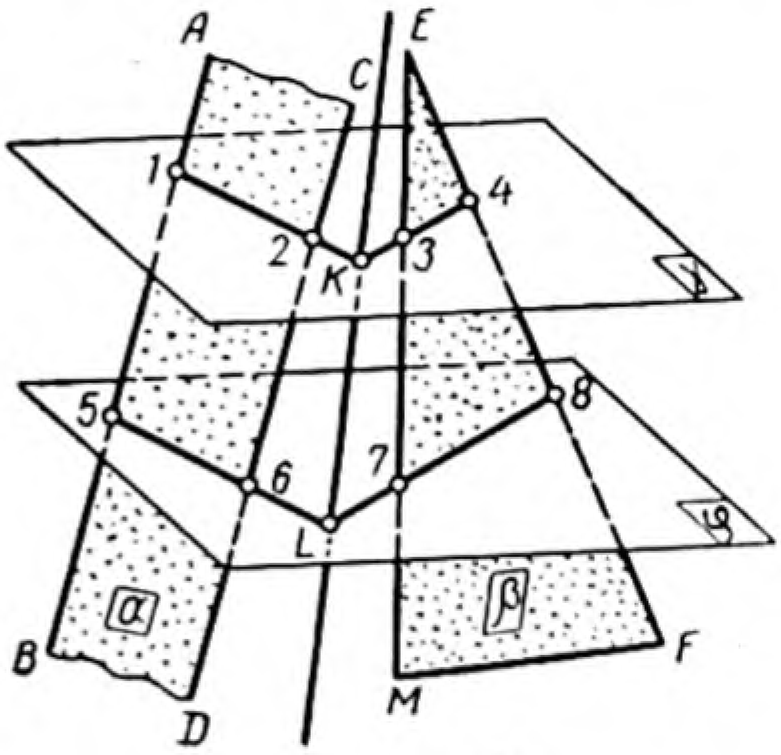


Fig. 149

The plane γ intersects plane α in the horizontal *1-2*, the frontal projection 1_2-2_2 of which coincides with trace γ_2. The horizontal projection 1_1-2_1 is easily found with the help of projectors 1_2-1_1 and 2_2-2_1. The projections of the horizontal *3-4*, along which the plane β is intersected by plane γ, are found in a similar manner. Point K_1 lies at the intersection of the horizontal projections $1_1 2_1$ and $3_1 4_1$ of the horizontals. Point K_1 is the horizontal projection of point *K* which belongs to the line of intersection of the given planes α and β. The frontal projection K_2 of point *K* is obtained by drawing through K_1 a projector to intersect the trace γ_2. This trace possesses collecting properties.

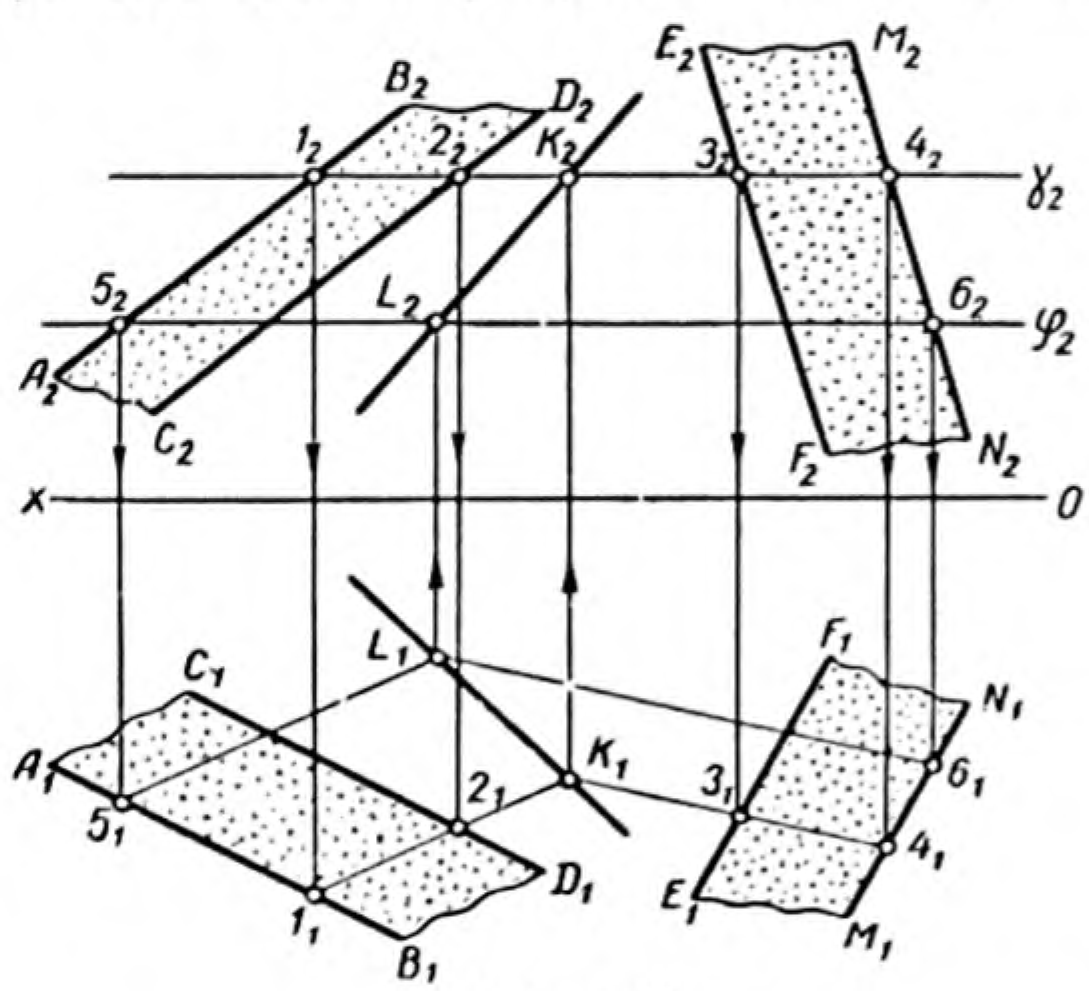


Fig. 150

To construct the second point on the line of intersection of planes α and β a second auxiliary plane φ is drawn. Its intersections with the given planes α and β are also horizontals. Since all the horizontals of one plane are parallel to each other, in order to construct the horizontals in which plane φ intersects planes α and β it is sufficient to find one point of each of these two horizontals. Having constructed points 5_1 and 6_1 in Π_1 draw through them the horizontal projections of the horizontals. They are respectively parallel to the projections 1_12_1 and 3_14_1 of the horizontals previously constructed. Point L_1 lies at the intersection of the lines drawn through 5_1 and 6_1. This is the horizontal projection of point L through which the line of intersection of planes α and β passes. The frontal projection L_2 of point L is found with the help of a projector drawn through point L_1 to the trace φ_2 of the second auxiliary plane. By joining up the corresponding projections of points K and L two projections of KL, the line of intersection of the given planes α and β, are found.

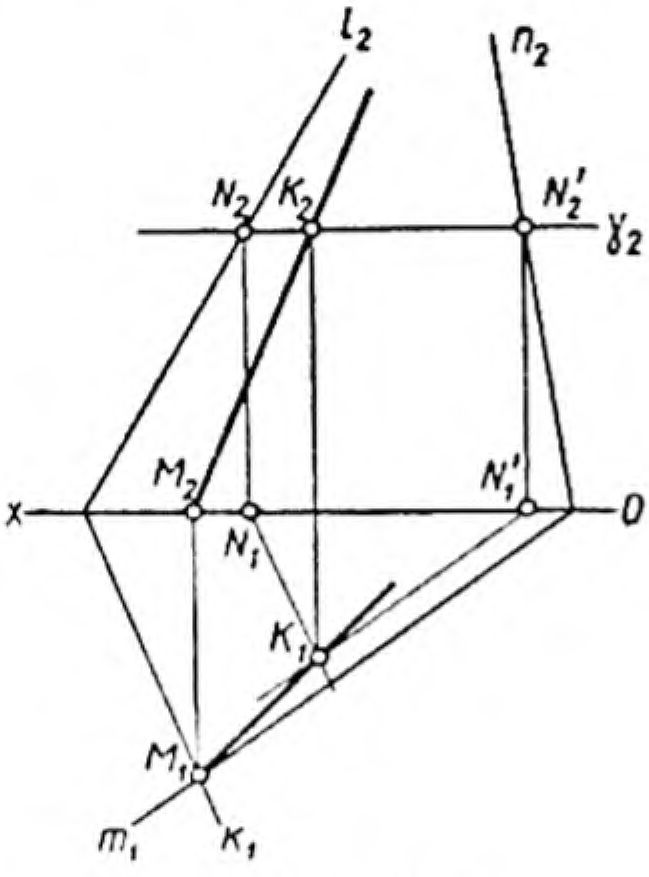


Fig. 151

It should be noted that, instead of auxiliary horizontal planes, any two projecting planes could be used without changing the sequence of the construction.

2. When the position of one of the points contained in the line of intersection of two planes is known the construction is simplified.

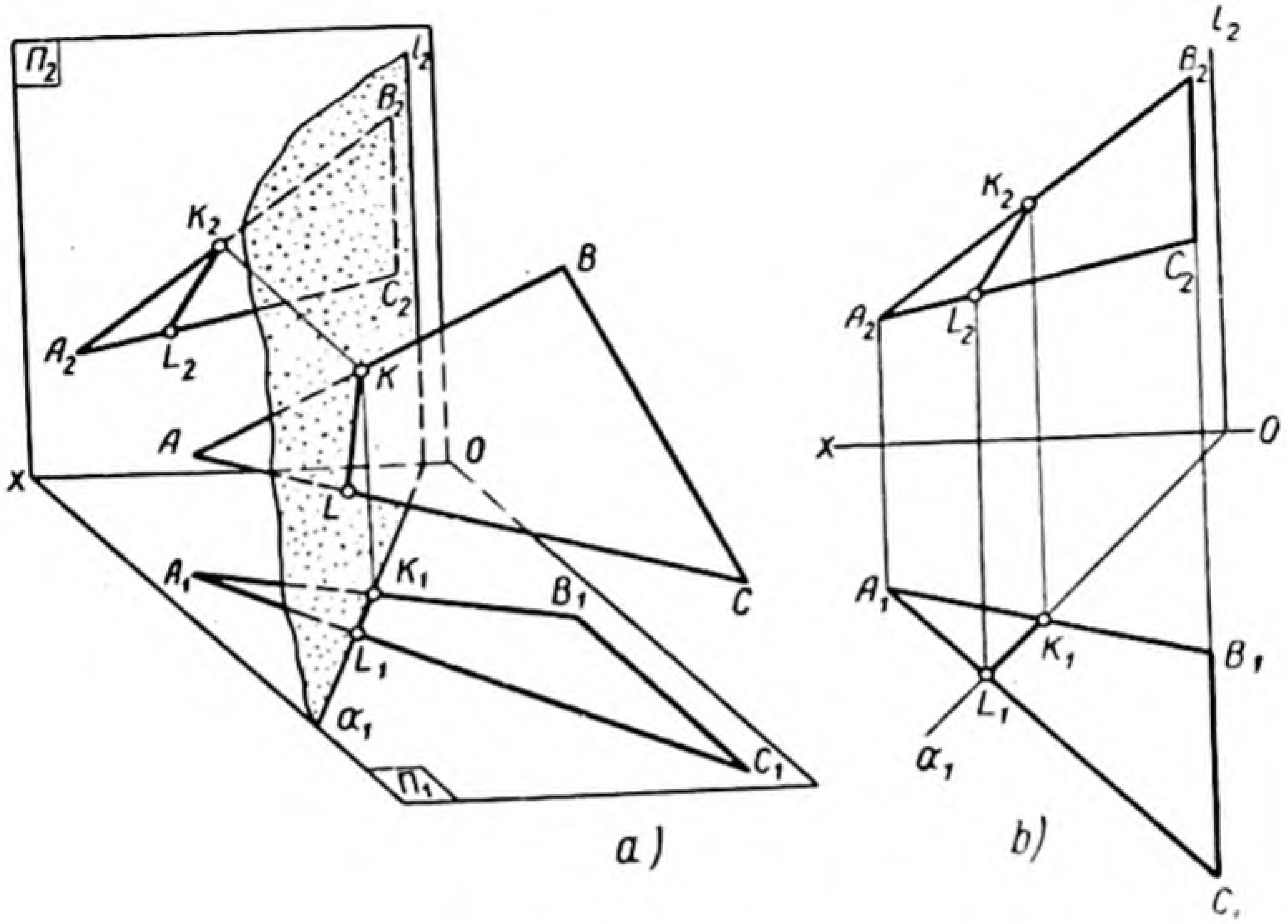
In Fig. 151 planes α and β are given. Their horizontal traces k_1 and m_1 intersect within the limits of the drawing. The point M, common to the planes α and β, is one of the points belonging to the line of intersection of these planes. The horizontal projection of this point coincides with the point itself ($M_1 \equiv M$). The frontal projection M_2 lies on ground line Ox.

To find the second point, common to both planes α and β, draw an auxiliary horizontal plane γ the intersection of which with each of the given planes will be a horizontal. The frontal projection of these horizontals will coincide with the frontal trace γ_2 of plane γ, while the horizontal projections will be parallel to the horizontal traces k_1 and m_1 of planes α and β, respectively, and pass through the horizontal projections N_1 and N_1' of their frontal traces N and N'.

The horizontal projection K_1 of point K contained in the line of intersection of planes α and β lies at the intersection of horizontal projections N_1K_1 and $N_1'K_1$ of the horizontals constructed. The frontal projection K_2 of point K is found on γ_2 by drawing a projector through K_1. The projections of the line of intersection MK of the planes α and β are found by joining the projections of point M with the corresponding projections of point K.

It should be noted that the use of an auxiliary horizontal plane

γ is convenient because the construction of the horizontals in planes α and β requires only one piercing point to be determined for each of those lines, since the horizontal projection of the horizontal of a plane is parallel to the horizontal trace of that plane — see § 17 and Fig. 104.



a) b)

Fig. 152

3. The construction of the line of intersection of two planes is further simplified if at least one of the planes is a projecting plane. Such a case is shown in Fig. 152, where an oblique plane given by triangle ABC intersects the horizontal-projecting plane α. Since the horizontal trace of the horizontal-projecting plane possesses collecting properties, the horizontal projection K_1L_1 of the intersection line KL for the given planes coincides with trace α_1 of plane α. Mark on this trace-projection the points K_1 and L_1 in which the horizontal projections A_1B_1 and A_1C_1 of the sides of the triangle intersect with α_1. Draw through these points projectors to intersect the frontal projections A_2B_2 and A_2C_2 of the sides of the triangle in points K_2 and L_2. The straight line joining these points is the frontal projection K_2L_2 of the line of intersection of the two given planes.

4. The intersection of planes α and β, given by their traces is shown in Fig. 153. The corresponding traces of these planes intersect within the limits of the drawing. The construction of the line of intersection MN consists merely in determining the second projections of points M and N in which the similar traces of the planes intersect. The frontal projection M_2 of point M and the horizontal projection N_1 of point N both lie on the ground line Ox, as they are projections of points situated in the planes of projections. The projections of the line MN, i. e., the line of intersection of planes α and β, are obtained by joining up the corresponding projections of points M and N.

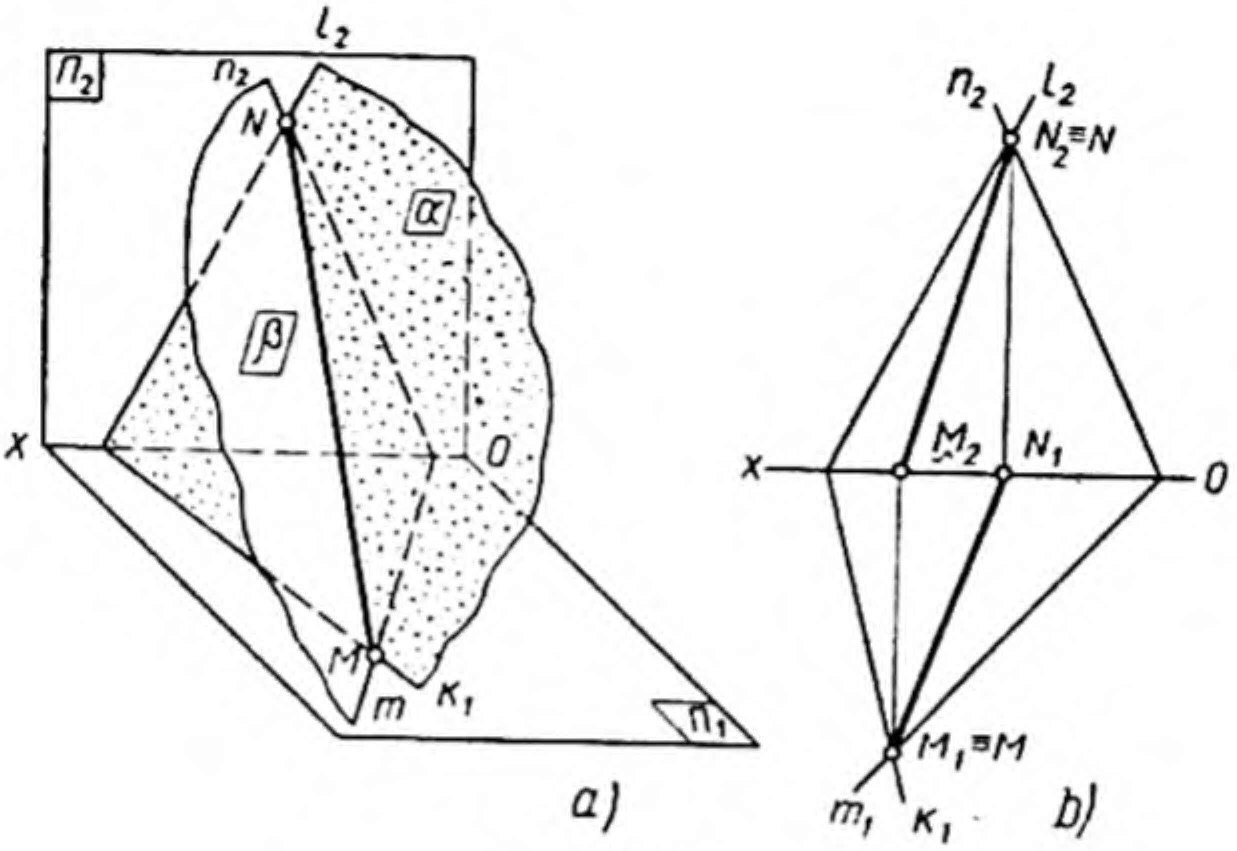

l_2
Π_2
n_2
N
α
β
x
0
M
m
K_1
Π_1
n_2
l_2
$N_2 \equiv N$
x
M_2
N_1
0
$M_1 \equiv M$
m_1
K_1
a) b)

Fig. 153

5. The construction of the line of intersection of the frontal plane with the oblique plane α (Fig. 154) consists in determining the frontal line common to both given planes. The point M where the horizontal traces k_1 and β_1 intersect is the horizontal trace of this frontal. Projection M_1 coincides with the point itself, whereas the frontal projection M_2 of point M lies on the ground line Ox. The line M_2L_2 parallel to trace l_2 of plane α is the frontal projection of line ML of intersection of planes α and β. The horizontal

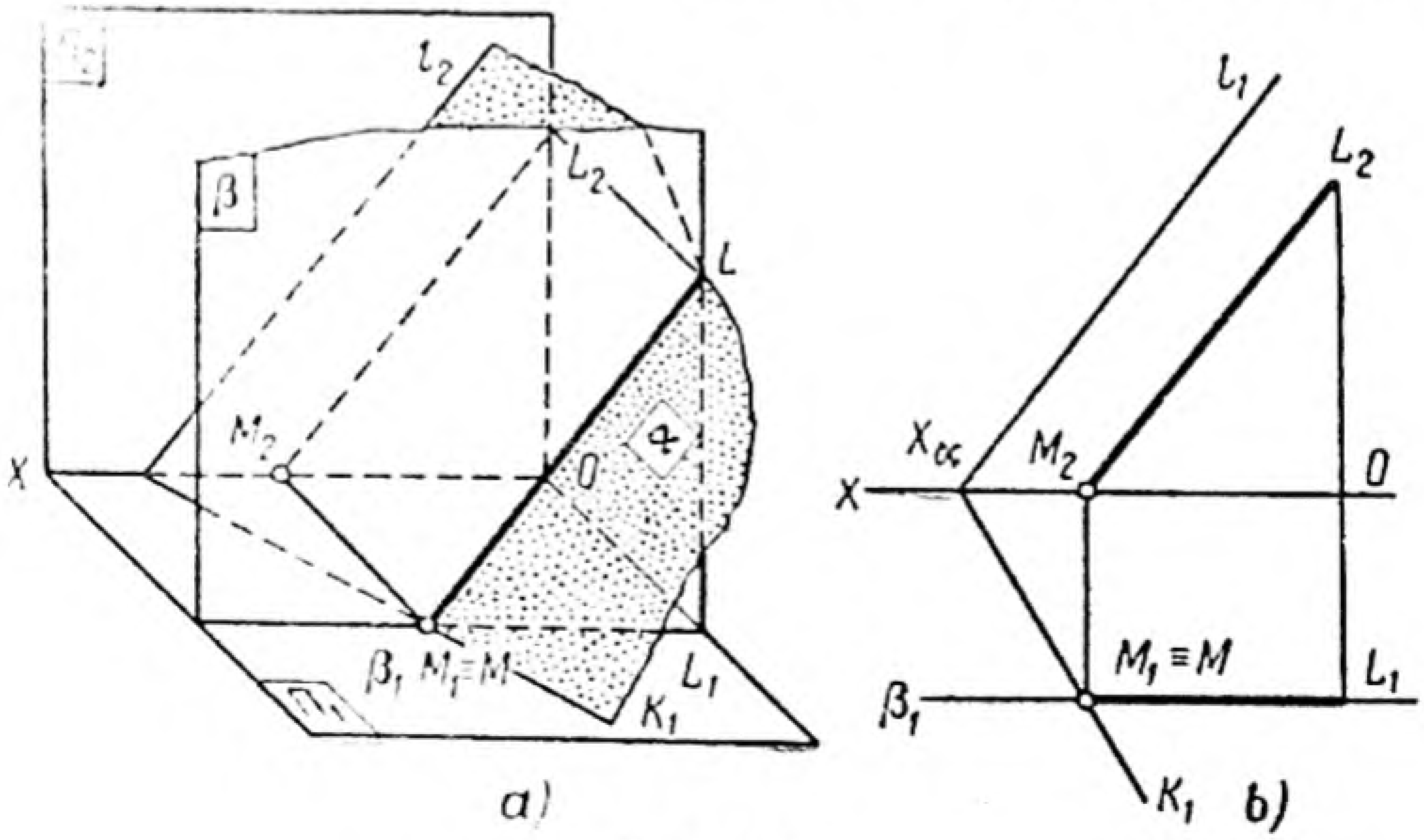

Π_2
l_2
β
L_2
L
M_2
x
0
α
β_1
$M_1 \equiv M$
L_1
K_1
Π_1
l_1
L_2
X_α
M_2
0
x
$M_1 \equiv M$
L_1
β_1
K_1
a) b)

Fig. 154

projection $M_1 L_1$ of the line of intersection of these planes coincides with trace β_1.

6. The cases of intersection of two planes considered above may be divided into two groups.

The first group consists of pairs of planes, the construction of the line of intersection of which requires the construction of two points contained in that line (Figs. 150, 152). The second group consists of pairs of planes whose lines of intersection are determined by one point and a known direction (Fig. 154).

It should be noted that when determining the direction of the lines of intersection of two planes an auxiliary plane parallel to one of those planes may be used.

For example, it is required to construct the line of intersection of two oblique planes α and β which are given by their traces. The horizontal traces k_1 and m_1 of these planes intersect within the limits of the drawing (Fig. 155).

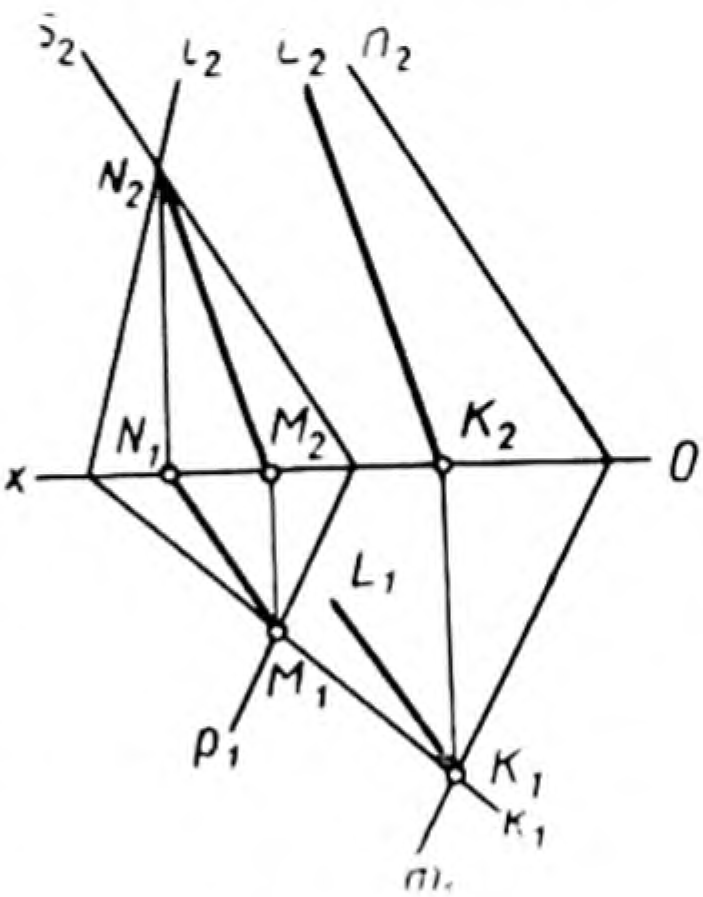


Fig. 155

Draw an auxiliary plane γ parallel to plane β so that its traces p_1 and s_2, which will be parallel to traces m_1 and n_2, respectively, intersect the corresponding traces of plane α. Having obtained points M and N, where the traces of planes α and γ intersect, construct the projections M_1N_1 and M_2N_2 of the line of intersection MN of these planes. Since the intersection lines of two parallel planes with a third plane are themselves two parallel lines, the line MN determines the direction to which the line of intersection of the given planes α and β will be parallel. Having found point K, common to both of these planes, i. e., the point in which their horizontal traces intersect, through it draw line $KL \parallel MN$. This line KL is the required line of intersection of planes α and β.

7. While working on problems of graphical construction the drawing should be kept as free as possible from auxiliary lines. The following example shows how to achieve this.

Plane α is given by its traces k_1 and l_2 and plane β by the triangle ABC. It is required to construct the projections of the line of intersection of these planes (Fig. 156).

Two auxiliary frontal planes γ and φ will be used. Each of these auxiliary planes intersects the given planes α and β in frontals. The point of their intersection will lie on line of intersection of planes α and β.

Let the first auxiliary plane γ pass through the vertex C of the triangle. This plane will cut plane α along the frontal MS and the plane of triangle ABC along the frontal CD. The intersection of the frontals MS and CD will determine the position of the first point K through which the required line of intersection of the given planes will pass.

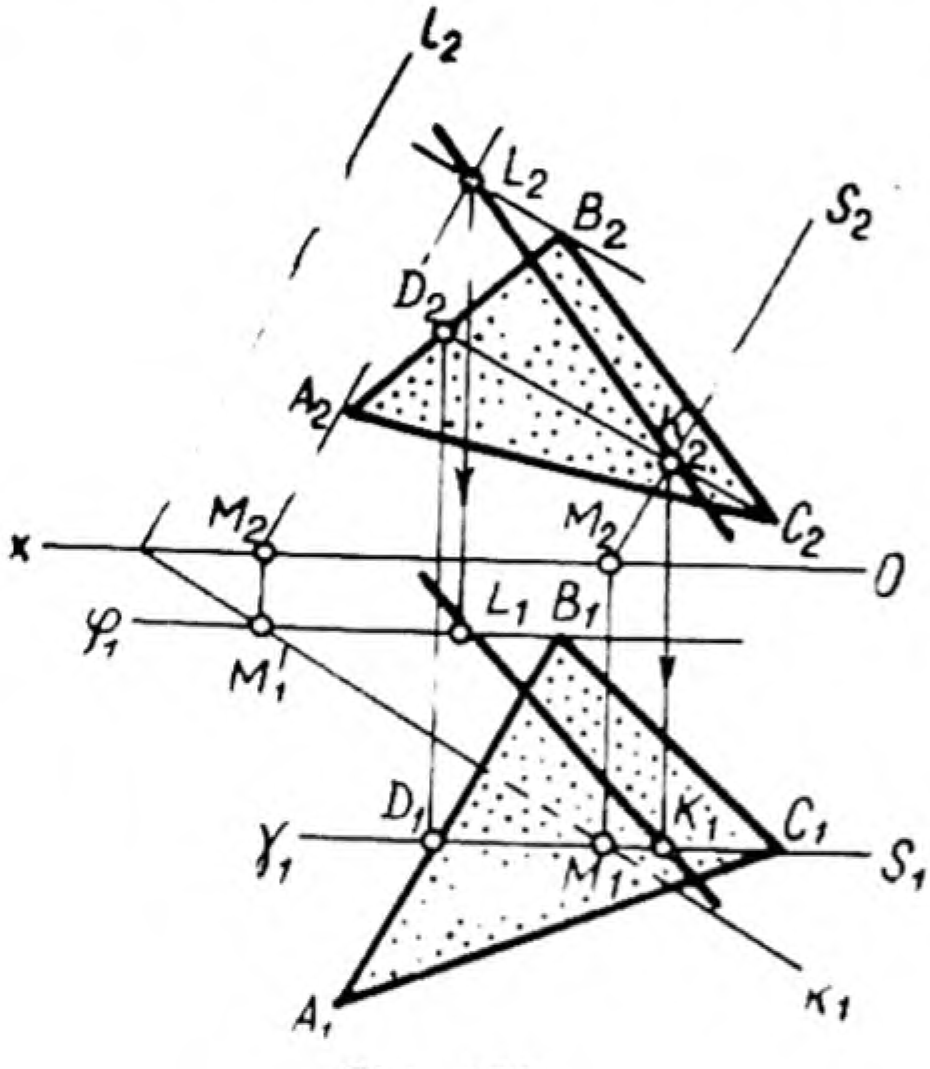
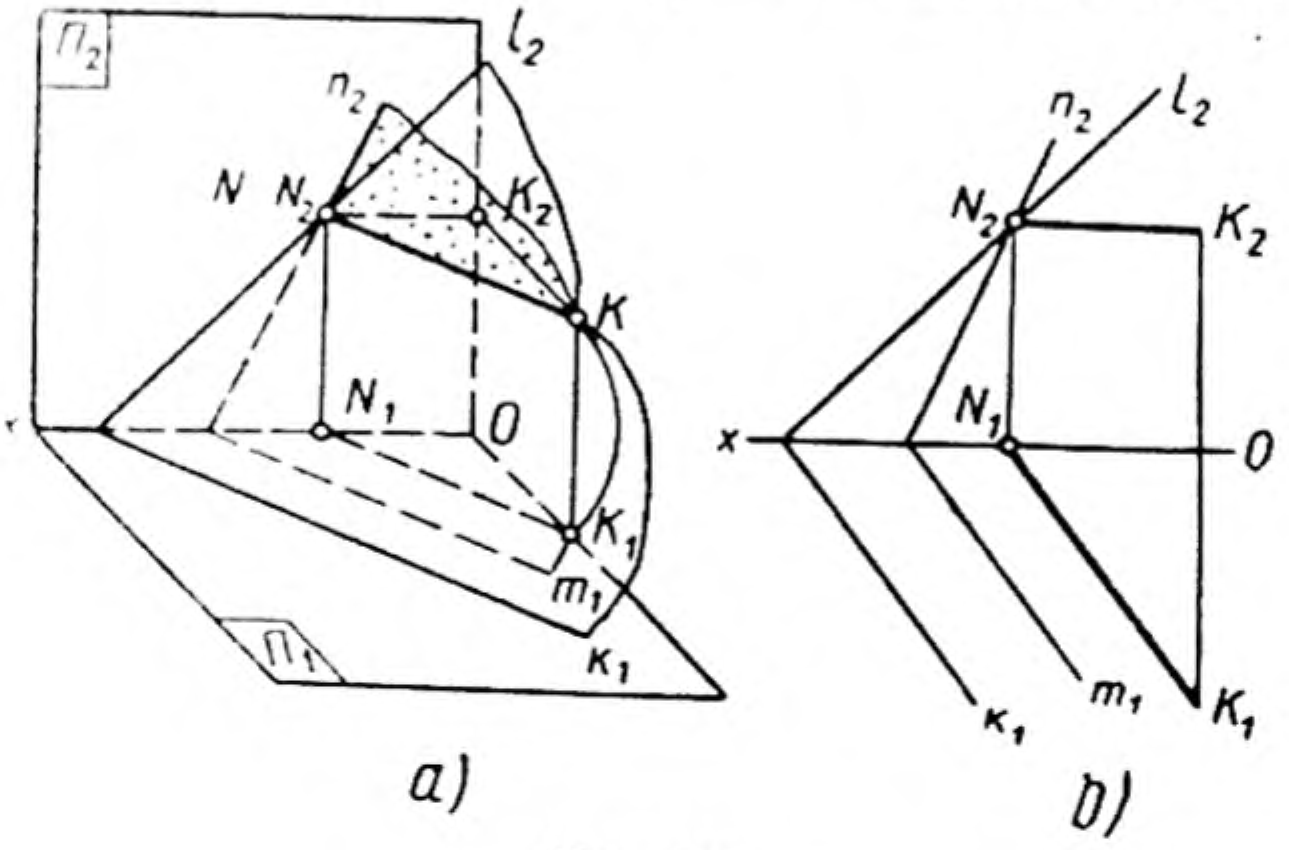


Fig. 156

The second auxiliary plane φ is drawn through point B parallel to the frontal plane of projection Π_2. The frontal projection, which is the line of intersection of planes φ and α, will pass through point M_2' and be parallel to trace l_2 of plane α. The frontal projection of the frontal, which is the line of intersection of the plane φ and the triangle ABC, will pass through point B_2 and be parallel to the projections C_2D_2 of the frontal CD previously constructed. The intersection of the frontals in which the plane φ intersects the planes α and β will define point L, lying on the line of intersection of planes α and β. The line KL is the required line of intersection of the two given planes.

It should be noted that, in order to simplify the construction in the drawing, plane γ was passed through one of the vertices of triangle ABC. So when constructing the frontal projection C_2D_2 of the frontal CD only one point D_2 has to be found. The second auxiliary plane φ is passed through the vertex B of triangle ABC.



a) b)

Fig. 157

again simplifying the construction, since the frontal projection of the second frontal of the plane of the triangle was then drawn through point B_2 parallel to line C_2D_2.

8. Finally attention should be paid to a particular case of intersection of two oblique planes when two of the corresponding traces of the planes are parallel.

In Fig. 157 the line of intersection NK of planes α and β is parallel to the horizontal traces of the given planes since these traces are parallel. In other words, the line of intersection is the horizontal common to both planes. Similarly, if the frontal traces of two intersecting planes are parallel, the line of intersection is the frontal common to both of the planes and, if the profile traces of the planes are parallel, the line of intersection is the profile line common to the planes.

Problems and Exercises

1. Why are auxiliary planes used when constructing the line of intersection of two planes?

2. Which planes are most frequently used as auxiliary planes?

3. Give an example of the use of an oblique plane as an auxiliary plane.

4. Where do the traces of a line lie when two planes intersect on that line?

5. Construct the line of intersection of two planes, one of which is given by the triangle ABC and the second by two parallel lines a and b (Fig. 158).

Hint. Use two auxiliary horizontal or frontal section planes. It is not advisable to use one horizontal plane and one frontal plane, since the solution of the problem would require an unnecessary number of construction lines which would overload the drawing.

6. Construct the lines of intersection of the frontal-projecting plane α with the oblique plane β (Fig. 159).

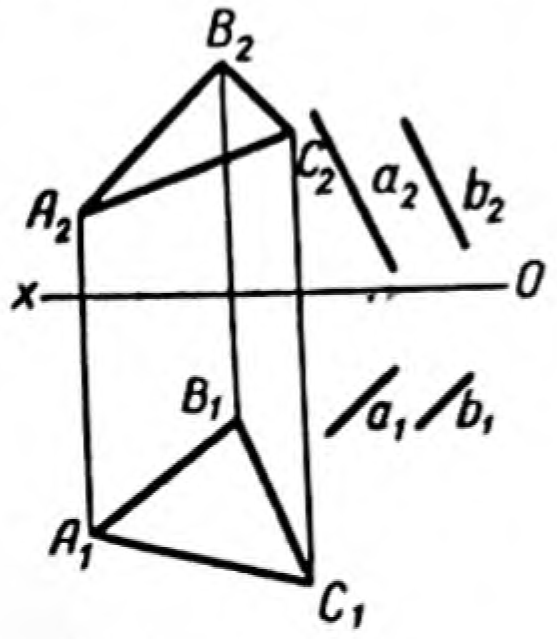


Fig. 158

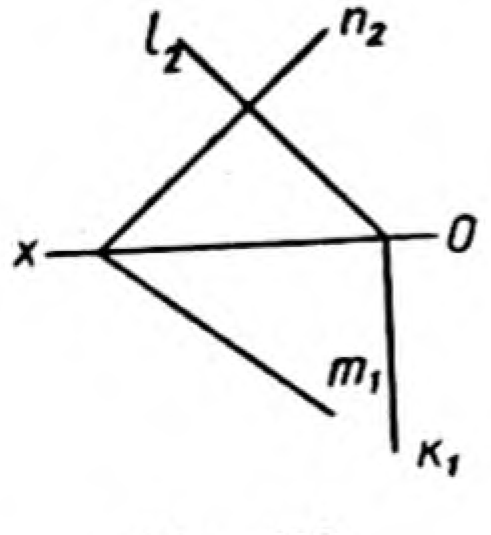


Fig. 159

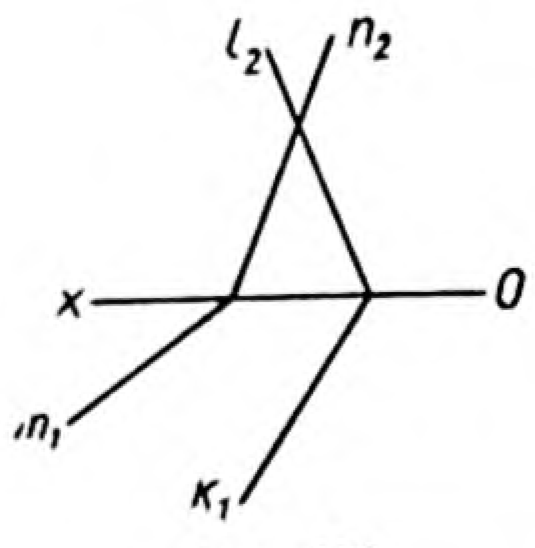


Fig. 160

7. Construct the line of intersection of two oblique planes (Fig. 160).

Hint. Use an auxiliary frontal section plane.

8. Find the horizontal traces of planes α and β whose line of intersection passes through point M (Fig. 161).

Hint. The line of intersection of planes, when their two corresponding traces are parallel to each other, is also parallel to the traces. In this case the line of intersection is the frontal passing through point A parallel to traces l and n. The required horizontal traces of the planes α and β will pass through the trace of the given frontal.

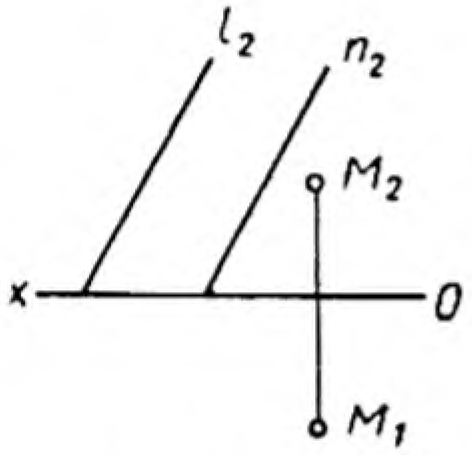


Fig. 161

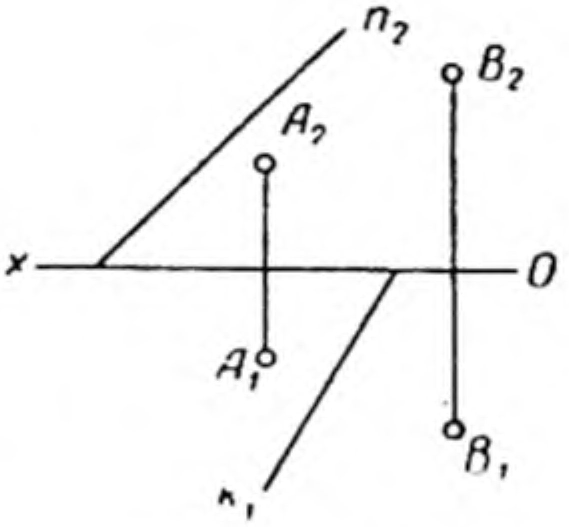


Fig. 162

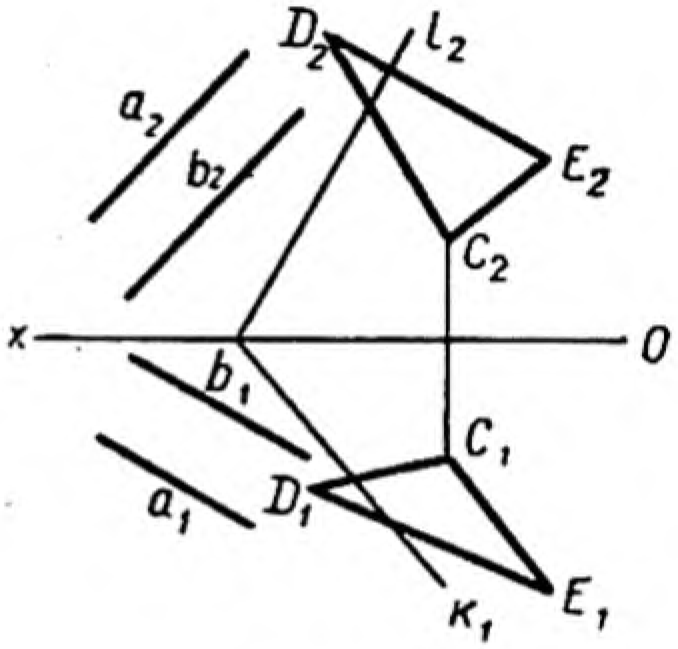


Fig. 163

9. Construct the line of intersection of planes α and β. The plane α is given by the horizontal trace k_1 and point A. Plane β is given by the frontal trace n_2 and point B (Fig. 162).

Hint. First find the traces of the given planes. The trace of plane α is found with the help of the horizontal of the plane passing through point A. The trace of plane β is found with the help of the frontal passing through point B.

10. Find the point common to three planes, one of which is given by triangle CDE, the second, by parallel lines a and b, and the third by traces k_1 and l_2 (Fig. 163).

Hint. This problem can be reduced to the construction of two lines of intersection of two pairs of planes. For example, the line of intersection of plane CDE and the plane given by traces k and l can be found. Then the line of intersection of the plane given by the traces k and l and the plane given by two parallel lines can be determined. The construction should be worked out with the help of two auxiliary horizontal planes, the first of which should pass through point E and the second, through point C.

11. Construct the line of intersection of two oblique planes the points of convergence of the frontal and horizontal traces of which converge in one point.

Hint. One point belonging to the line of intersection of the planes is the point of convergence of the four traces. Another point is found with the help of horizontal and frontal auxiliary sections.

§ 20. The Intersection of Lines and Planes

1. Plane α and a line *AB* intersecting it are shown in Fig. 164. If a plane β is drawn through line *AB* and the line *MN* is the line of intersection of planes α and β, then the point *K* at the intersection of lines *AB* and *MN* will be the point in which line *AB* pierces plane α. This is so because line *MN* is the locus of points common to both planes α and β, and point *K*, at the intersection of lines *AB* and *MN* which lie in these planes, is a point common to line *AB* and plane α.

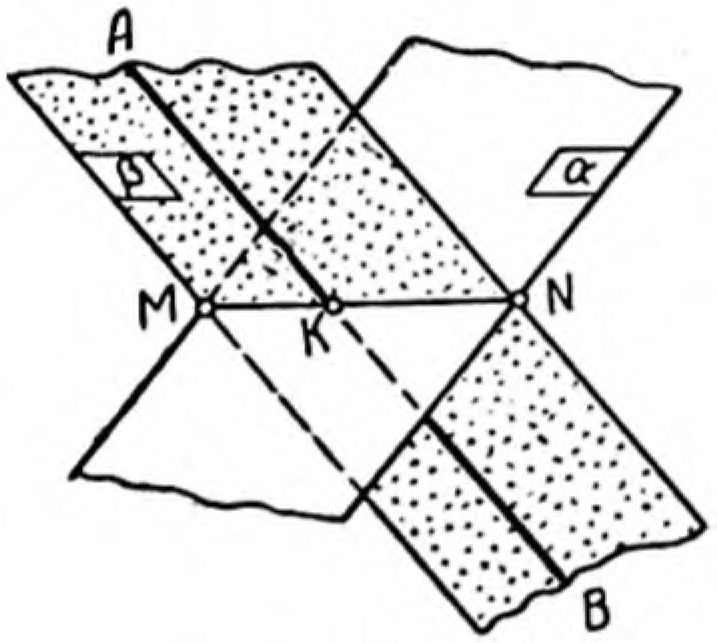


Fig. 164

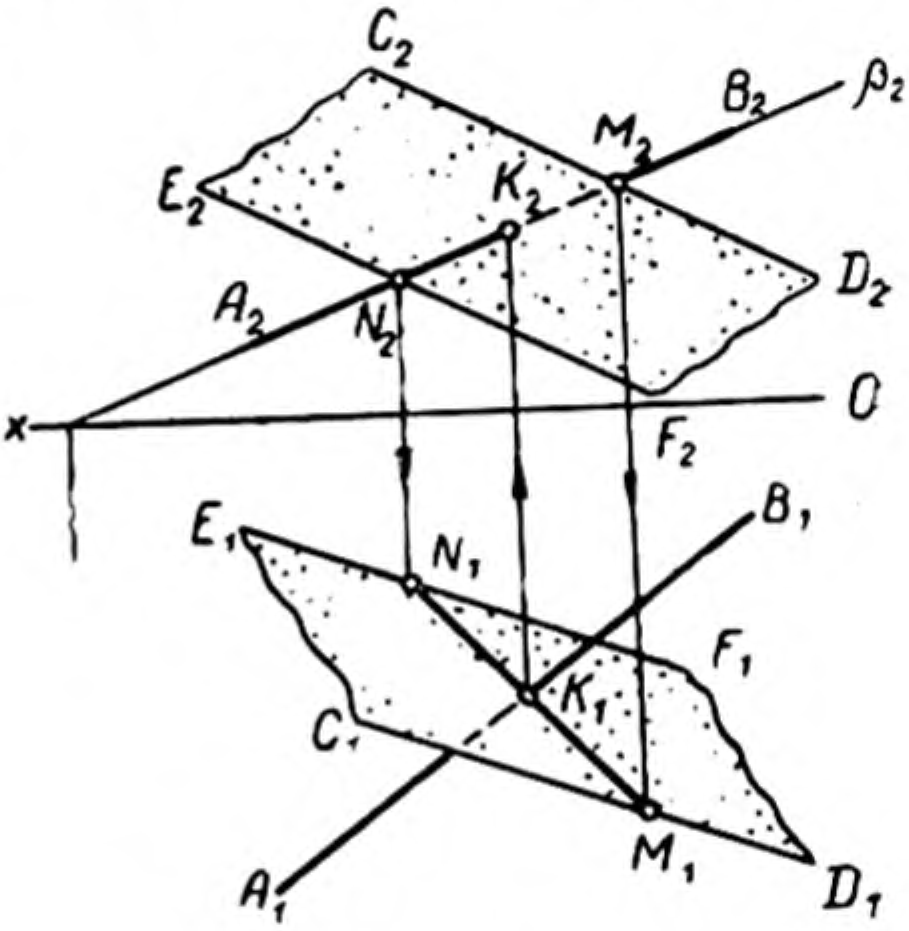


Fig. 165

The construction of the piercing point of a line in a plane generally consists of three operations:

1. The construction of an auxiliary plane through the line.
2. The determination of the line of intersection of the given plane with the auxiliary plane.
3. Finally, the determination of the point of intersection of the given line with the constructed line of intersection of the two planes.

Projecting planes are generally used as auxiliary planes because of the simplicity with which these planes, passing through straight lines, may be shown on the drawing, due to the collecting properties of the traces of such planes.

A line *AB* and a plane α, given by parallel lines *CD* and *EF*, are shown in Fig. 165. It is required to find the meeting point of line *AB* with plane α. The following operations are necessary:

1. Draw an auxiliary frontal-projecting plane β through line *AB*. Its frontal trace-projection merges with the frontal projection

A_2B_2 of line AB. Its horizontal trace is perpendicular to the ground line Ox.

2. Construct the line MN of intersection of planes α and β. To do this find points M and N in which lines CD and EF pierce the auxiliary plane.

3. The intersection of lines AB and MN gives the required piercing point K of line AB in plane α. The horizontal projection K_1 of this point lies at the intersection of projections A_1B_1 and M_1N_1. Draw a projector from point K_1 to obtain the frontal projection K_2 of point K contained in the frontal projection A_2B_2 of line AB.

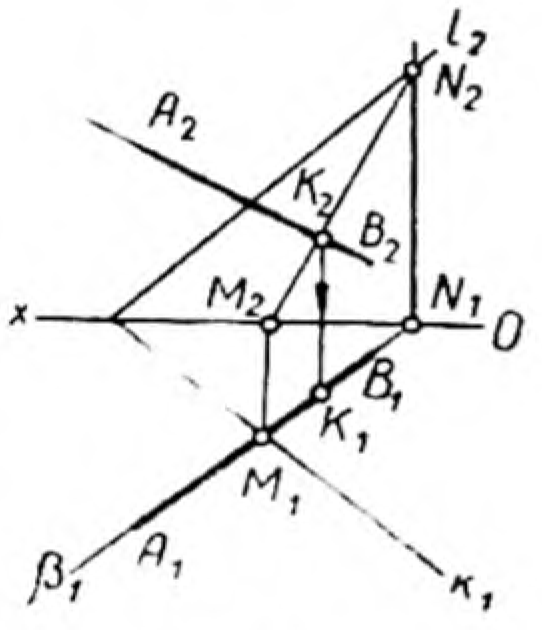


Fig. 166

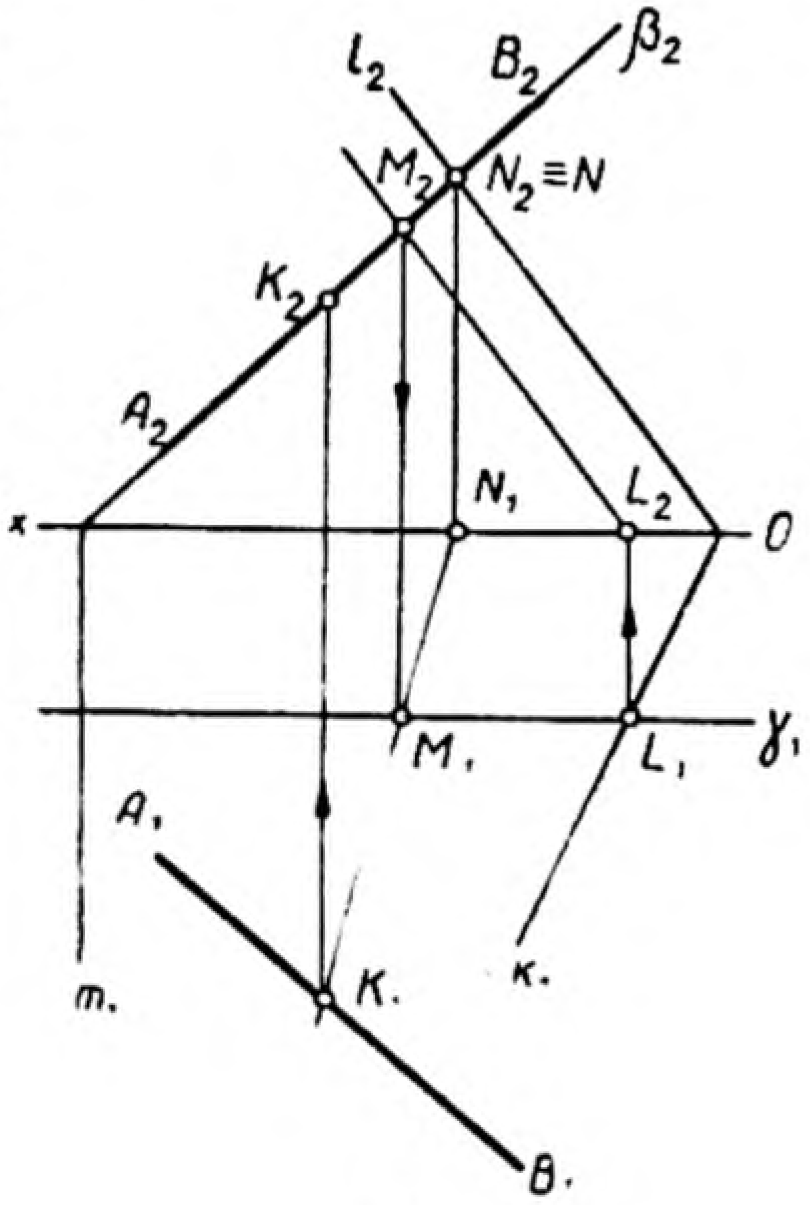


Fig. 167

The construction of the piercing point of line AB in plane α, given by traces k_1 and l_2, is shown in Fig. 166. The auxiliary plane β passed through line AB is perpendicular to the plane of projection Π_1. Trace β_1 coincides with projection A_1B_1. The line MN is the line of intersection of planes α and β. The intersection of projections A_2B_2 and M_2N_2 determines the frontal projection K_2 of point K, which is the piercing point of line AB in plane α. The horizontal projection K_1 of point K is found by drawing through K_2 a projector to intersect line A_1B_1.

2. When determining the piercing of a line with a plane, the traces of which are given, it may happen that one pair of corresponding traces of the given and the auxiliary plane do not intersect within the limits of the drawing. In that case it is necessary to introduce a second auxiliary cutting plane (see Fig. 151). Such a case is considered in Fig. 167.

When an auxiliary frontal-projecting plane β is drawn through line AB and the frontal trace N of the line of intersection of planes

α and β has been determined, it may be seen that the horizontal trace m of plane β does not intersect the horizontal trace k_1 of plane α within the limits of the drawing. So in order to find the second point M contained in the line MN of intersection of planes α and β, draw an auxiliary plane γ parallel to Π_2. The intersections of plane γ with α and β will be frontals. The projection on plane Π_2 of the frontal of plane α will pass through point L_2 and lie parallel to trace l_2, while the frontal projection of the frontal plane β will coincide with trace β_2 which possesses collecting properties. The intersection of the frontal projections of these frontals gives the frontal projection M_2 of point M. The horizontal projection M_1 of point M will lie at the intersection of projector M_2M_1 with trace γ_1. This trace possesses collecting properties. The line of intersection of planes α and β passes through points M and N. This determines the piercing point K of the line AB in plane α.

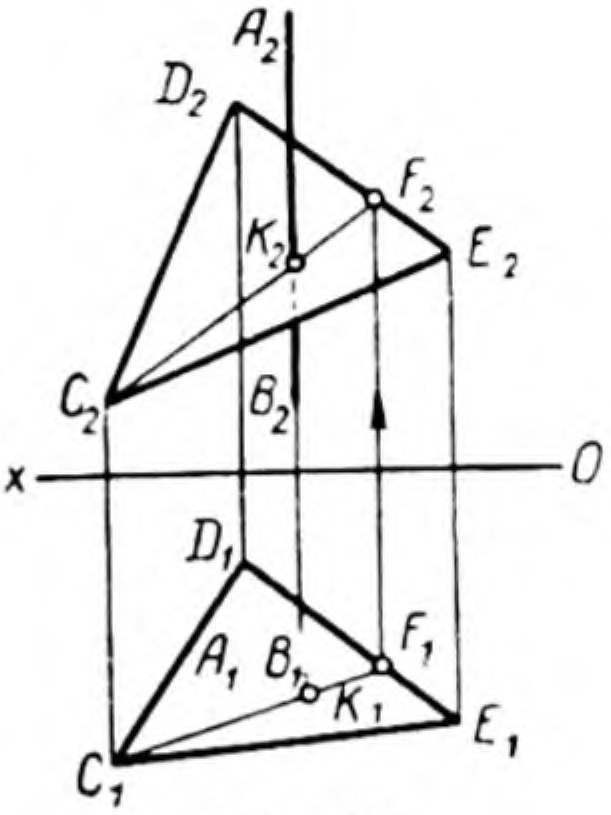


Fig. 168

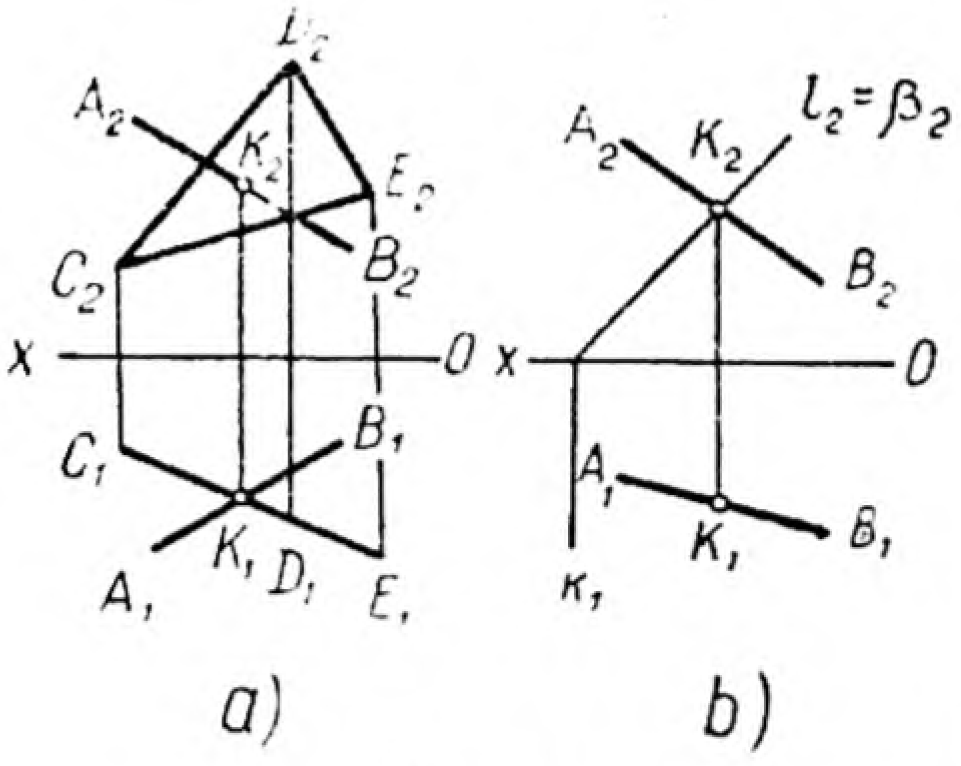
3. When the line piercing a given plane is perpendicular to the plane of projection the projection of the point of intersection on the plane to which the line is perpendicular coincides with the projection of the point on that same plane. So the problem may be reduced to finding the second projection of the point of intersection of the line with a plane by means of one known projection of the point.

This case is illustrated in Fig. 168, where the line AB is perpendicular to plane Π_1 and pierces the plane of triangle CDE. The horizontal projection K_1 of the piercing point K in plane CDE coincides with point $A_1 \equiv B_1$ which is the projection of line AB on plane Π_1.

To construct the frontal projection K_2 of piercing point K through point K_1 draw the auxiliary line $C_1K_1F_1$ and, assuming it to be the horizontal projection of line CKF contained in the plane of triangle CDE, by means of projection C_1F_1 determine the frontal projection C_2F_2 of the line. The frontal projection K_2 of the piercing point of line AB in the plane of the triangle CDE lies at the intersection of lines C_2F_2 and A_2B_2.

4. The piercing point of a line with a projecting plane is determined directly without auxiliary constructions.

For instance, in Fig. 169, *a* the horizontal projection K_1 of the piercing point K of line AB in the horizontal-projecting plane α, which is given by the triangle CDE, lies at the intersection of the projection-trace of that plane on Π_1 with the projection of the line AB on the same plane of projection Π_1. The frontal projection K_2 of point K is found by passing a projector through point K_1. In Fig. 169, *b* the projections K_2 and K_1 of piercing point K of line AB in the frontal-projecting plane β are shown.



L_2
A_2
K_2
E_2
C_2
B_2
x
0
C_1
B_1
K_1
D_1
A_1
E_1
$l_2=\beta_2$
A_2
K_2
B_2
x
0
A_1
K_1
B_1
K_1

a) b)

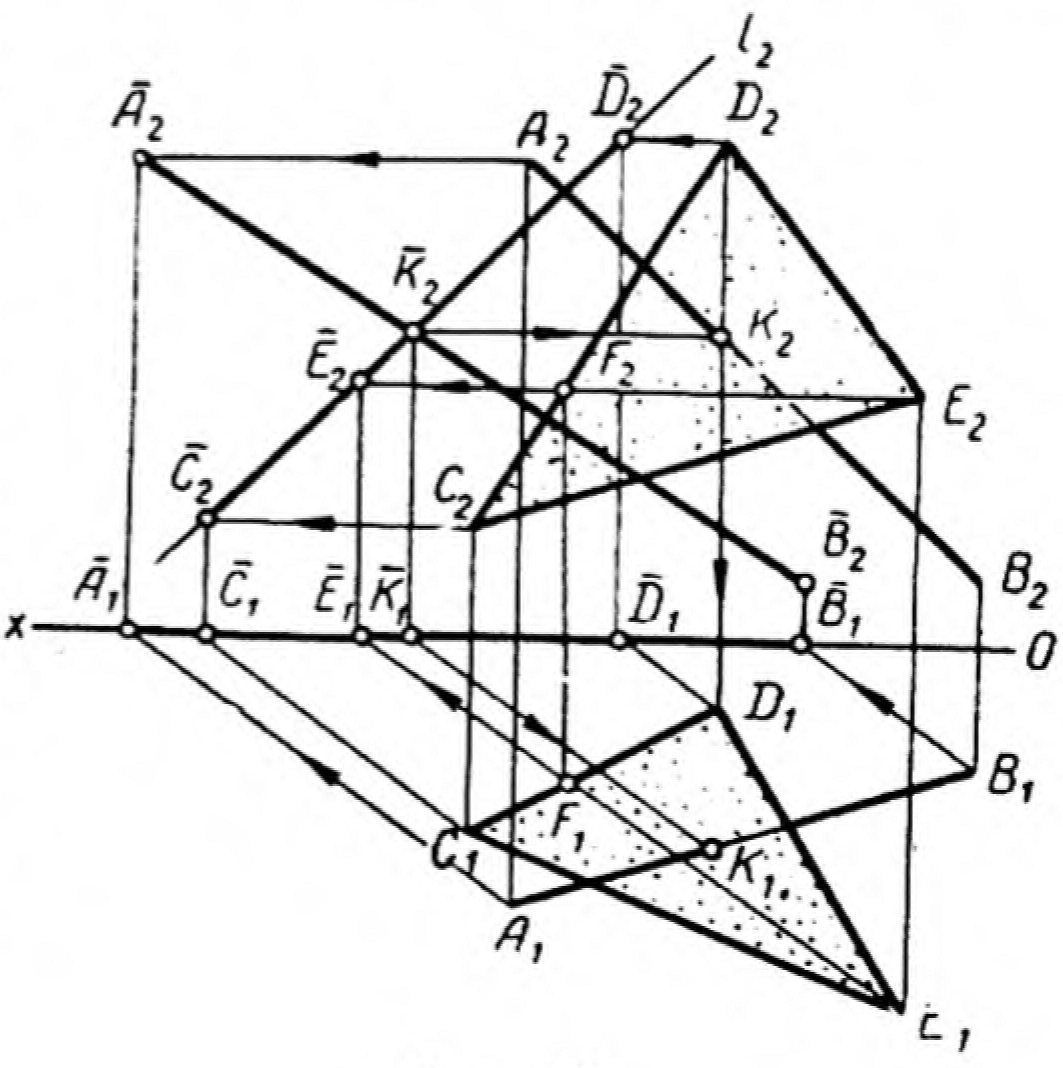
Fig. 169

5. Next let us examine the constructions required to determine the piercing point of a line with a plane by means of **the method of oblique projections.**

The construction of oblique projections in the system of planes of projection Π_1 and Π_2 is similar to the construction of traces of lines and planes (§ 11 and 15, 7). The straight line AB and plane α which is given by the triangle CDE, are shown in Fig. 170. It is required to determine the piercing point of line AB in plane α.

Project the plane and line in a direction parallel to the horizontal EF of triangle CDE. The oblique projections of the vertices of the triangle C, D and E will be found as traces of horizontals drawn in the plane of the triangle through these points. The oblique projections $\overline{C}_1$, $\overline{D}_1$ and $\overline{E}_1$ (read as "C line prime," etc.) lie on ground line Ox and the projections $\overline{C}_2$, $\overline{D}_2$ and $\overline{E}_2$ lie on the frontal trace l_2 of plane α.

By drawing horizontal lines parallel to the horizontal EF through points A and B and having found the traces of these lines we obtain oblique projections $\overline{A}_1$, $\overline{A}_2$ and $\overline{B}_1$, $\overline{B}_2$ of points A and B. The oblique projection $\overline{A}_1\overline{B}_1$ of line AB coincides with ground line Ox and its oblique projection $\overline{A}_2\overline{B}_2$ will cut the oblique projection of triangle $\overline{C}_2\overline{D}_2\overline{E}_2$, i. e., trace l_2 in point $\overline{K}_2$.



l_2
$\overline{A}_2$
A_2
$\overline{D}_2$
D_2
$\overline{K}_2$
K_2
$\overline{E}_2$
F_2
E_2
$\overline{C}_2$
C_2
$\overline{B}_2$
B_2
$\overline{A}_1$
$\overline{C}_1$
$\overline{E}_1$
$\overline{K}_1$
$\overline{D}_1$
$\overline{B}_1$
x
0
D_1
B_1
F_1
C_1
K_1
A_1
E_1

Fig. 170

The point $\bar{K}_2$ is the oblique projection on Π_2 of point K which is the piercing point of line AB in the plane of triangle CDE. The oblique projection K_1 of this point lies on ground line Ox.

By carrying out the reverse projection, namely by drawing through points K_1 and K_2 the projections of a line parallel to the horizontal EF, at their intersections with the corresponding projections of line AB we find the projections K_1 and K_2 of K, that is the piercing point of line AB in the plane of triangle CDE.

Problems and Exercises

1. Which planes are generally used as auxiliary planes when determining the piercing point of a given line in a plane?
2. What constructions are required to determine the piercing point of a line in a plane?
3. When is the piercing point of a line in a plane determined directly, without auxiliary constructions? How is this done?
4. How is the piercing point of a line in a plane found by oblique projection?
5. Find the piercing point of line AB in a plane given by the intersecting lines CD and DE (Fig. 171).

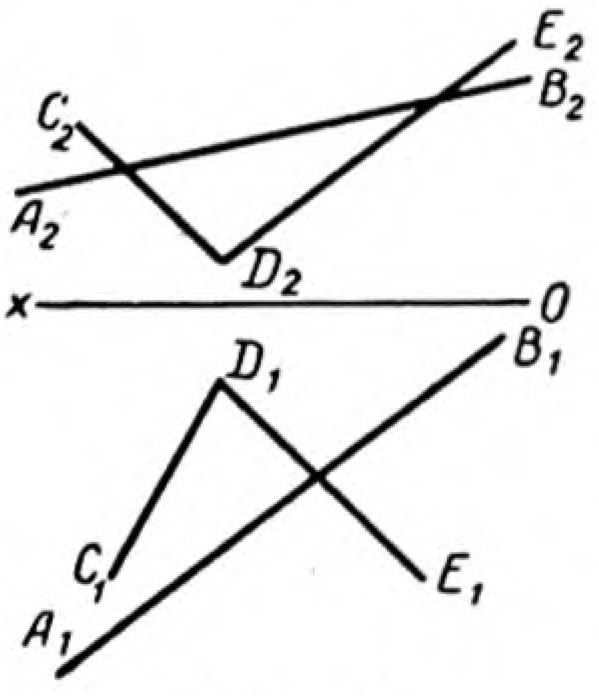


Fig. 171

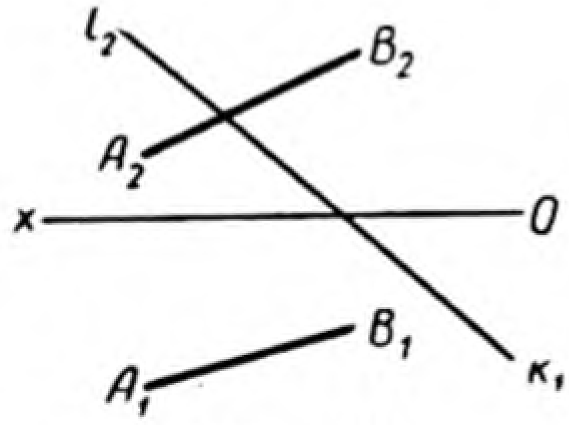


Fig. 172

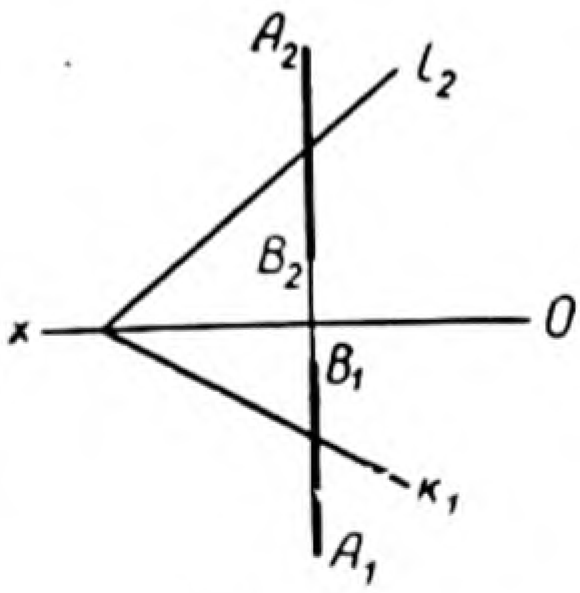


Fig. 173

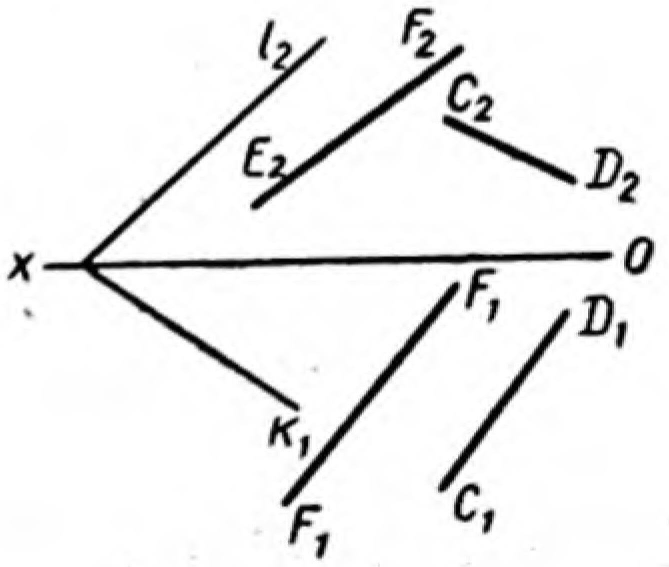


Fig. 174

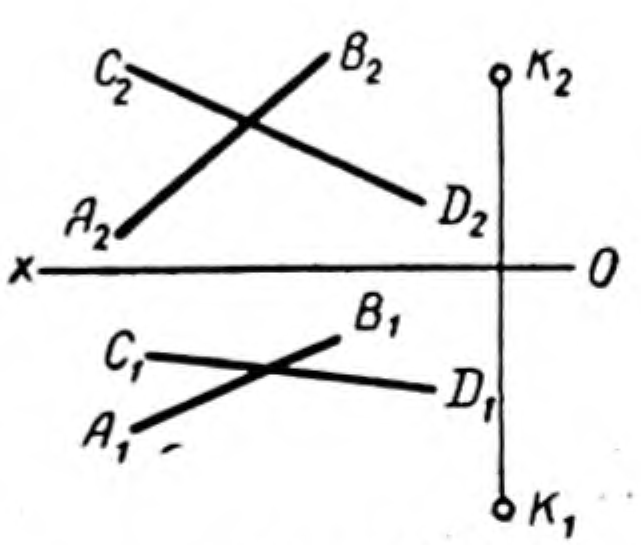


Fig. 175

6. Find the meeting point of line AB with the oblique plane α (Fig. 172).

Hint. Frontal projecting planes permit the most direct approach.

7. Find the meeting point of the profile line AB with plane α which is given by traces k and l, using the method of oblique projections (Fig. 173). (See Fig. 170 and text.)

8. Draw a line AB parallel to plane α and cutting the non-intersecting lines CD and EF (Fig. 174).

Hint. First, at any suitable distance, construct a plane parallel to plane α. Then determine the piercing points of lines CD and EF in that plane. The line joining the piercing points will be the required line.

9. Draw a line through point K to cut the non-intersecting lines AB and CD (Fig. 175).

Hint. Join point K with two points contained in one of the given lines, for example, with points A and B. Then, find the piercing point of the second line CD in the plane of triangle ABK. The straight line joining point K, the piercing point is the required line.

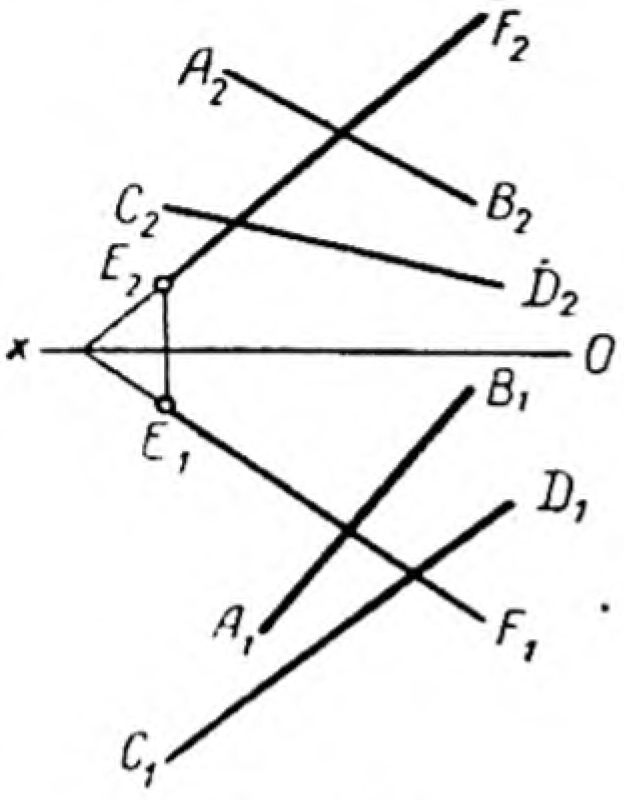


Fig. 176

10. Draw a straight line to intersect lines AB, CD and EF (Fig. 176).

Hint. Through any point on one of the straight lines and two points of another pass a plane, for example, CDF. Then, find the piercing point of the third line, AB, with this plane. The line drawn through point F and this piercing point of intersection is the required line.

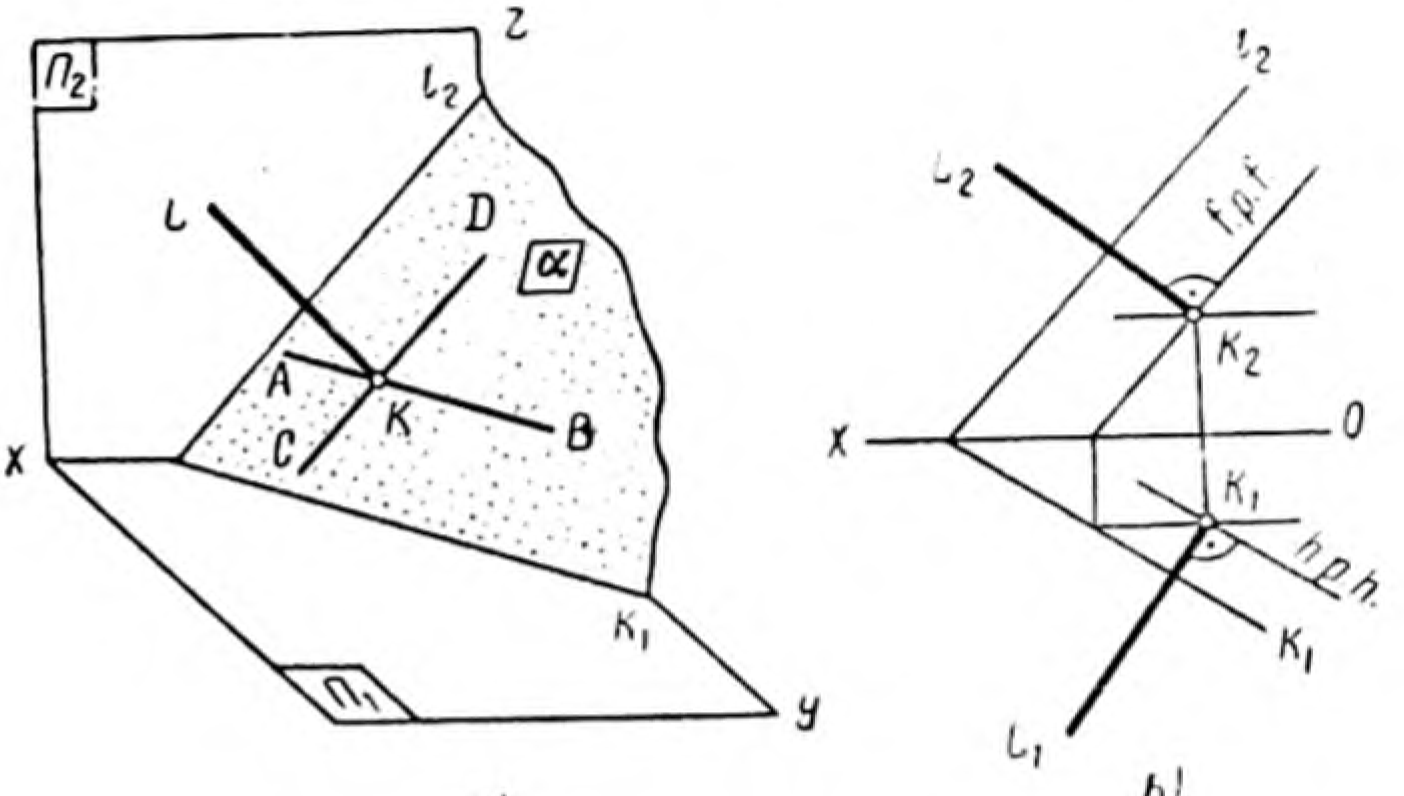
§ 21. Straight Lines Perpendicular to Planes

1. Any line perpendicular to a plane is an intersecting or non-intersecting perpendicular with respect to any line drawn in the plane.

A line KL perpendicular to plane α and intersecting it in point K is shown in Fig. 177. If we draw through this point in plane α a horizontal AB and a frontal CD, they will form right angles with KL. Since line AB is parallel to plane Π_1, the right angle which it forms with line $KL \perp \alpha$ is projected onto plane Π_1 without distortion (§ 13).

For the same reason an angle formed by the frontal CD and line $KL \perp \alpha$ is projected onto plane Π_2 without distortion.

Taking into account the fact that the horizontal projection of the horizontal (marked by the letters *h.p.h.* in Fig. 177, *b*) and the frontal projection of the frontal *(f.p.f.)* of a plane are parallel to the corresponding traces of the plane, it follows that in the multiview drawing the projections of a line perpendicular to a plane, are perpendicular to the corresponding traces of the plane and also to corresponding projections of any horizontal and frontal of the plane.



a) *b*)

Fig. 177

This being so the construction of a perpendicular to the plane α, given by triangle *BCD* (Fig. 178) and passing through a point *A*, consists of the following: 1) the construction of a horizontal *DE* and a frontal *BF* of the plane, and 2) the drawing of a perpendicular from A_1 to D_1E_1 and another from A_2 to B_2F_2, i. e., $A_1K_1 \perp D_1E_1$ and $A_2K_2 \perp B_2F_2$.

2. The determination of the distance of a point from a plane. The problem of determining the distance of a point from a plane may be subdivided as follows: 1) the construction of a perpendicular to the plane from a given point; 2) the determination of the piercing point of this perpendicular in the plane; 3) the determination of the true length of the perpendicular line the length of which is the distance required.

It is required to determine the distance of point *D* from the plane of triangle *ABC* (Fig. 179).

1. In the plane of triangle *ABC* draw the horizontal *AL* and frontal *CF*. From point D_1 draw a perpendicular to the horizontal projection A_1L_1 of the horizontal and from point D_2 draw a perpendicular to the frontal projection C_2F_2 of the frontal. Line *DE* is then the perpendicular drawn from point *D* to the plane *ABC*.

2. The piercing point of perpendicular DE with plane ABC is found with the help of a frontal-projecting plane γ passed through DE. The frontal trace γ_2 of this plane coincides with the frontal projection D_2E_2 of the perpendicular DE. The horizontal trace m_1 is perpendicular to ground line Ox. Find the line MN of intersection of planes ABC and γ. The intersection of lines DE and MN gives point K where the perpendicular DE pierces the plane ABC. The length of line DK, whose projections D_1K_1 and D_2K_2 have been found, is the distance from point D to plane ABC.

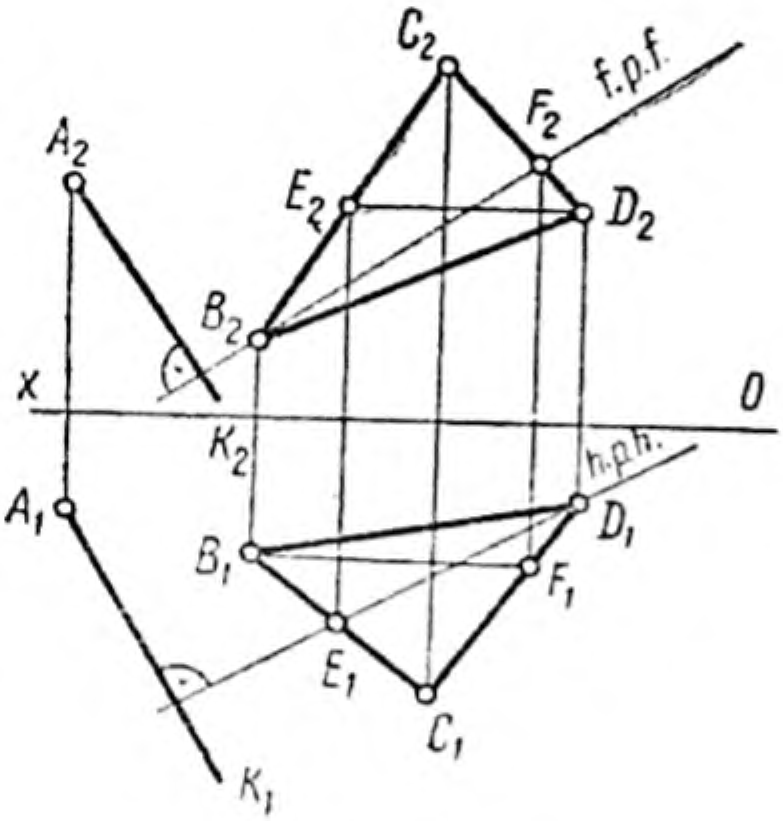


Fig. 178

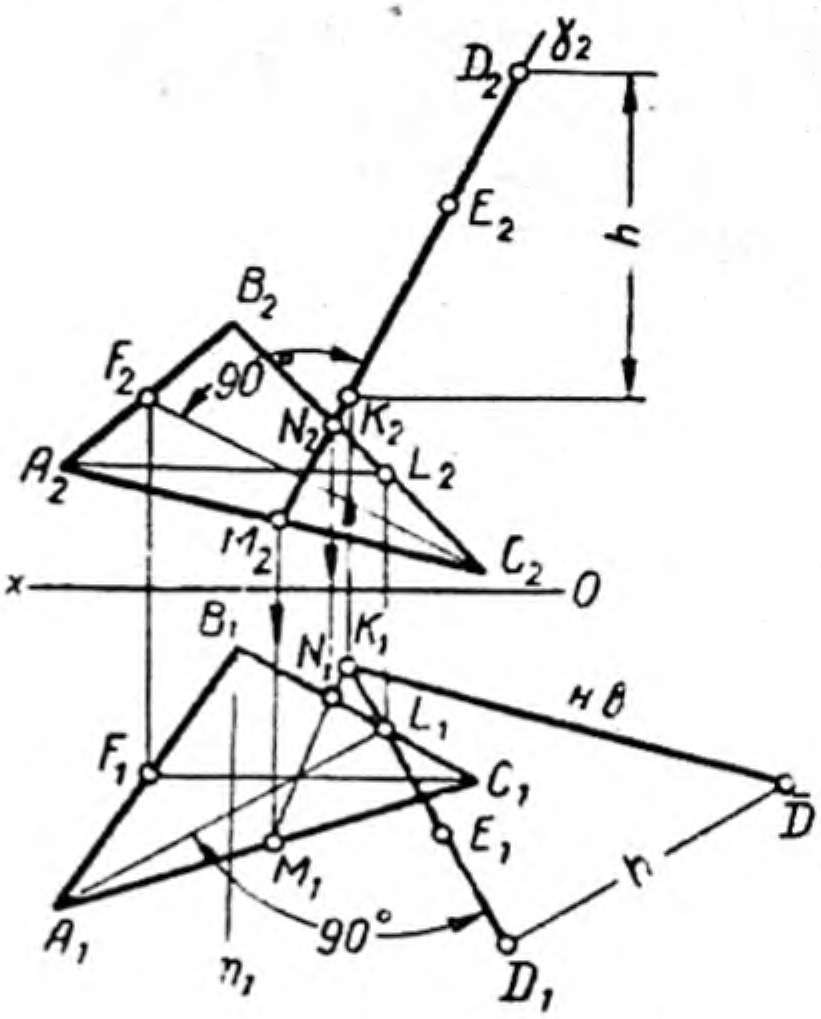


Fig. 179

3. The true length of this line DK is found by the right triangle method. Having obtained the difference h of the distances of points D and K from plane Π_1, construct the right triangle $D_1K_1\bar{D}$, one side of which is the horizontal projection D_1K_1 of line DK. A second side is equal in length to h. The hypotenuse $K_1\bar{D}$ gives the true value of the required distance from point D to the plane of triangle ABC.

3. If the plane is one of the projecting planes, as in Fig. 180, where the plane α is a frontal-projecting plane, the determination of the distance of the plane from the given point A is considerably simplified. This is so because the frontal projection K_2 of the piercing point K of the perpendicular drawn from point A to plane α lies on the trace l_2, while the perpendicular AK is parallel to plane Π_2 and so is projected onto Π_2 in its true length.

4. In order to pass a plane horizontal to line BC through a point A, a horizontal or frontal at right angles to the given line is constructed (Fig. 181). By drawing through point A the horizontal AN, whose horizontal projection A_1N_1 is

perpendicular to B_1C_1 and, having found trace N of this horizontal, draw through point $N_2 \equiv N$ the trace $l_2 \perp B_2C_2$. Through the point of convergence of the traces draw the trace $k_1 \perp B_1C_1$. The plane α so constructed is the required plane, since it passes through point A and is perpendicular to line BC, moreover its traces k_1 and l_2 are perpendicular to the corresponding projections of the given line.

5. The problem of determining the distance of a point from a line may be solved with the help of a plane passed through the point perpendicular to the line.

The solution of this problem is as follows (Fig. 182):

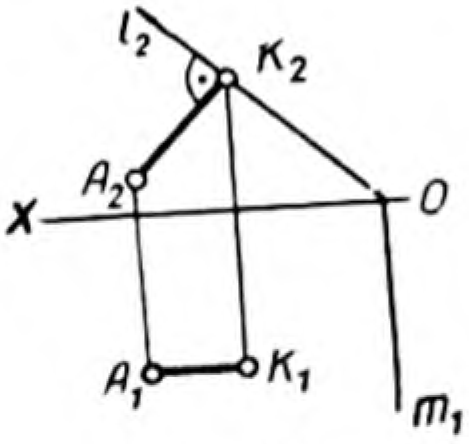


Fig. 180

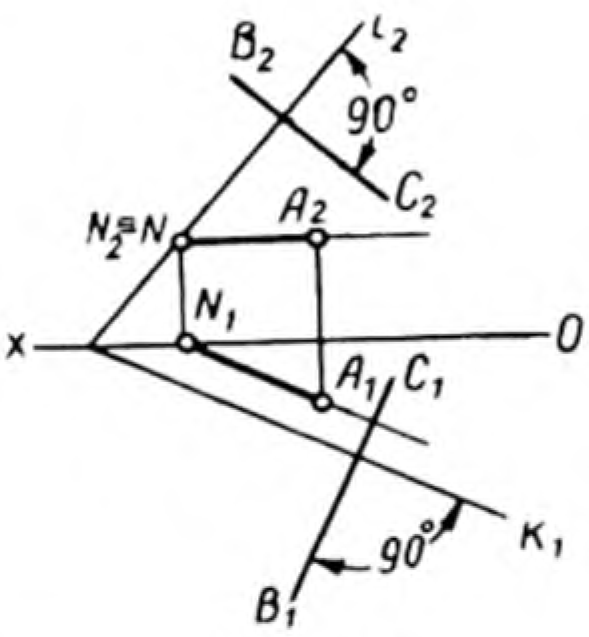


Fig. 181

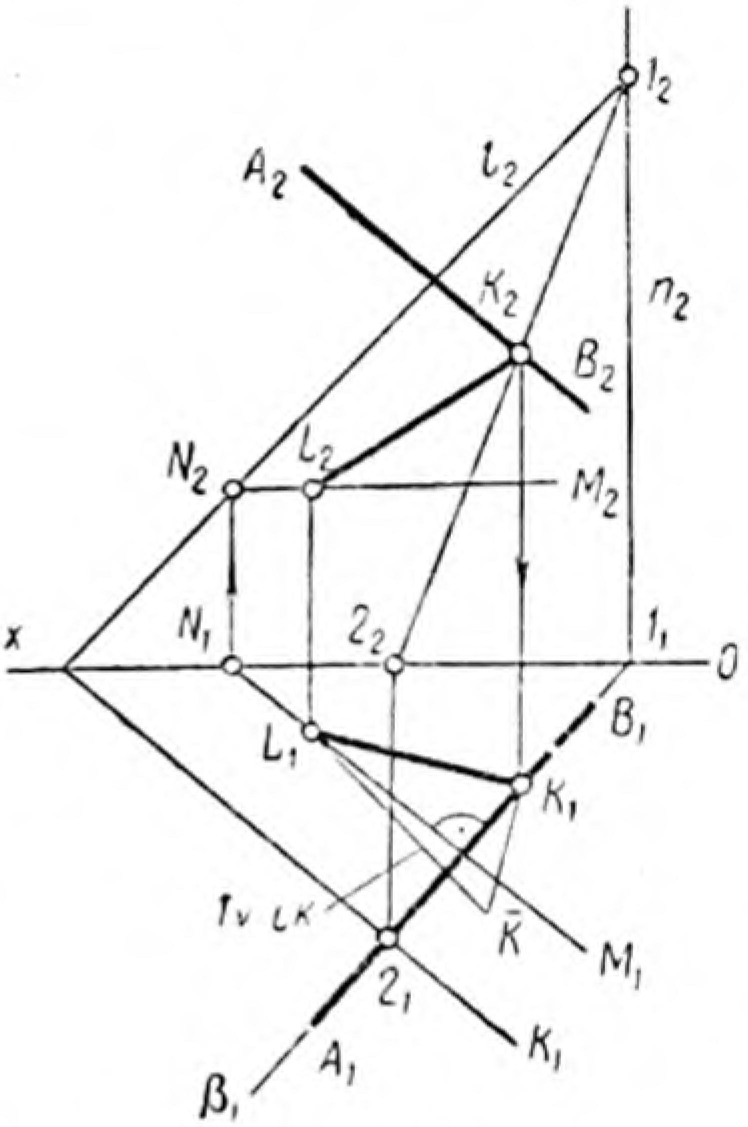


Fig. 182

1. Construct the plane $\alpha \perp AB$ with the help of the horizontal MN drawn through point L perpendicularly to AB (see Fig. 181).

2. Find the meeting point K of line AB with plane α using plane $\beta \perp \Pi_1$ passed through AB.

3. Join up points K and L. The perpendicular KL located in the plane $\alpha \perp AB$ enables the distance of point L from line AB to be determined.

4. The true length $\bar{K}L_1$ of line KL is found by constructing the right-angle triangle $K_1L_1\bar{K}$.

§ 22. Perpendicular Planes

1. Two planes are mutually perpendicular if in each plane a line may be drawn perpendicular to the other.

It follows from this that, in order to pass a plane perpendicular to the triangle ABC through some point M (Fig. 183), the first step is to construct a line MN perpendicular to the given plane. To do so the horizontal AL and the frontal CF are drawn. The horizontal projection M_1N_1 of the perpendicular passing through M will be perpendicular to the horizontal projection A_1L_1 of the horizontal AL. The frontal projection M_2N_2 of the perpendicular will, in turn, be perpendicular to the frontal projections C_2F_2 of the frontal CF. A plane passing through line MN will be perpendicular to the plane of triangle ABC.

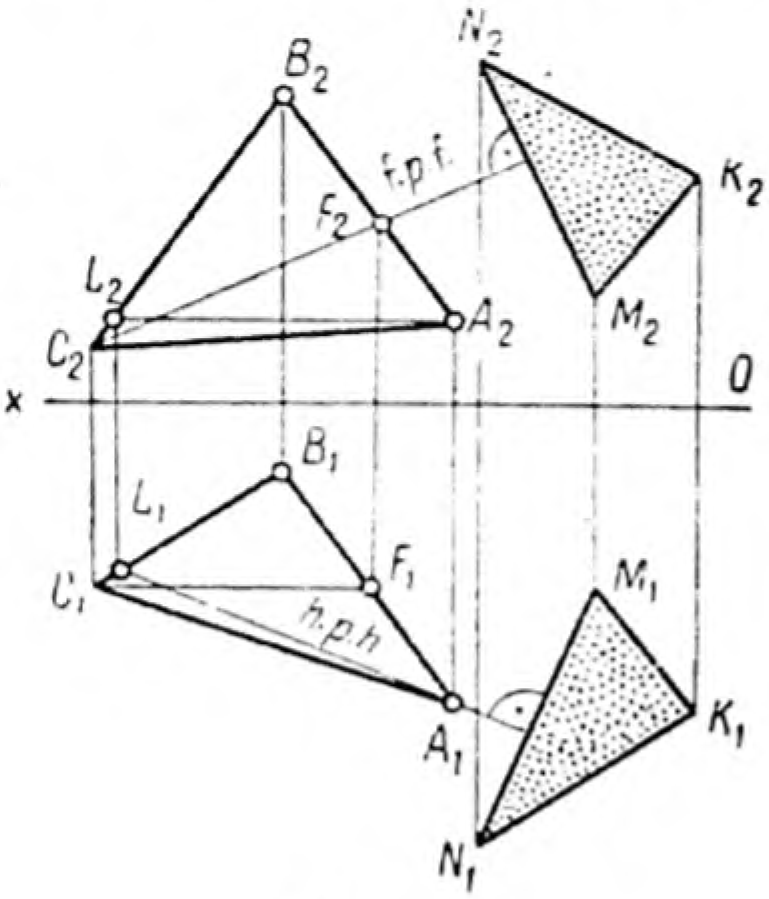


Fig. 183

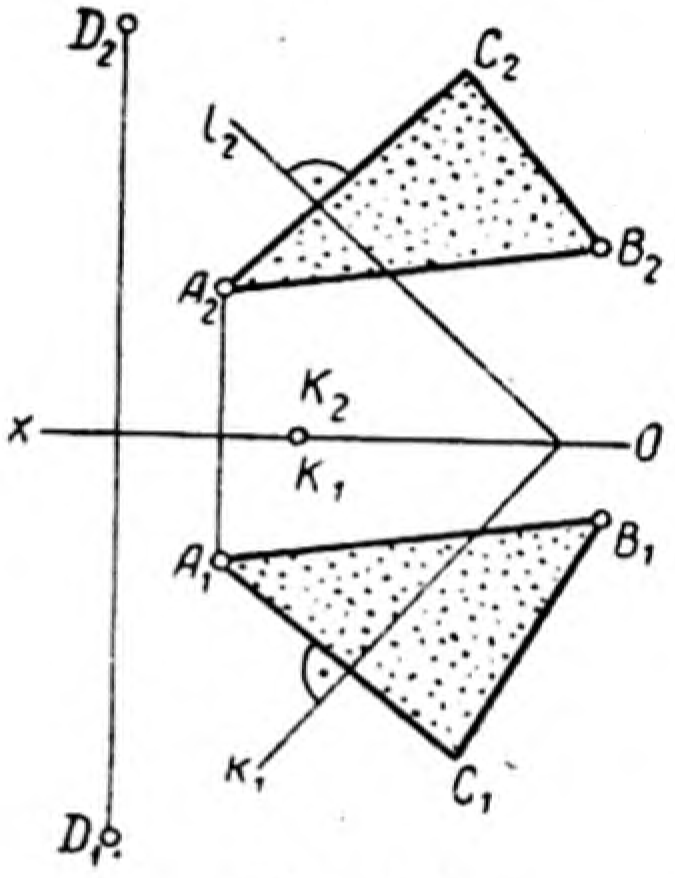


Fig. 184

An infinite number of planes perpendicular to plane ABC could be drawn through line MN. The plane determined by triangle MNK in Fig. 183 is one of them.

In Fig. 184 the plane of triangle ABC is perpendicular to plane α given by traces k_1 and l_2, since the side AC of this triangle is perpendicular to plane α ($A_1C_1 \perp k_1$ and $A_2C_2 \perp l_2$).

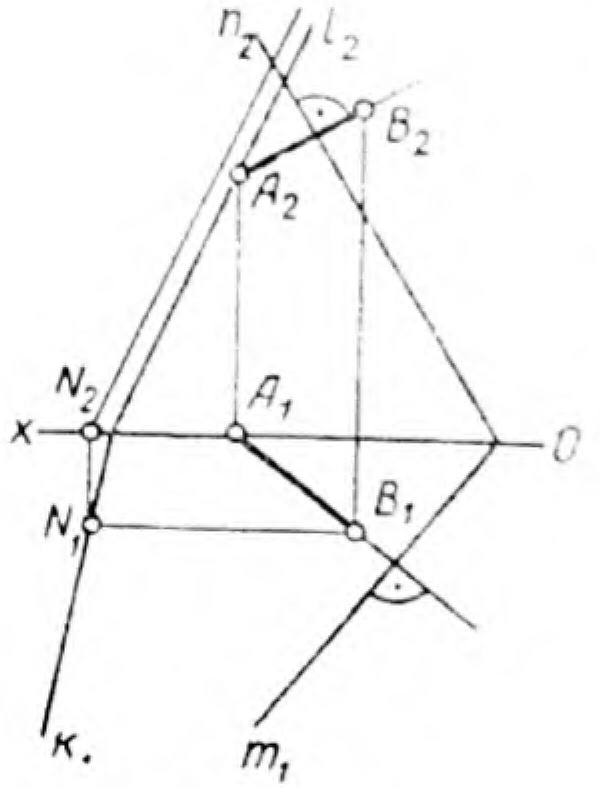


Fig. 185

2. The problem whether two given planes are perpendicular to each other is solved by drawing through a point taken in one of these planes a line perpendicular to the other plane. If that perpendicular is contained in the first plane, then the planes are perpendicular to each other.

The planes α and β, depicted in Fig. 185, are not perpendicular, since the perpendicular AB drawn to plane β from a point A, taken on trace l_2 of plane α, does not lie in plane α. This may be proved by drawing an auxiliary line in plane α. Such a line is the frontal BN in Fig. 185.

It is clear that frontal BN does not intersect the perpendicular AB and is not parallel to it.

3. Planes α and β whose corresponding traces are perpendicular, i. e., $k_1 \perp m_1$ and $l_2 \perp n_2$, are not perpendicular to each other. This is so because a

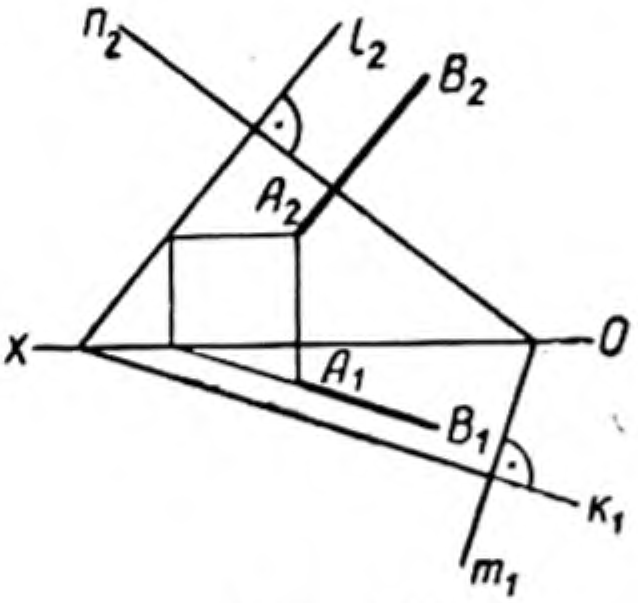


Fig. 186

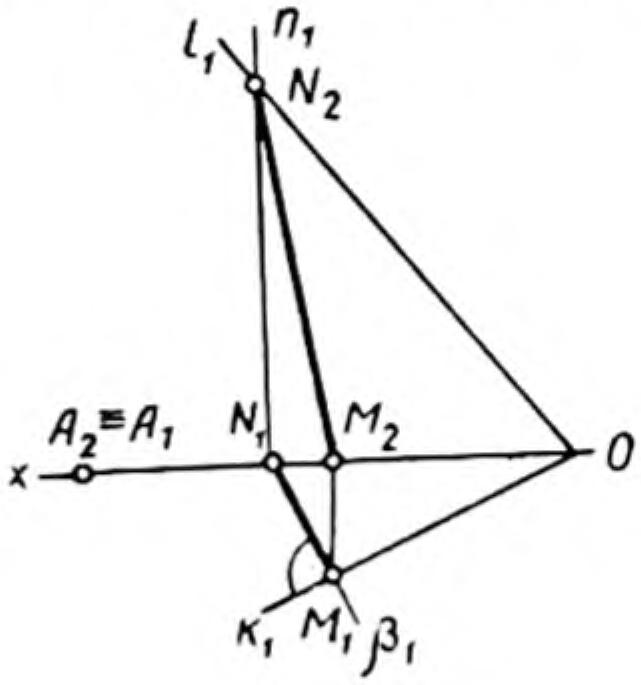


Fig. 187

perpendicular AB drawn to plane β from point A, taken in plane α, is not contained in plane α, since its projections are parallel to the corresponding traces of this plane ($A_1B_1 \| k_1$ and $A_2B_2 \| l_2$).

4. Attention should be paid to the fact that the mutually perpendicular disposition of corresponding traces of two oblique lines, such as those in Fig. 186, is a sign that such planes do not intersect at right angles.

If at least one of two intersecting planes is a projecting plane and if the angle formed by the trace-projection of the projecting plane and the similar trace of the other plane is a right angle, then the planes are mutually perpendicular.

Such a case is shown in Fig. 187. The oblique plane is perpendicular to the horizontal-projecting plane β since the horizontal trace k_1 of plane α is perpendicular to plane β. This is so because $k_1 \perp \beta_1$ and the frontal trace n_1 is perpendicular to the ground line Ox with which the frontal projection of trace k ($k_2 \equiv Ox$) coincides.

If one of two mutually perpendicular planes is a profile plane and the other is parallel to ground line Ox or passes through it, then the corresponding traces of such planes in the Π_1-Π_2 system of planes of projection are mutually perpendicular.

It should be noted that the line of intersection MN of the mutually perpendicular planes α and β (Fig. 187) is perpendicular to trace k_1, since $M_1N_1 \perp k_1$. Therefore MN is at the same time the line of maximum inclination of plane α to the horizontal plane of projections Π_1. It follows that the line of maximum inclination of a given plane may be constructed by intersecting it by a projecting plane perpendicular to the trace of the given plane (for comparison see § 17).

Problems and Exercises

1. What are the positions in the multiview drawing of the projections of a line which is perpendicular to a plane given by its traces?

2. How is the true distance to be found of a point from a) a horizontal-projecting plane, b) an oblique plane?

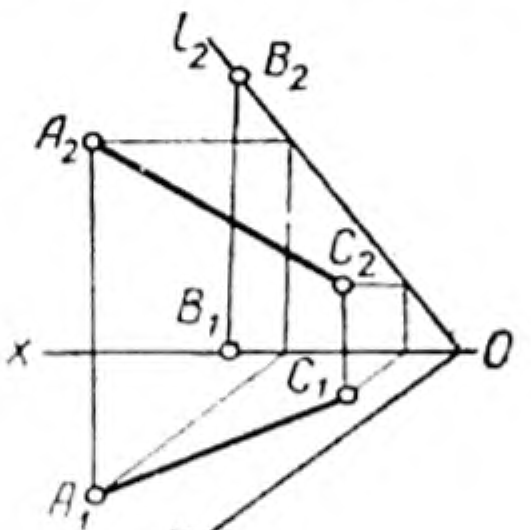


Fig. 188

3. How are the frontal traces of two mutually perpendicular planes seen in the multiview drawing if one of them is an oblique plane and the other is a frontal-projecting plane?

4. Are oblique lines perpendicular to each other if their corresponding traces in Π_1 and Π_2 are mutually perpendicular?

5. What planes should be made to intersect a plane in order to obtain in it a line of maximum inclination a) to plane Π_1, b) to plane Π_2, and c) to plane Π_3?

6. Find the distance of point D from the plane of triangle ABC (Fig. 184).

7. Find the distance of point K contained in ground line Ox from plane α (Fig. 184).

8. Determine whether the planes given by triangles ABC and DEF are perpendicular (Fig. 197).

Hint. From any point taken in one of the planes drop a perpendicular to the other plane and check whether the perpendicular belongs to the first plane (see Fig. 185 and text).

9. Through a point C taken on line AB draw a line CL perpendicular to AB to intersect line DE.

Hint. With the help of level lines perpendicular to AB pass through point C a plane perpendicular to AB. Find the point of intersection L of line DE with the plane drawn and join points C and L.

10. Construct a rectangle $ABCD$ by means of two projections of its side AB and the horizontal projection of side BC.

Hint. With the help of level lines perpendicular to AB draw an auxiliary plane perpendicular to AB and passing through point B. The vertex C_1 will lie in this plane. Having found projection C_1 find projection C_2 and join up points C_2 and B_2.

The projections of the other two sides of the rectangle are parallel and equal to the corresponding projections of its sides AB and BC.

11. In plane α construct a parallelogram $ABCD$ given its diagonal AC and its vertex B (Fig. 188).

12. By means of the frontal projection C_2D_2 construct the horizontal projection C_1D_1 of a line CD which is to be perpendicular to a given line AB.

Hint. Draw a plane perpendicular to line AB and in it construct an auxiliary line parallel to line CD. The required projection C_1D_1 will be parallel to the horizontal projection of this auxiliary line.

13. Through the piercing point of line AB in an oblique plane α draw a line perpendicular to AB lying in the plane.

14. Construct a plane β, parallel to a given plane α and situated 20 mm from it.

§ 23. The Determination of Visibility of Geometrical Elements in Multiview Drawings

1. In order that a drawing may be as clear as possible the visible contours of the object represented are usually shown by continuous lines and hidden lines, by broken lines. Sometimes invisible parts are not shown at all.

The methods of determining the visibility of elements in the multiview drawing are given below.

Of two given points lying on a line perpendicular to a given plane of projection, the point nearer to the eye of the observer will be the visible point. It is assumed that the observer is standing in the first octant looking along the line perpendicular to the plane of projection to which the two points belong.

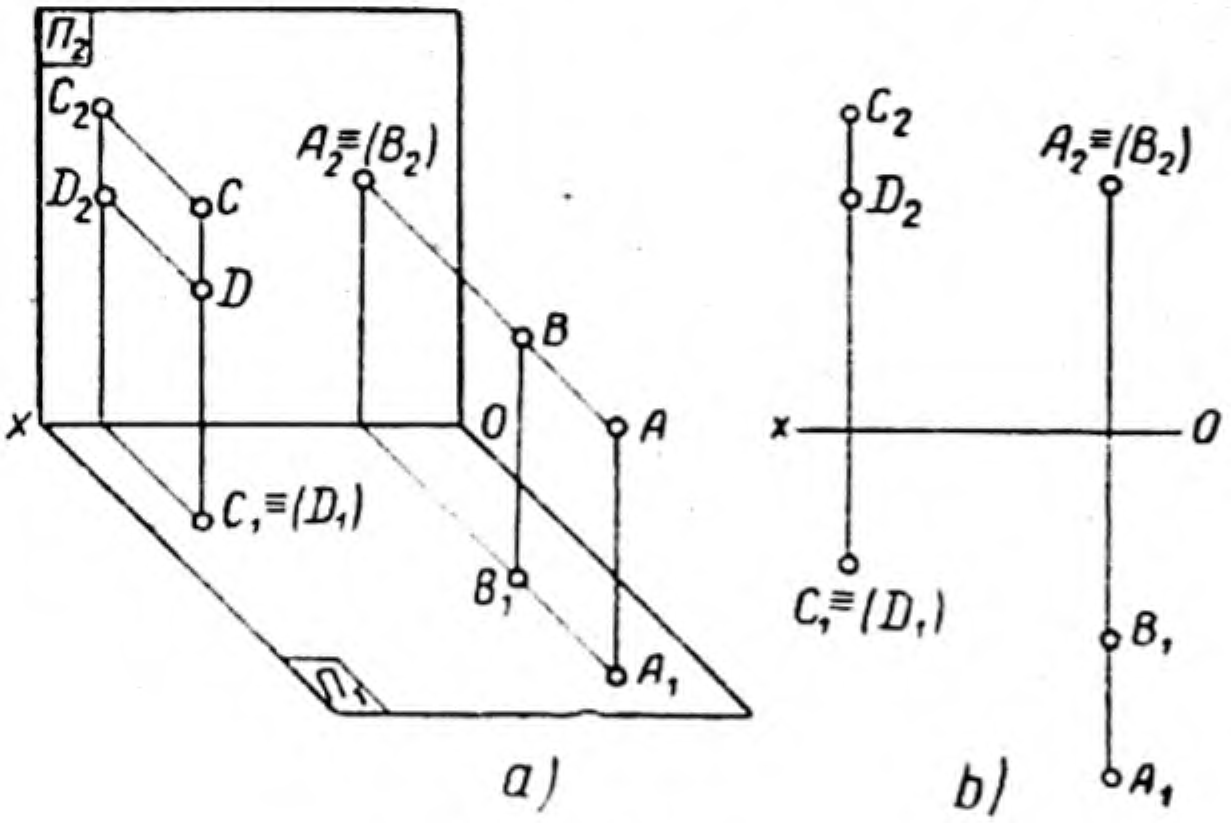
It follows that, of two points lying on a line perpendicular to a plane of projection, the one farther away from the plane is the visible point.

The points A and B (Fig. 189, *a*) are situated on the same projector perpendicular to plane Π_2. Point A being farther away from plane Π_2 covers point B. So, when projected onto Π_2, point A is visible and point B is hidden. When projected onto plane Π_1 both points A and B are visible because they lie on different lines perpendicular to plane Π_1.

In the multiview drawing (Fig. 189, *b*) the frontal projections of points A and B coincide, $A_2 \equiv (B_2)$, since these points in space lie on the same frontal projector. The horizontal projection A_1 of point A lies farther away from ground line Ox than the horizontal projection B_1 of point B, so it follows that point A is farther away from plane Π_2 and, being projected on this plane Π_2, covers point B.

Points C and D (Fig. 189, *a*) lie on the same line perpendicular to plane Π_1, point C being higher than point D, that is to say, farther away from plane Π_1, so point C, when projected onto plane Π_1, will be visible and point D will be hidden.

In the multiview drawing, Fig. 189, *b*, the horizontal projections of points C and D coincide, $C_1 \equiv (D_1)$. The frontal projection C_2 of point C is farther from ground line Ox than the frontal projection D_2 of point D, so point C is farther from plane Π_1 and, being projected onto that plane, will hide point D.



a) b)

Fig. 189

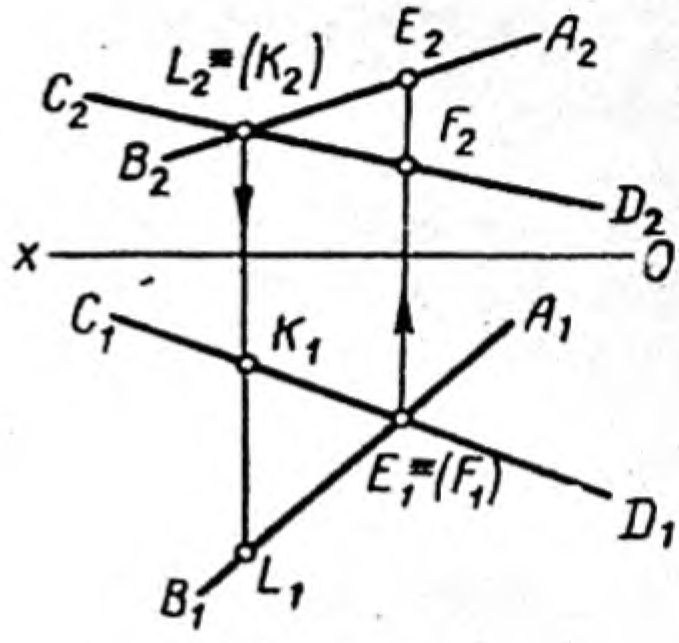


Fig. 190

2. Visibility of points on non-intersecting lines.

Two skew lines AB and CD are shown in Fig. 190. Point E, contained in line AB, and point F, contained in line CD, are situated on the same line perpendicular to plane Π_1. The horizontal projections of points E and F merge as one point, where the horizontal projections of the given lines AB and CD cross. Point E is farther from plane Π_1 than point F, and so, when projected on plane Π_1, will be visible.

Point L, contained in line AB, and point K, contained in line CD, are situated on one perpendicular to plane Π_2. The frontal projections of points L and K merge into the point, where the frontal projections of the given lines AB and CD cross.

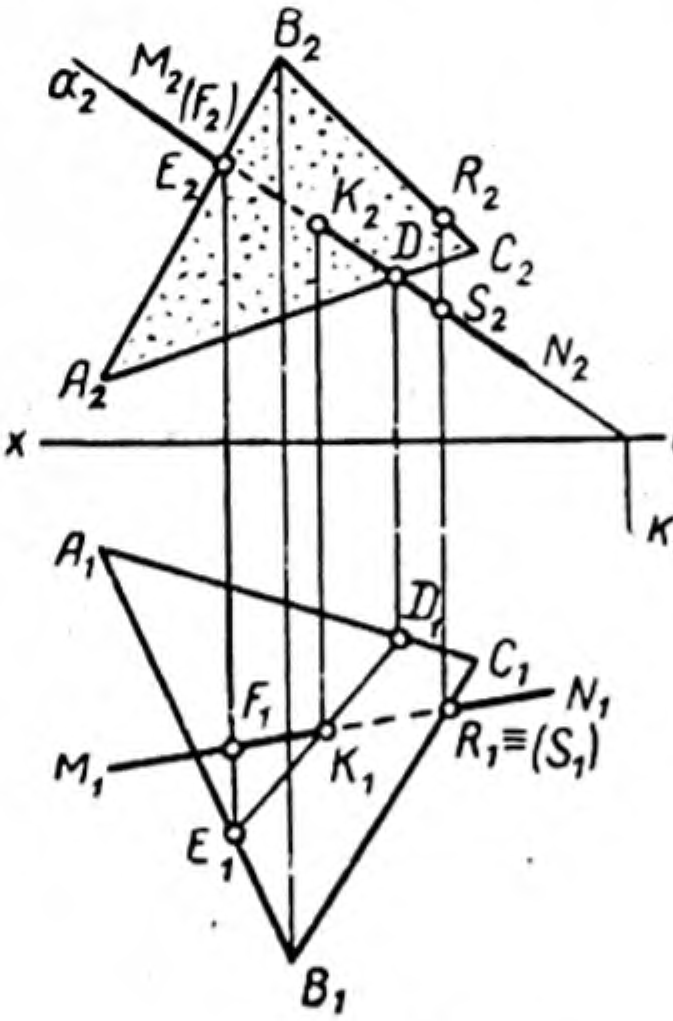
Point L lies farther away from plane Π_2 than point K and, when projected onto plane Π_2, will be visible.

3. Next let us consider the visibility of the parts of the projections of a line MN intersecting an opaque triangular plate ABC (Fig. 191).

By passing through MN an auxiliary frontal-projecting plane, point K, where line MN pierces triangular plate ABC, is found. This point K is the boundary of visibility of the parts of line MN. Let us determine the visibility of separate parts of line MN when it is projected onto plane Π_2. Take a point E on the side AB of the triangle and a point F on the line MN, so that points E and F

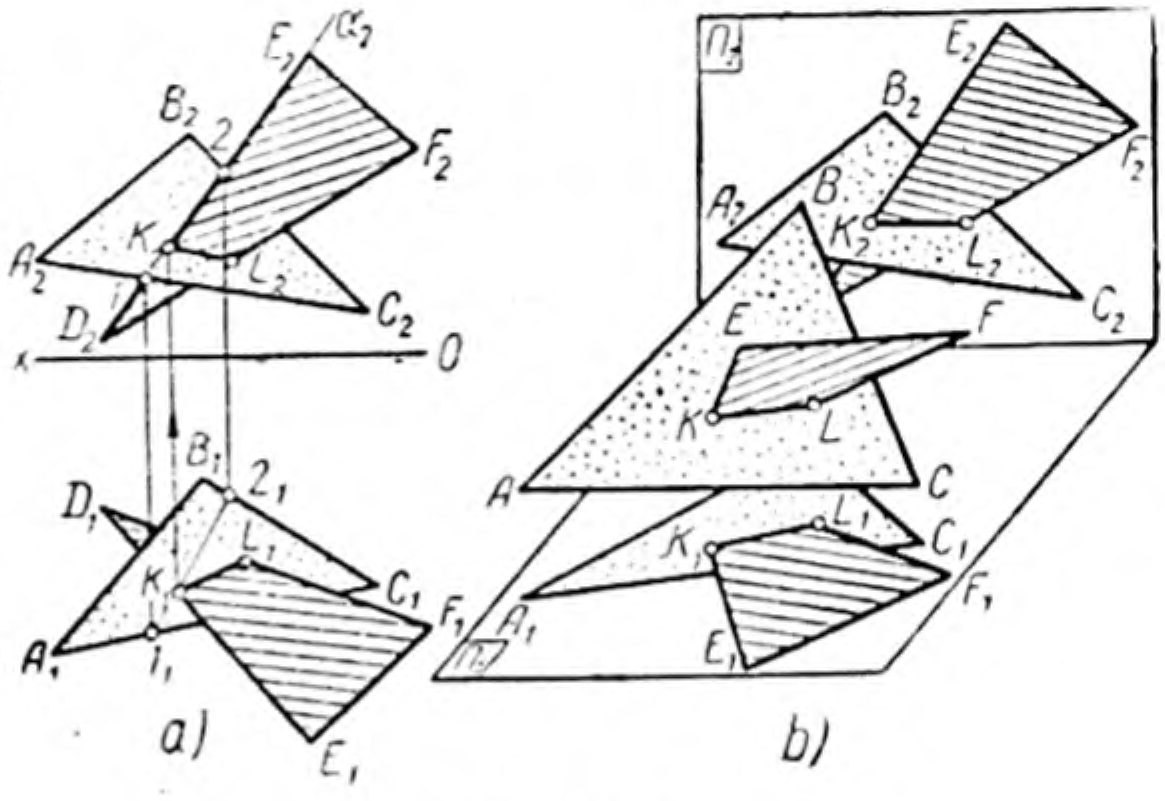
project onto plane Π_2 as one point $E_2 \equiv F_2$. The horizontal projection E_1 of point E lies farther away from ground line Ox than the horizontal projection F_1 of point F. In other words, the value of y for point E is greater than that of y for point F. Therefore point E, when it is projected onto plane Π_2, hides point F.

The part FK of line MN when projected onto plane Π_2 will not be visible. It is hidden by the opaque triangular plate ABC, whereas part KN remains visible since it lies in front of the plate.



α_2
$M_2(F_2)$
B_2
E_2
K_2
R_2
D_2
C_2
S_2
N_2
A_2
x
0
K_1
A_1
D_1
C_1
N_1
F_1
M_1
K_1
$R_1 \equiv (S_1)$
E_1
B_1

Fig. 191



E_2
α_2
B_2
2
F_2
A_2
K_2
L_2
C_2
D_2
x
0
B_1
2_1
D_1
L_1
K_1
C_1
F_1
A_1
1_1
E_1
a)
Π_2
E_2
B_2
A_2
B
K_2
L_2
F_2
C_2
E
F
K
L
A
C
K_1
L_1
C_1
F_1
A_1
E_1
Π_1
b)

Fig. 192

The visibility of parts of line MN, when projected onto plane Π_1, may be determined by comparing the values of z for points R and S. The horizontal projections of these points merge into one point $R_1 \equiv S_1$ and are located at the intersection of lines M_1N_1 and B_1C_1. Since the distance z of point R, which belongs to side BC of the triangle, is greater than that of point S which belongs to line MN, the part KS of line MN lies below the triangular plate ABC and so, when projected onto plane Π_1, is invisible. The part SN of line MN when projected on plane Π_1 is visible since it is not covered by the plate ABC.

4. Now let us find the line of intersection of the opaque triangular plates ABC and DEF. The visible part of triangle ABC will be shown by dots and that of triangle DEF by hatching (Fig. 192).

To construct the line of intersection KL of these triangles find the piercing point K of side ED with triangle ABC. This is done with the help of the frontal-projecting plane α and point L in which side FD pierces triangle ABC. This construction is not shown in Fig. 192. Next join points K and L.

The visibility of the separate parts of the triangles *ABC* and *DEF* when projected onto Π_1 and Π_2 may be established by determining the visible parts of the sides *ED* and *FD*, as was done with line *MN* in Fig. 191. The visible parts of triangles *ABC* and *DEF* are shown by dots and hatching respectively. The parts of the sides of the triangle that are hidden are not shown. The space drawing of the intersection of the triangles *ABC* and *DEF* and the projections of these triangles on planes Π_1 and Π_2 are shown in Fig. 192, *b*.

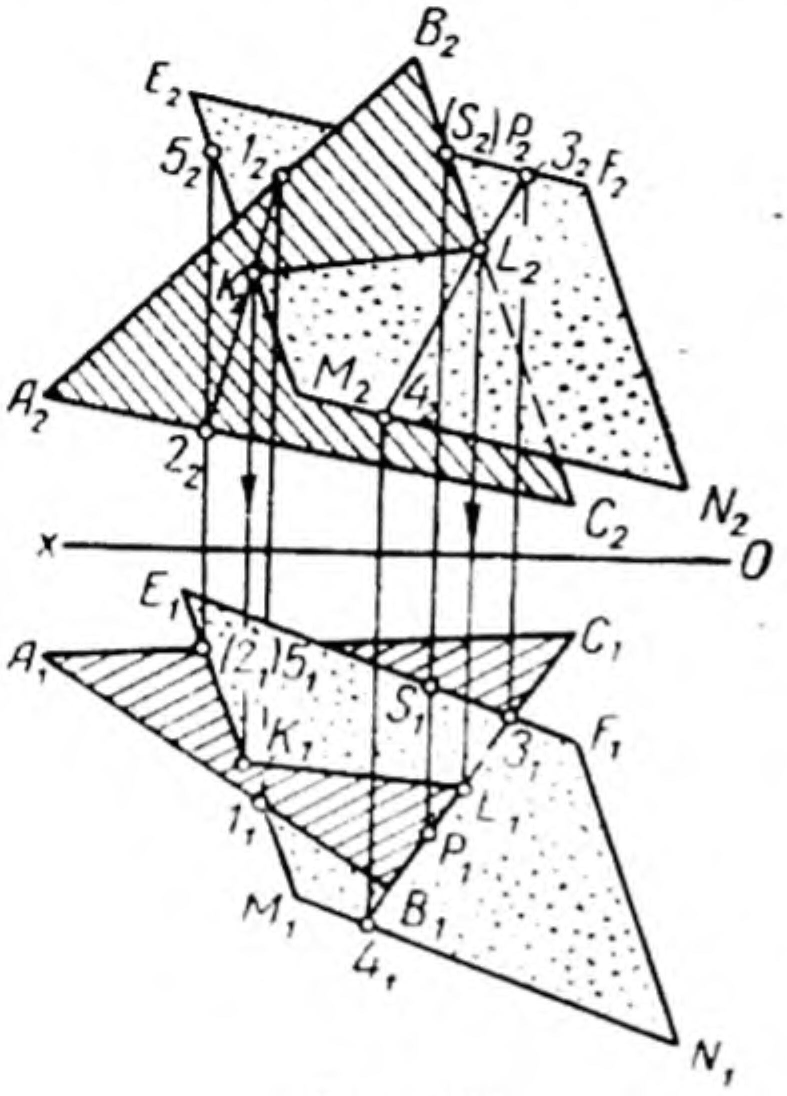


Fig. 193

5. Let us construct the line of intersection of triangle *ABC* with parallelogram *EFNM*, both of which are assumed to be opaque plates (Fig. 193).

In order to determine the line of intersection *KL* of the given figures, first find point *K* in which the side *EM* of the parallelogram pierces the plane of the triangle, and point *L*, where side *BC* of the triangle pierces the plane of the parallelogram. Points *K* and *L* are then joined up.

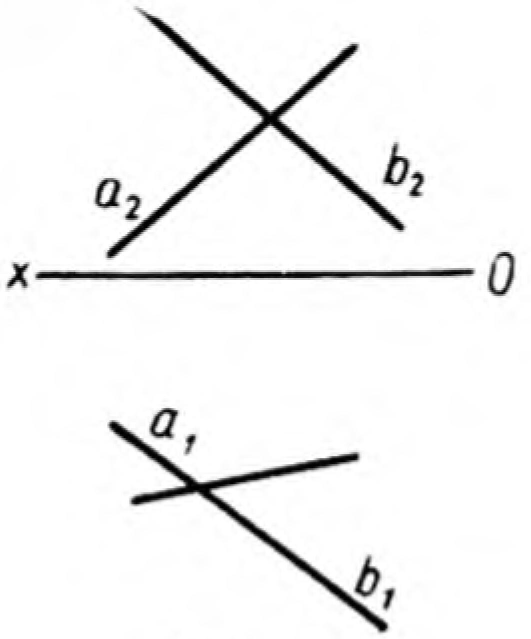
The visibility of the various parts, when projected onto plane Π_2, is determined with the help of points *S* and *P* taken on side *EF* of the parallelogram and on the side *BC* of the triangle, respectively. By comparing the values of y for points *S* and *P* it can be seen that point *P* is situated farther away from plane Π_2 than point *S*. It follows that part *BL* of side *BC* of the triangle will be visible when projected onto plane Π_2. Beyond the boundary of visibility, point *L*, the side *BC* is covered by the parallelogram and again becomes visible in the vicinity of the vertex *C* since this part is not hidden by the plane of the parallelogram.

The visibility of the various parts, when projected onto plane Π_1, is established by comparing the values of z for points *2* and *5* taken on sides *AC* and *EM* of the given figures. The value of z for point *2* is smaller than that of point *5*. It follows that point *5*, together with the corresponding part of side *EM* of the parallelogram up to point *K*, when projected onto plane Π_1, is visible. Beyond point *K* the side *EM* passes under the triangle *ABC* and again becomes visible when it emerges from the plane of triangle *ABC*.

Problems and Exercises

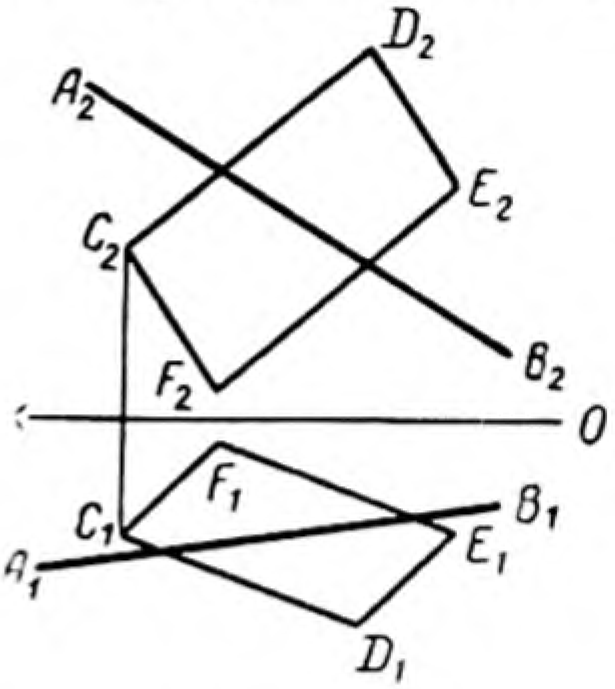
1. If two points are situated on one perpendicular to a plane of projections, how is their visibility determined?

2. When will a line which lies behind an opaque plate and is projected onto plane Π_1, be visible if it is projected onto plane Π_2?



a_2
b_2
x
0
a_1
b_1

Fig. 194

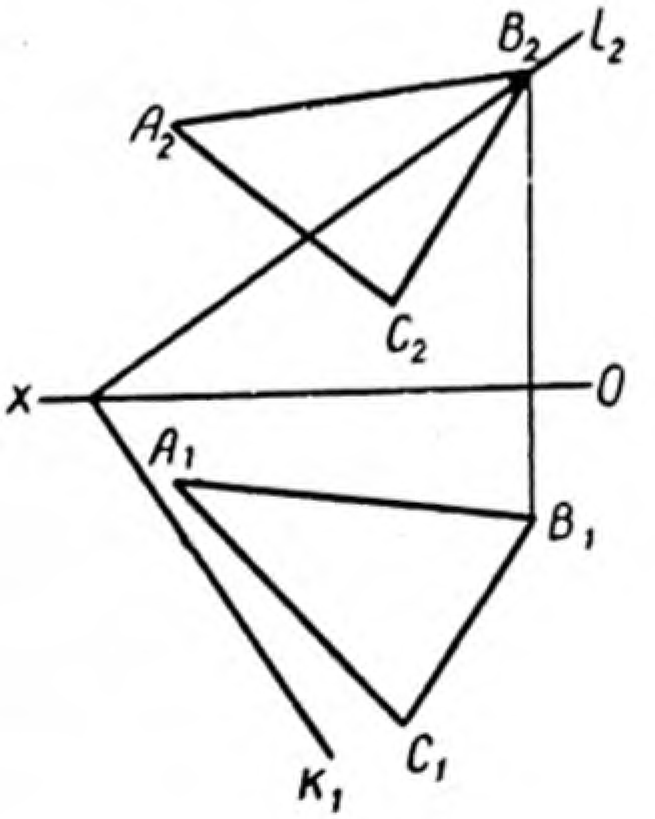


D_2
A_2
E_2
C_2
F_2
B_2
x
0
F_1
C_1
B_1
E_1
A_1
D_1

Fig. 195

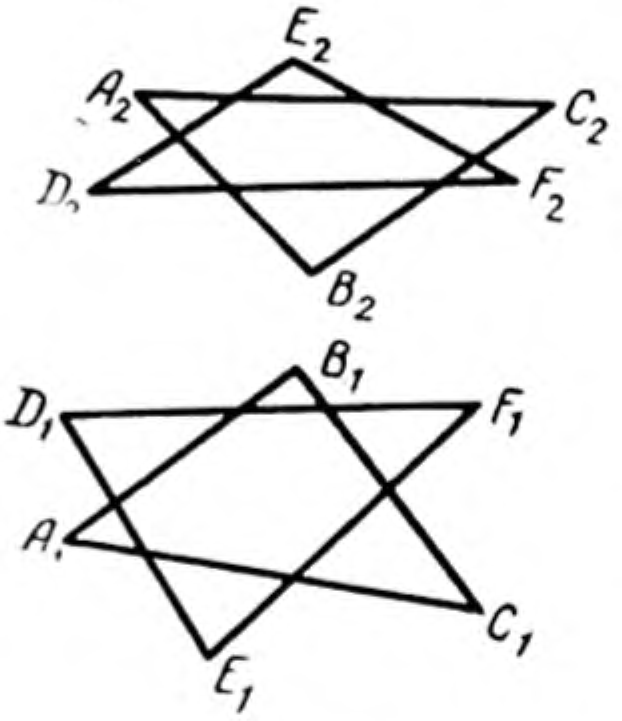
3. On the two non-intersecting lines a and b find the points, which are not visible when projected a) onto Π_1, b) onto Π_2, c) onto Π_3 (Fig. 194).

4. Determine the visible and hidden parts of line AB when it pierces the plane of parallelogram $CDEF$ (Fig. 195).



B_2
l_2
A_2
C_2
x
0
A_1
B_1
C_1
K_1

Fig. 196



E_2
A_2
C_2
D_2
F_2
B_2
B_1
D_1
F_1
A_1
C_1
E_1

Fig. 197

5. Construct the line of intersection of plane α, given by traces k_1 and l_2, with the triangle ABC. Show its visible part by hatching (Fig. 196).

6. Construct the line of intersection of triangles ABC and DEF, showing visible parts of the triangle ABC by hatching and those of the triangle DEF by dots (Fig. 197).

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Chapter VI

THE TRANSFORMATION OF PROJECTIONS

If straight lines or plane figures are parallel or perpendicular to the planes of projection, then distances, angles and the relative positions of individual geometrical elements in space, as seen in the multiview drawings, may be measured without any additional construction. That is to say true views are obtained. For example, the true distance of a point from a line is given by the projection on the horizontal plane Π_1 if the given line is perpendicular to that plane. Any figure lying in a horizontal plane will also be projected onto plane Π_1 in its true size.

The determination of the true dimensions of oblique lines and planes requires special constructions transforming the projections into others enabling the true views to be determined directly.

Any course of descriptive geometry must outline various methods of transforming projections. These are the substitution of one or more planes of projection by others, the method of rotating the planes of projection and the method of coincidence.

§ 24. Substituting Planes of Projection

1. A point A is given relative to the pair of planes of projection Π_1-Π_2 (Fig. 198, *a*). Instead of the frontal plane Π_2 an additional plane of projection Π_4, perpendicular to plane Π_1, may be introduced. In the case of such a substitution the profile plane of projection in the principal system of projection planes Π_1-Π_2-Π_3 continues to be referred to as Π_3. The planes Π_1 and Π_4 constitute a new pair of mutually perpendicular planes of projection and the line s_{14} will be the new ground line.

Let us construct the orthographic projections of point A in the new system of planes of projection Π_1-Π_4. The horizontal projection A_1 of point A in the new system Π_1-Π_4 will still be the horizontal projection of point A since plane Π_1 remains unchanged. To obtain the projection A_4 of point A on plane Π_4 draw a perpendicular to that plane from point A together with the rectangle $AA_1A_{14}A_4$, of which one side is the line AA_1 and another the perpendicular A_1A_{14} drawn from point A_1 to ground line s_{14}.

Since A_4A_{14} and AA_1 are the opposite sides of rectangle $AA_1A_{14}A_4$, they are equal. But $AA_1=A_2A_{12}$, so $A_4A_{14}=A_2A_{12}$. In other words, the distance of the new projection A_4 from the new ground line s_{14} is equal to the distance of the original projection A_2 from ground line Ox.

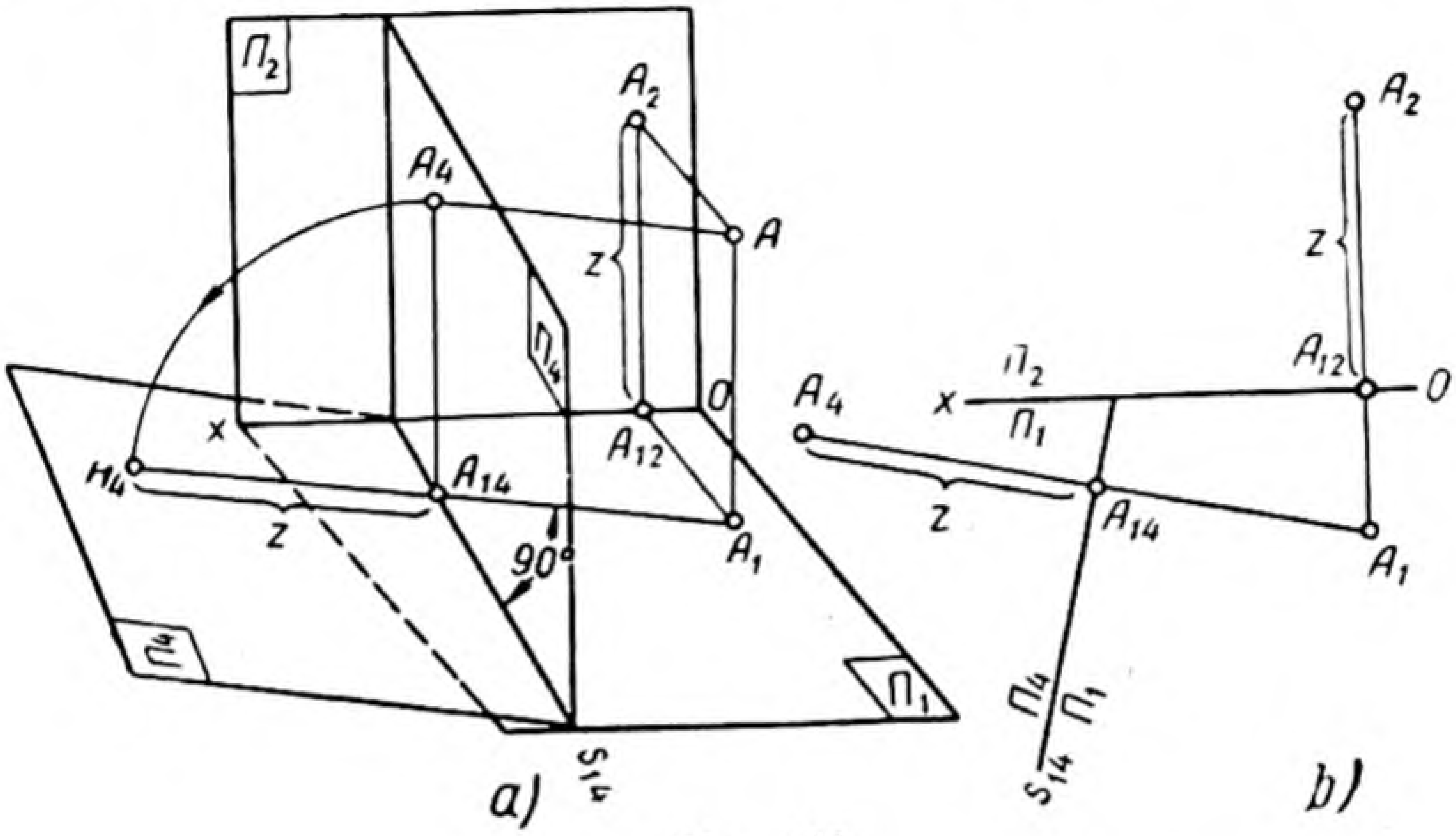


Fig. 198

In order to pass from the three-dimensional representation to the multiview drawing plane Π_4 is rotated about ground line s_{14} to coincide with plane Π_1. Projection A_4 of point A will then lie on the same perpendicular to the ground line s_{14} as point A_1.

Planes Π_1 and Π_4 may be brought into coincidence by rotating Π_4 in either direction. However, the direction of rotation should be such that the projections A_1 and A_4 of point A lie on opposite sides of ground line s_{14}.

The construction of projection A_4 of point A in the multiview drawing (Fig. 198, *b*) is carried out in the following order: 1) draw ground line s_{14} of the new system of planes of projection Π_1-Π_4; 2) from point A_1 draw a perpendicular to the line s_{14}; 3) on this perpendicular from the line s_{14} lay off distance A_4A_{14} equal to the length of line A_2A_{12}. The point A_4 obtained in this way is the orthographic projection of point A on plane Π_4.

2. The construction of the multiview drawing of point B (Fig. 199, *a*), when the horizontal plane of projection Π_1 is changed for the new plane Π_4, also perpendicular to plane Π_2, is carried out in the same sequence. First, the ground line s_{24} of the Π_2-Π_4 system of planes of projection is drawn. Then, from point B_2 a perpendicular is drawn to line s_{24}. Finally, on this perpendicular the distance $B_4B_{24}=B_1B_{12}$ is laid off from line s_{24}. In this case the distance of projection B_4 of point B from the new ground line s_{24} is the same as the distance of the original projection B_1 from the ground line Ox.

The construction of point B_4 when substituting the plane of projections Π_1 by Π_4 is also shown in Fig. 199, *b* where plane Π_4,

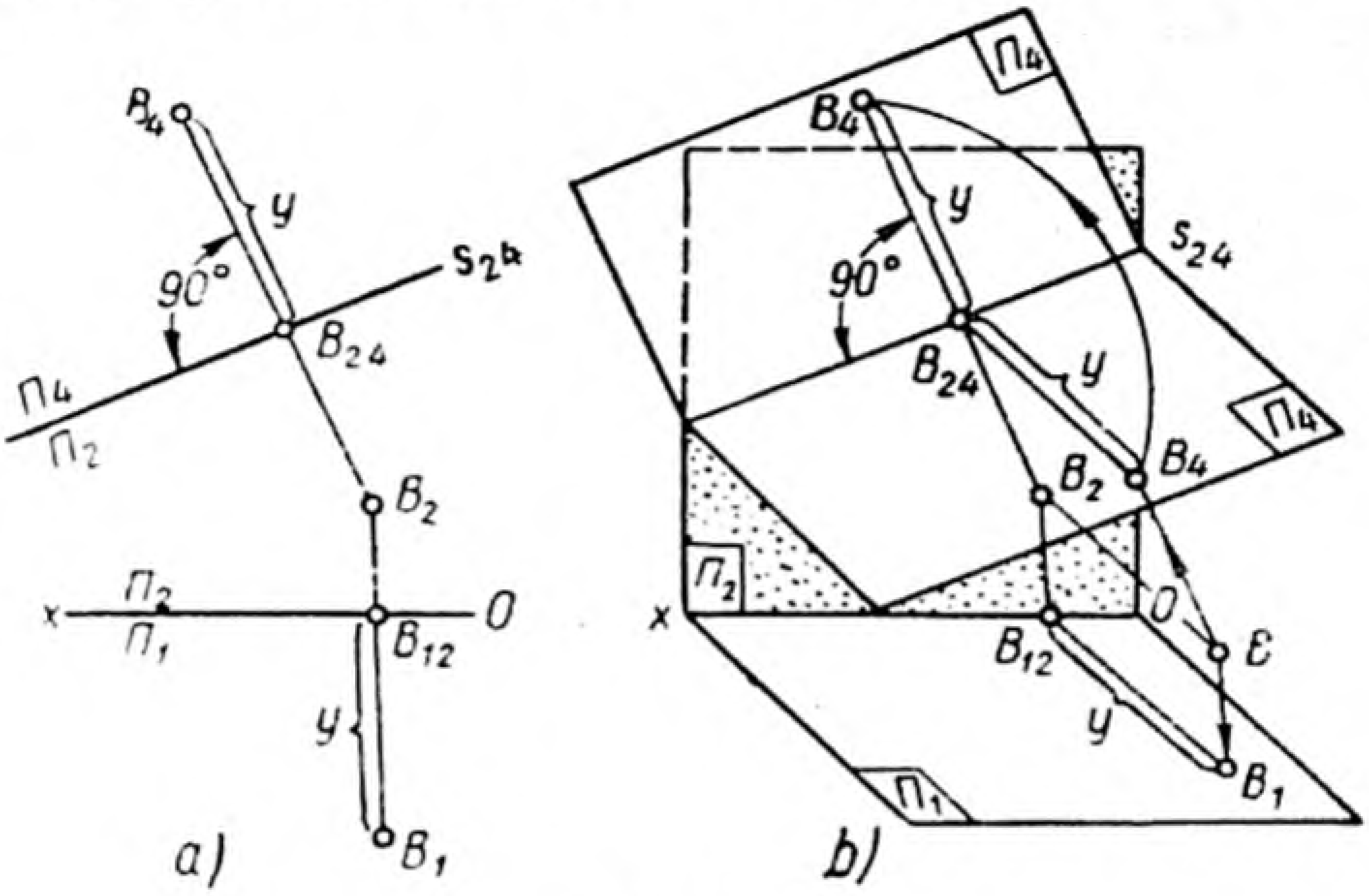


Fig. 199

which is also perpendicular to plane Π_2, is rotated about line s_{24} together with projection B_4 until it coincides with plane Π_2.

3. The projection of a line, when one of the planes of projection is changed, is received by projecting two of its points onto the new plane of projection and joining up their projections.

Given the projections of a line AB, see Fig. 200, the sequence in which its new projection on a plane Π_4, perpendicular to the horizontal plane of projection Π_1 and parallel to AB, may be found, is as follows. Since plane Π_4 has substituted plane Π_2, the distances of points A_4 and B_4 from line s_{14} are respectively equal to the distances of points A_2 and B_2 from ground line Ox.

The new projection A_4B_4 of line AB is a true length of the line, so the length of A_4B_4 is its real length. It should be recalled that since the plane Π_4 is parallel to AB the ground line s_{14} is drawn parallel to the horizontal projection A_1B_1 of segment AB.

Line AB, as seen in Fig. 201, is projected onto the new plane Π_4, drawn parallel to AB itself and perpendicular to plane Π_2. The distances from points A_4 and B_4 to line s_{24} are respectively equal to the distances from points A_1 and B_1 to line Ox. The new projection A_4B_4 gives the true dimension of line AB since this line is parallel to Π_4.

The angles α and β, shown in Figs. 200 and 201, are the true angles of inclination of line AB to the planes of projection Π_1 (angle α) and Π_2 (angle β).

Hint. It should be noted that the angle of inclination α of the given line to the horizontal plane of projection Π_1 is given by the angle between the new axis of projection s_{14}, or of any line parallel to it, and the true length of the line in the system Π_1-Π_4 which includes the horizontal projection of line A_1B_1.

The true angle of inclination β of the given line to frontal plane of projection Π_2 is seen in the new system Π_1-Π_4 which includes the original frontal projection of the line.

Finally, it should be understood that when determining the true angle of inclination γ of the given line to the profile plane of projection, it is necessary to construct a new system of planes of projection which must include the profile plane of projection Π_3.

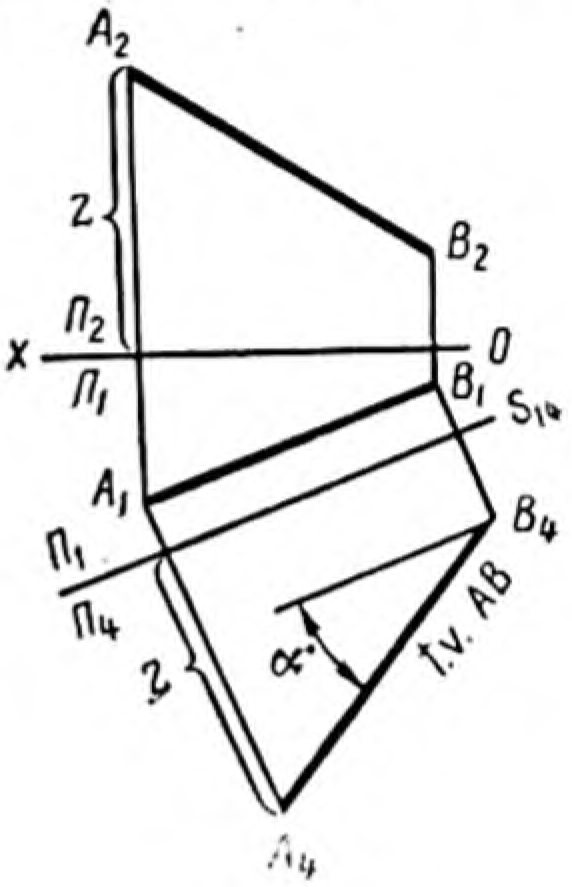


Fig. 200

4. It must be remembered that, although the new planes of projection parallel to line AB in the transformations analyzed above (see Figs. 200 and 201) may lie at any distance from AB, the ground line of the new system of planes of projection must necessarily in all cases be parallel to that projection of the line which remains unchanged. In that case the new projection will give the true length of the line.

In Fig. 202, when determining the true length of AB by the

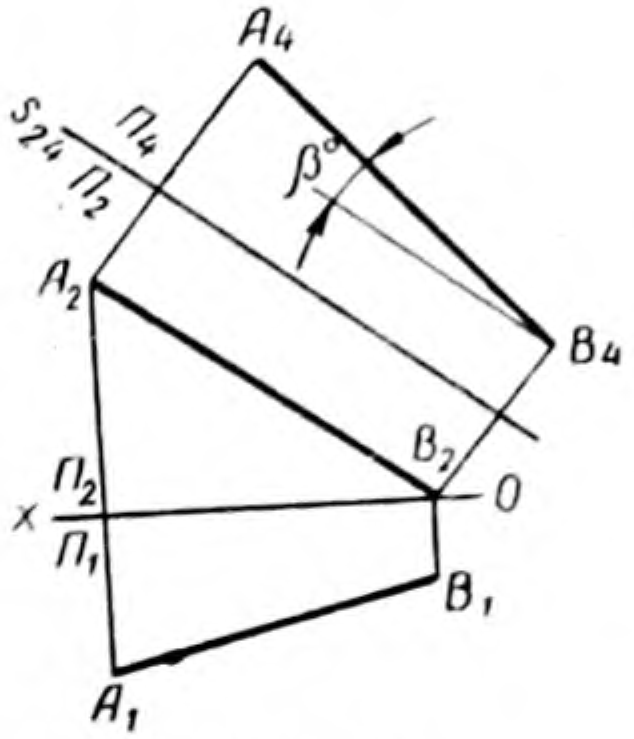


Fig. 201

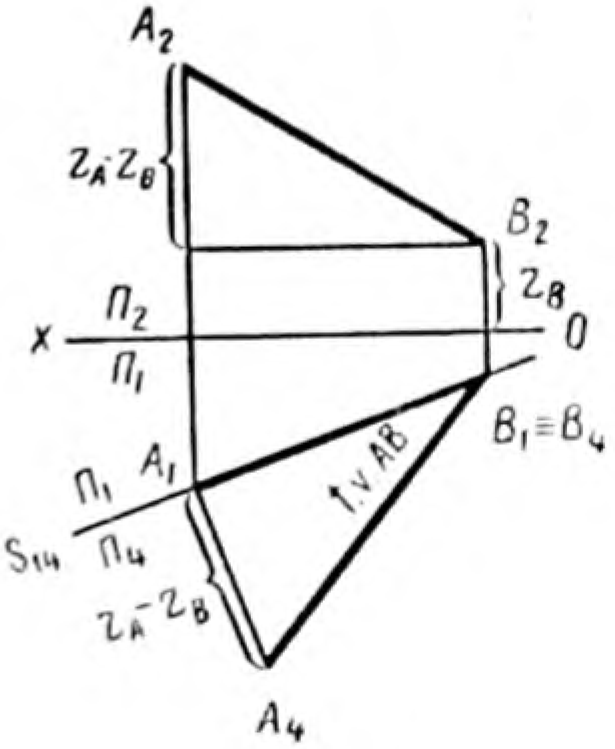


Fig. 202

method of planes of projection, in order to keep the drawing as small as possible, the ground line s_{14} of the Π_1-Π_4 system of planes of projection is drawn through the projection A_1B_1. The size of the drawing may also be limited by laying off from line s_{14} not the real values of z_A and z_B for points A and B, but these dimensions less some value. In the case in point z_B is the value subtracted, and for point B we receive $z_B - z_B = 0$.

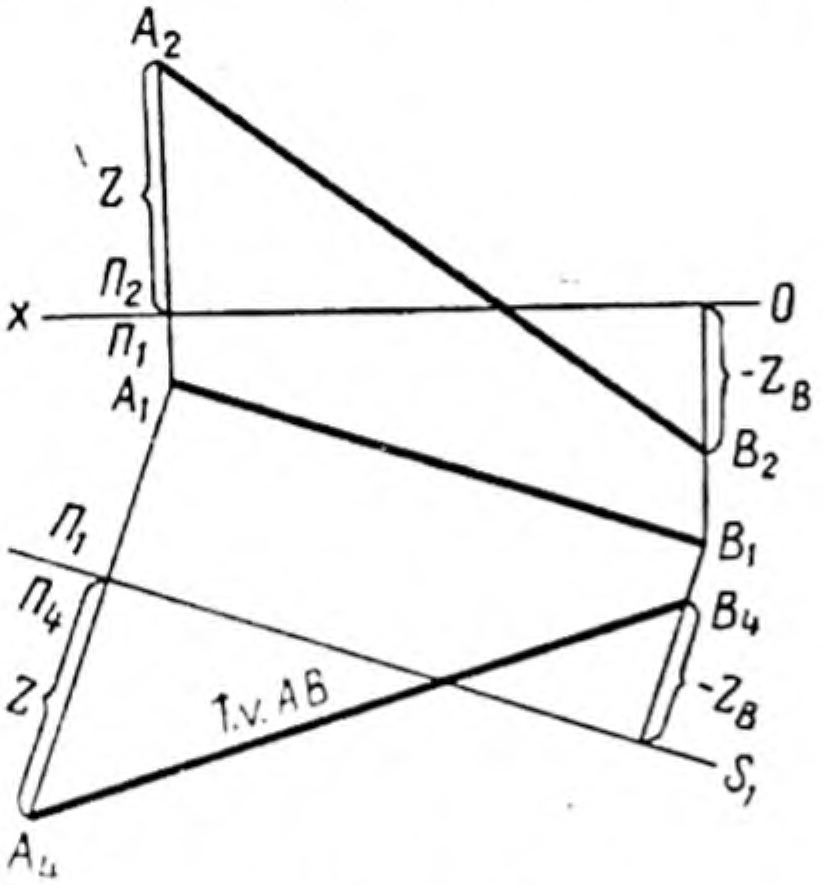


Fig. 203

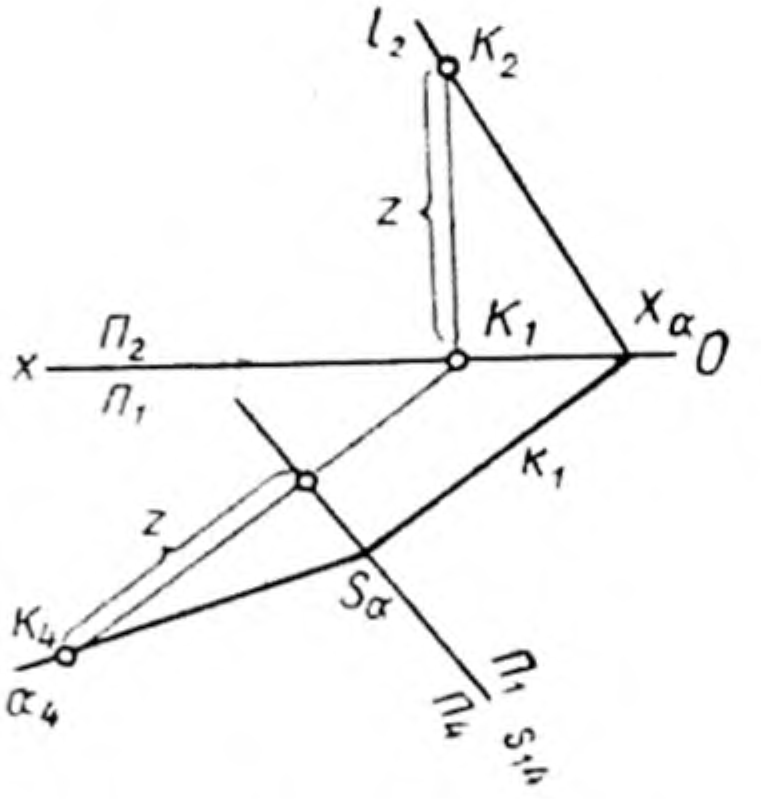


Fig. 204

In this case point B_4 coincides with point B_1 and projection A_4B_4 will be the hypotenuse of the right triangle $A_1A_4B_4$, one side of which is the horizontal projection A_1B_1 of the given line and the second side is the difference between distances at which its limiting points A and B are situated from plane Π_1, in which the side A_1B_1 of the triangle lies. Thus the right triangle method is actually the conversion of the multiview drawing by the method of changing planes of projection.

5. When constructing a new projection of a line situated in different quadrants the positive coordinates of points should be laid off on one side of the new ground line and the negative coordinates on the other.

A line AB which contains point B in the fourth quadrant is given in the drawing (Fig. 203). When constructing a new projection of this line in the Π_1-Π_4 system of planes, since plane Π_4 is perpendicular to plane Π_1, the negative value of z_B for point B is laid off upwards from the ground line s_{14}, whereas the positive coordinate z_A of point A is laid off downwards from it.

The projection A_4B_4 gives the true value of the line AB, since line s_{14} is drawn parallel to the unchanged projection A_1B_1, so plane Π_4 is also located in space parallel to line AB.

6. The transformation of an oblique plane α into a projecting plane by the method of substituting a plane of projection is shown in Fig. 204.

The ground line s_{14} of the new system of planes of projection Π_1-Π_4 is drawn perpendicular to trace k_1 of plane α (see § 16). The point of intersection of trace k_1 with line s_{14} is a new point of convergence of traces S_α of plane α.

In order to find the trace-projection α_4, which possesses collecting properties since the planes α and Π_4 are mutually perpendicular, on trace l_2 any convenient point K is taken and projected onto plane Π_4. Then points S_α and K_4 are joined by a straight line. The lines k_1 and α_4 are traces of plane α in the system of planes of projection Π_1-Π_4.

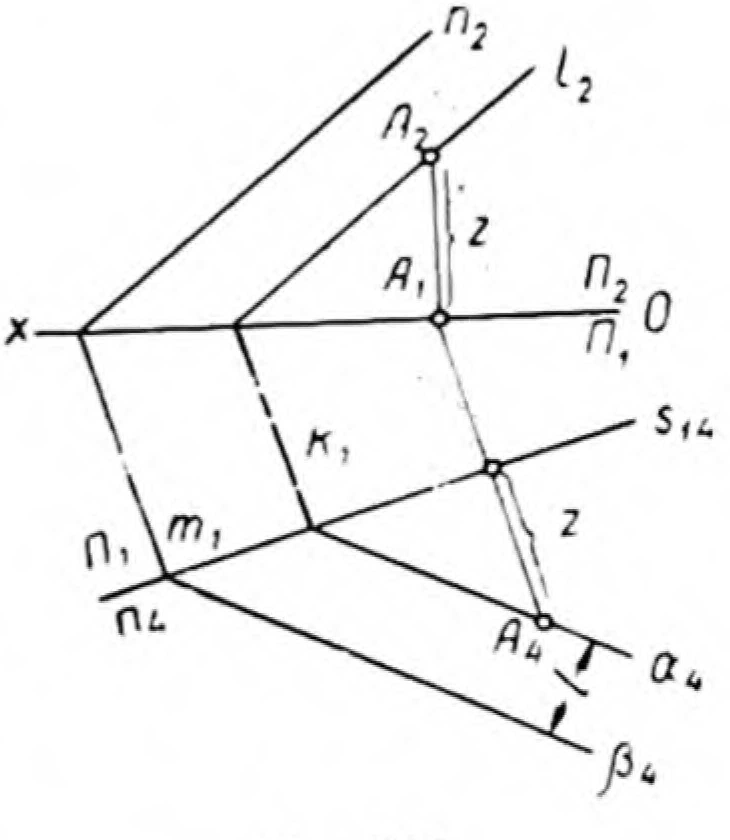


Fig. 205

If the oblique plane α were given not by traces, but by some other means, for example, by a triangle, then its transformation into a projecting plane could be carried out with the help of horizontals or frontals of plane α. The ground line of the new system of planes of projection in this case is drawn perpendicular either to the horizontal projection of the horizontal or to the frontal projection of the frontal.

7. The transformation of an oblique plane into a projecting plane simplifies the solution of many problems. Some examples are given below.

Example 1. It is required to determine the distance between the parallel planes α and β (Fig. 205).

Let us introduce a new plane of projection Π_4 perpendicular to planes Π_1, α and β. The ground line s_{14} of the Π_1-Π_4 system in the multiview drawing should be perpendicular to the traces k_1 and m_1. Having constructed the trace α_4 of plane α with the help of point A, draw the trace β_4 parallel to it. The distance between the trace projections α_4 and β_4 is the true distance l between the planes α and β.

Example 2. It is required to determine the distance of point F from the plane of a triangle ABC (Fig. 206).

Let us transform the oblique plane given by triangle ABC into a projecting plane by the method of substituting a plane of projection (Fig. 206. *a*). In order to do this in plane ABC construct the horizontal CD and draw the new plane of projection Π_4 perpendicular to it. The ground line s_{14} will be perpendicular to the projection C_1D_1 of the horizontal CD. Having constructed the projections A_4, B_4 and C_4 of the vertices of the triangle, we see that these points lie on straight line $A_4B_4C_4$, which is the trace-projection of the plane of triangle ABC on Π_4, i. e., the trace of a plane which has collecting properties.

Having found the projection F_4 of point F on the plane Π_4, from it draw a perpendicular to the trace-projection $A_4B_4C_4$ of plane ABC to obtain point K_4 (Fig. 206, *b*). The length of F_4K_4 is the true distance of point F from plane ABC.

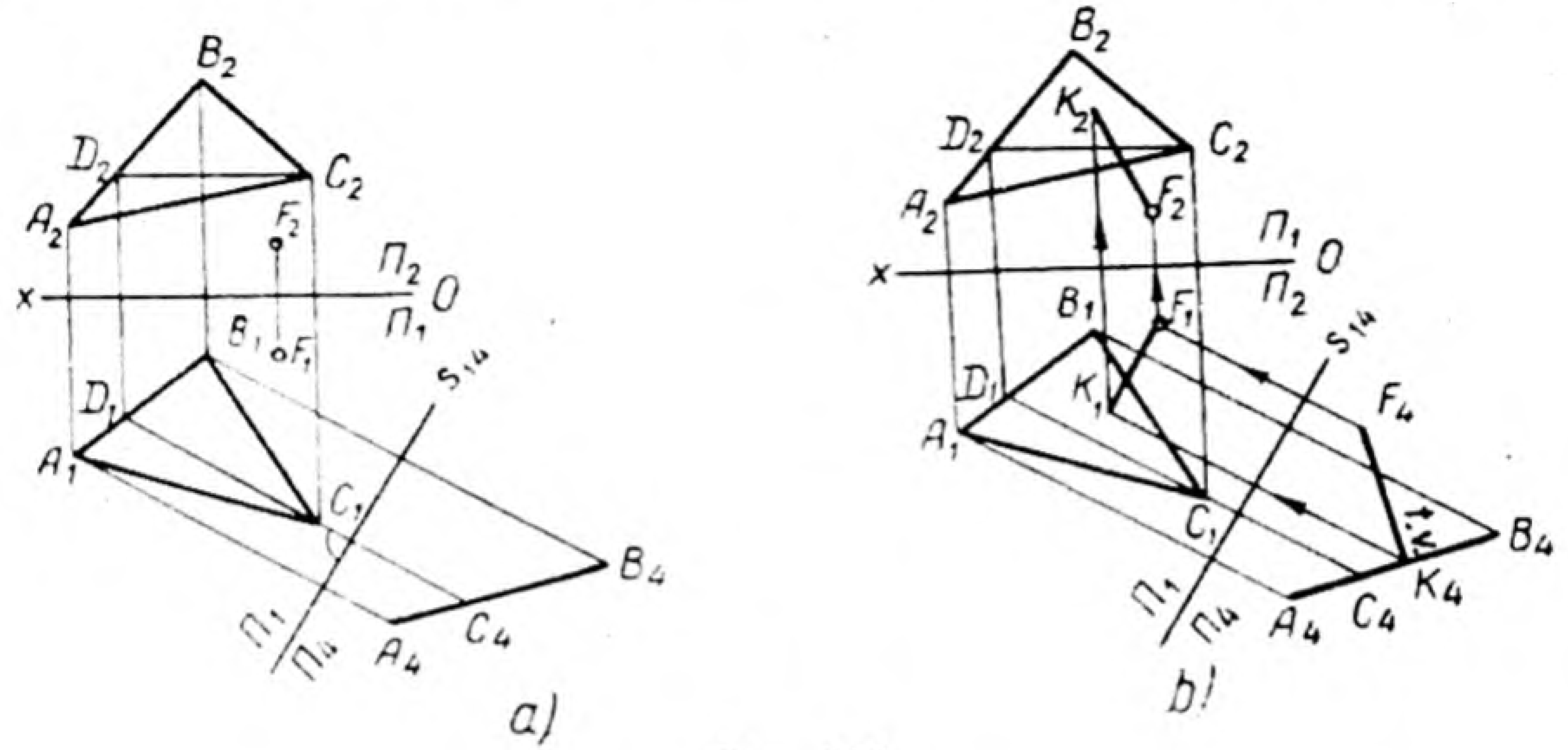


Fig. 206

When determining the projection of line FK on plane Π_1 and Π_2 it should be remembered that its horizontal projection F_1K_1 must be parallel to ground line s_{14} since line FK is parallel to plane Π_4. In other words, F_1K_1 must be perpendicular to projection C_1D_1 of the horizontal of the plane of triangle ABC, since FK is perpendicular to plane ABC (see § 21).

The frontal projection K_2 of point K is found on the perpendicular projector drawn from K_1. Point K_2 lies from Ox at a distance equal to the distance of point K_4 from line s_{14}.

When a plane is not given by a triangle but by its traces, the procedure remains unchanged. The ground line of the new system of planes of projection is drawn at right angles to one of the traces of the plane.

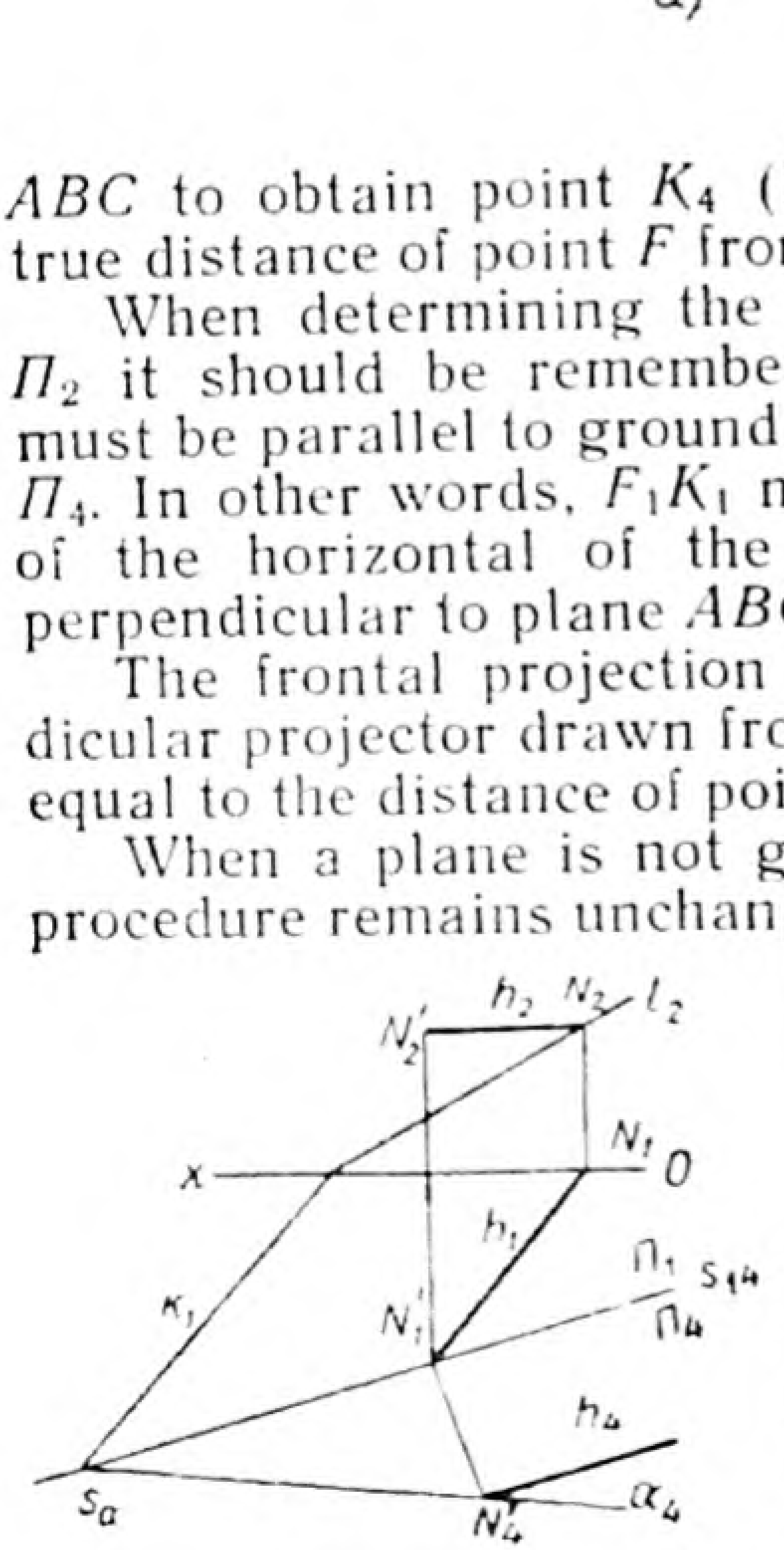
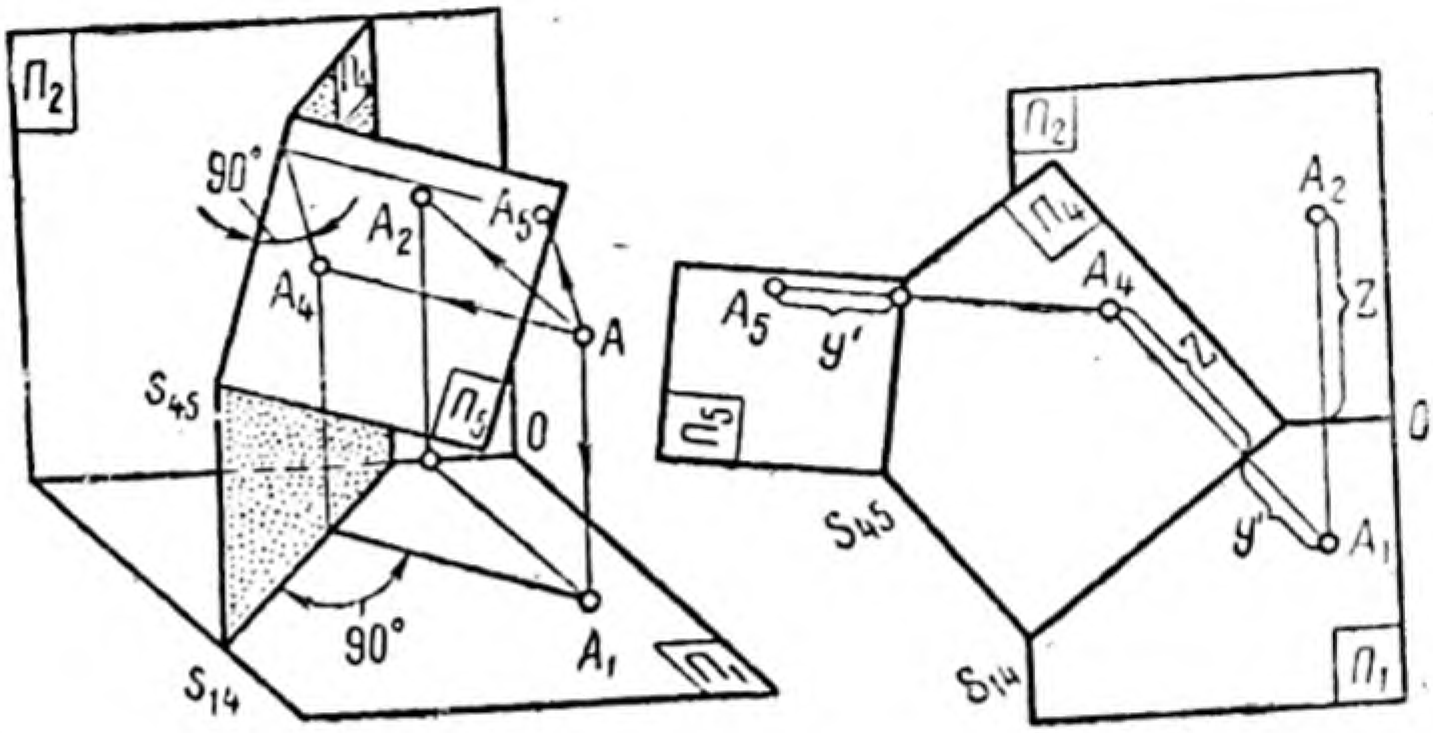


Fig. 207

8. The transformation of an oblique plane, which in the new system of planes Π_1-Π_4 continues to be an oblique plane, is seen in Fig. 207. The ground line s_{14} of the new system is inclined to trace k_1. The point of convergence of the traces of plane α in the system of planes Π_1-Π_4 is point S_α. This is the point of intersec-

tion of the horizontal trace k_1 with the ground line s_{14}. To find the second point through which trace α_4 of plane α will pass, in the plane α draw a convenient horizontal h. Construct its projection h_4 on plane Π_4 ($N'_4N'_1=N_1N_2$). This gives trace N'_4 of this horizontal in plane Π_4. The trace of plane α in plane Π_4 is the straight line joining points S_α and N'_4.



a)

b)

Fig. 208

9. The transformation of projections by substituting two or more planes of projection requires constructions similar to those used when substituting one plane of projection.

The substitution of the principal planes of projection Π_1 and Π_2 by the additional planes Π_4 and Π_5 is shown in Fig. 208. The plane Π_4 is perpendicular to plane Π_1, whereas plane Π_5 is perpendicular to Π_4. For the first transformation plane Π_2 is substituted by plane Π_4. Thus a new projection A_4 of the point A is obtained. The distance of projection A_4 from ground line s_{14} is equal to the distance of the substituted projection A_2 from ground line Ox, i. e., equal to the value z for point A. In the second conversion plane Π_1 is substituted by plane Π_5. In plane Π_5 we obtain the second new projection A_5 of point A. The distance of projection A_5 from the ground line s_{45} of the second system of planes of projection Π_4-Π_5 is equal to the distance y' of the substituted horizontal projection A_1 from the ground line s_{14} of the Π_1-Π_4 system of planes of projection.

The multiview drawing is then constructed as follows. Draw line s_{14} of the first new system of planes of projection Π_1-Π_4. Then, having drawn a perpendicular from A_1 to this line, lay off on it from line s_{14} the value z for point A. Point A_4 is then the projection of point A on plane Π_4. After that draw line s_{45} of the second new system of planes of projection Π_4-Π_5 and, on the perpendicular drawn from A_4 to line s_{45}, lay off from this new ground line distance

y', so finding point A_5. The points A_4 and A_5 are the new projections of point A on planes Π_4 and Π_5.

It should be noted that, when the second substitution of a plane of projection was carried out, the dimension y', i. e., the distance of projection A_5 to line s_{45} was taken as equal to the distance from projection A_1 to line s_{14} of the intermediate system of planes of projection Π_1-Π_4. The distance y' is not equal to the distance of projection A_1 from line Ox of the principal system of planes of projection Π_1-Π_2. The distance y' in plane Π_5 is the length of the fourth line of the four lines situated between the ground lines and projections A_5, A_4 and A_1.

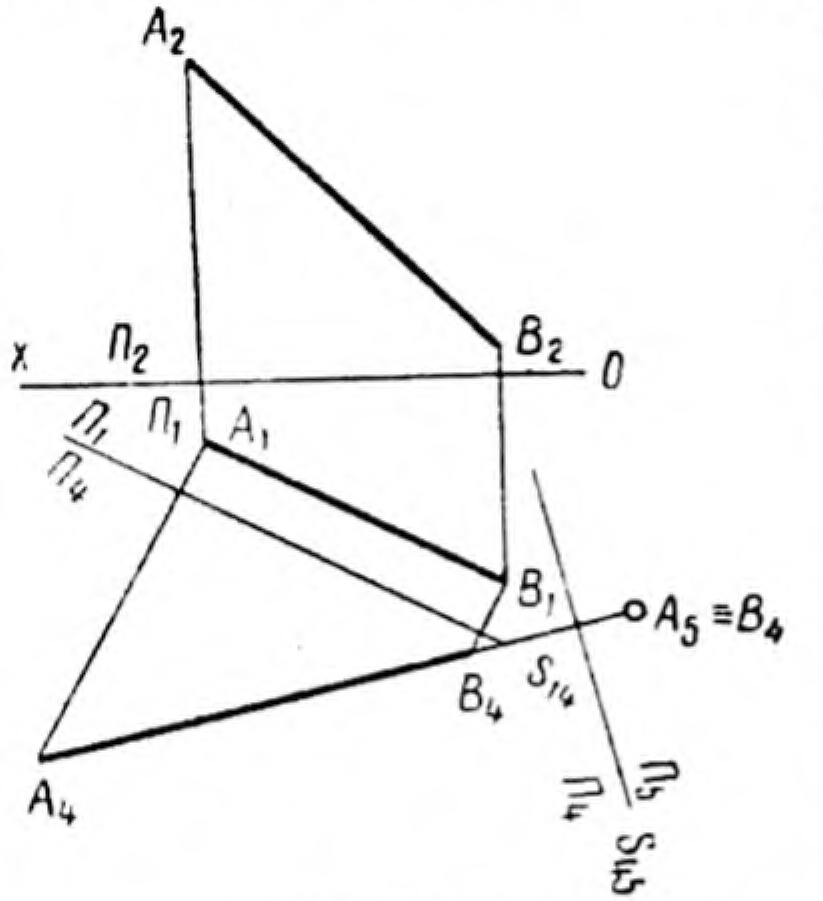


Fig. 209

These lines are the projections of four projecting lines in the main and new systems of planes of projection.

When constructing other projection systems by the method of substituting planes of projection, each new projection of a point will occupy the fourth place, counting from the projection being substituted. For example, in Fig. 218 point D_4 which substitutes the horizontal projection D_1 in the system Π_2-Π_4, D_4 lies at the end of the fourth line, counting 1) from point D_1 to ground line Ox, 2) from this line to point D_2, 3) from point D_2 to axis s_{24}, and finally 4) from axis s_{24} to point D_4.

10. An oblique line AB is given in the system of the principal planes of projection Π_1-Π_2 (Fig. 209). It is required to find a new system of planes of projection in which the given line is projected as a point.

A line is projected as a point when it is perpendicular to the plane of projection. Therefore the new plane of projection must be perpendicular to the given line AB. It is not possible in one operation to choose a plane of projection at the same time perpendicular both to line AB and to one of the original planes of projection, since AB is an oblique line. An intermediate plane of projection parallel to AB has to be introduced as the first step. The line AB is then projected onto this plane.

The second plane of projection to be introduced must be perpendicular to both line AB and the intermediate plane of projection.

In Fig. 209 the intermediate plane of projection Π_4 is perpendicular to plane Π_1 and parallel to line AB. The line AB is projected onto Π_4 in its true length, i. e., the length of projection

$A_4B_4 = AB$. The second plane of projection Π_5 introduced is perpendicular to AB and to plane Π_4. So, on the drawing the ground line s_{45} is perpendicular to projection A_4B_4. By extending line A_4B_4 to cut line s_{45} and laying off from line s_{45} a distance equal to the distance at which both points A_1 and B_1 lie from the intermediate ground line s_{14}, we obtain a point $A_5 \equiv B_5$ which is the projection of line AB on plane Π_5.

This construction is widely used in transforming projections by the method of substituting planes of projection.

11. Examples of the constructions used in descriptive geometry for determining true dimensions by substituting planes of projection are given below.

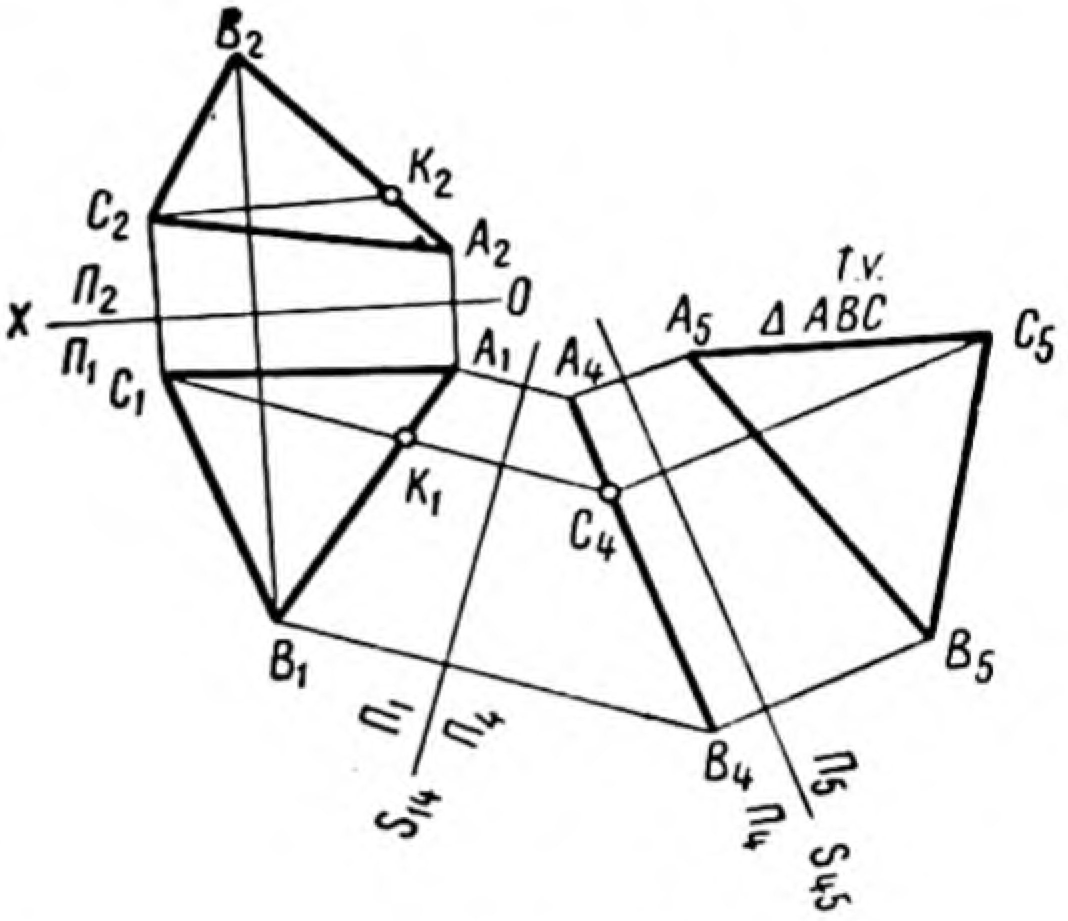


Fig. 210

Example 1. Find the true size of triangle ABC (Fig. 210). First project triangle ABC onto plane Π_4 perpendicular to plane Π_1 and to the plane of the triangle. To do this in the plane of triangle ABC draw the horizontal KC. The ground line s_{14} of the intermediate system of planes of projection must be taken as perpendicular to the projection K_1C_1 of the horizontal. The triangle will be projected onto plane Π_4 as line $A_4C_4B_4$.

Let us now introduce the second new plane of projection Π_5 parallel to the plane of triangle ABC and perpendicular to plane Π_4. Line s_{45} must be drawn parallel to the projection $A_4C_4B_4$. Having measured the distances at which the horizontal projections A_1, B_1 and C_1 of the vertices of the triangle lie from line s_{14}, lay off these distances on the extensions of the perpendiculars to ground line s_{45} drawn respectively through points A_4, B_4 and C_4. Having found points A_5, B_5 and C_5 join them by straight lines. The triangle $A_5B_5C_5$ constructed is identical with the original triangle ABC. In other words, a true view is obtained.

Example 2. Find the shortest distance between the nonintersecting lines AB and CD (Fig. 211.).

Here a double substitution of planes of projection is used. The first new plane Π_4 is drawn parallel to one of the lines, say, to AB. Then the line AB is projected onto this plane in its true length ($A_4B_4 = AB$). When line CD is projected onto plane Π_4 its length is distorted, since in the Π_1-Π_4 system of planes of projection it remains an oblique line.

The projections A_4, B_4, C_4 and D_4 of points A, B, C and D on plane Π_4 are obtained by taking the values of z for these points from the principal system of planes of projection Π_1-Π_2.

The second new plane of projection Π_5 is drawn perpendicular to plane Π_4 and to line AB. Then AB will be projected onto plane Π_5 as a point $A_5 \equiv B_5$, and the projection of line CD onto plane Π_5 will be line C_5D_5.

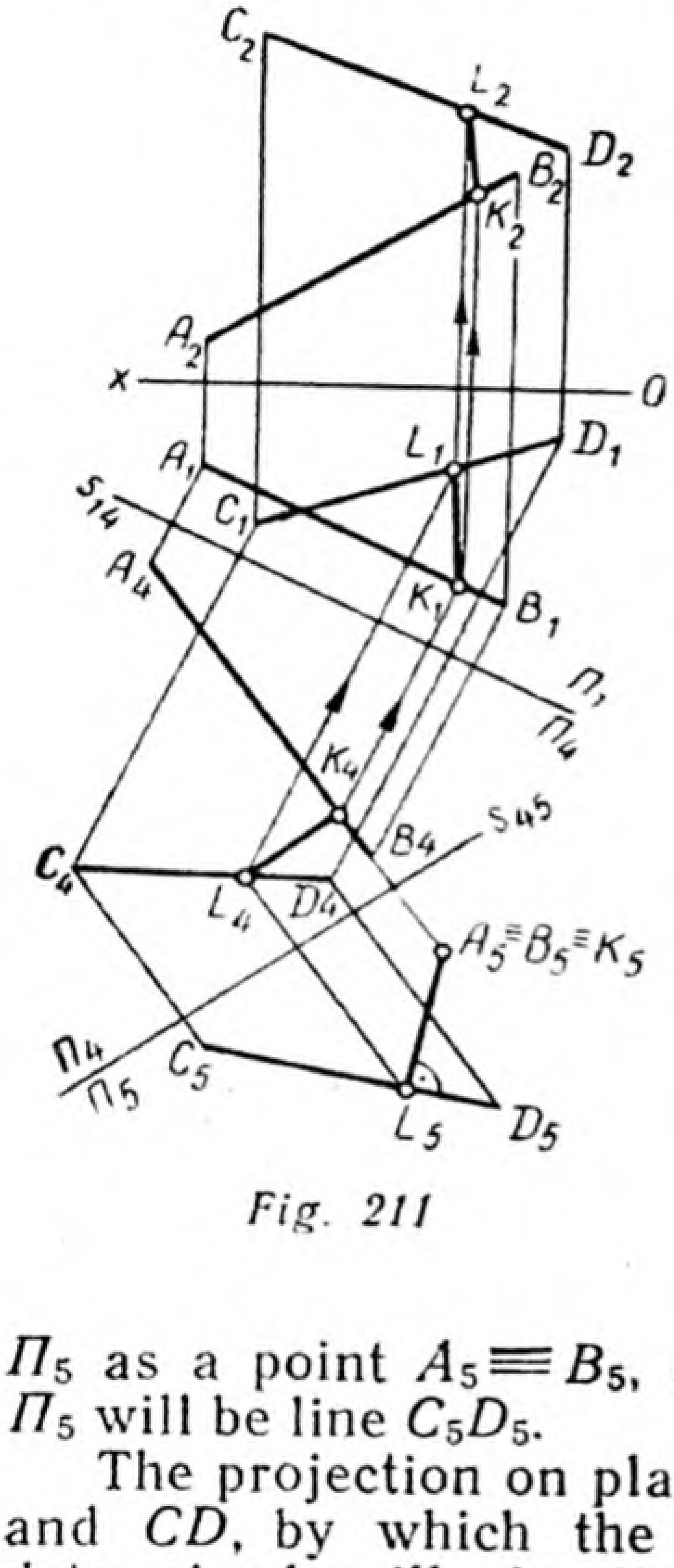

C_2
L_2
D_2
B_2
K_2
A_2
x
0
A_1
L_1
D_1
C_1
S_{14}
A_4
K_1
B_1
Π_1
Π_4
K_4
B_4
S_{45}
C_4
L_4
D_4
$A_5 \equiv B_5 \equiv K_5$
Π_4
Π_5
C_5
L_5
D_5

Fig. 211

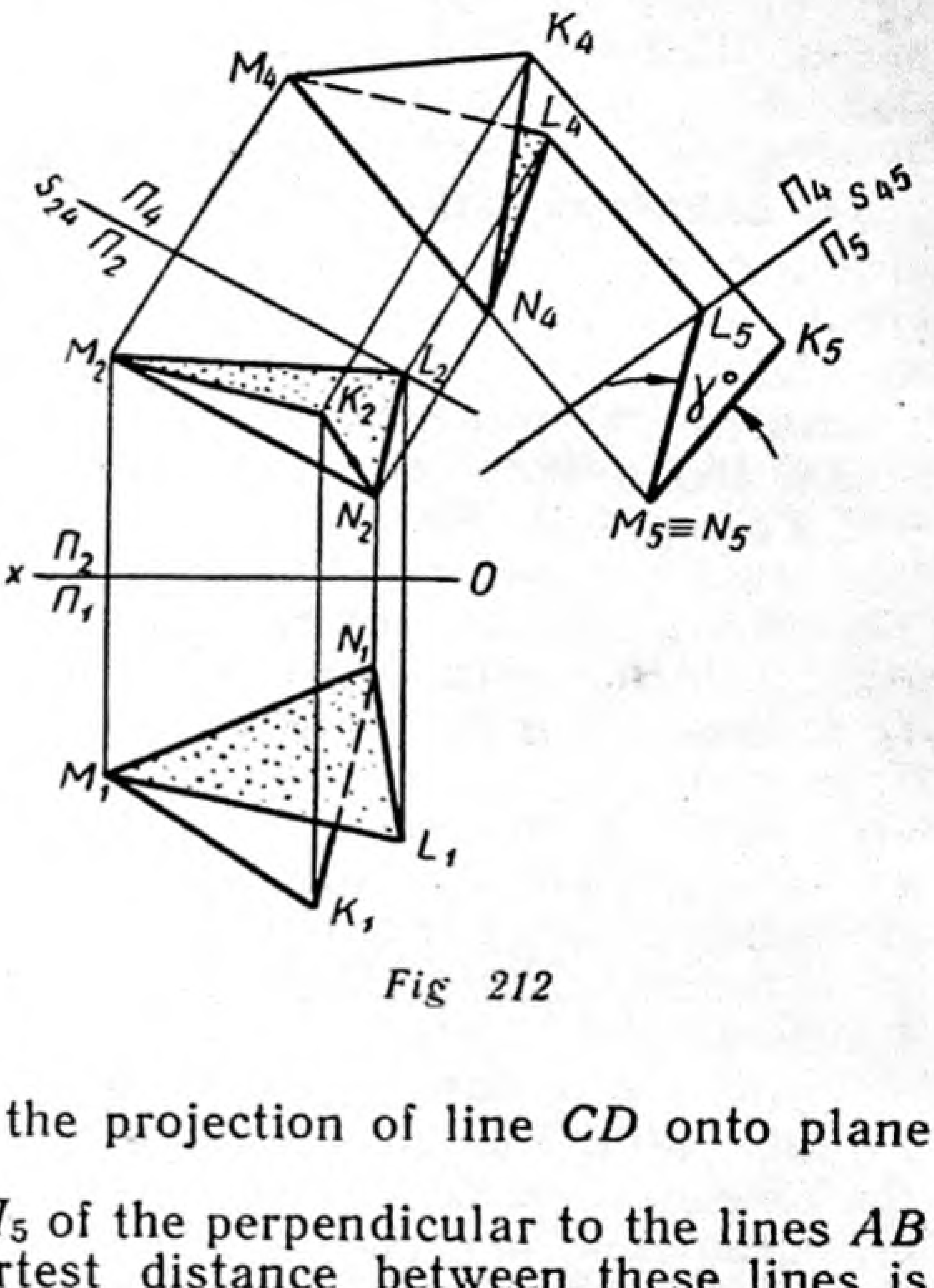

K_4
M_4
L_4
S_{24}
Π_4
Π_2
Π_4 S_{45}
Π_5
N_4
M_2
L_2
K_2
L_5
K_5
$\gamma°$
$M_5 \equiv N_5$
N_2
Π_2
Π_1
x
0
N_1
M_1
L_1
K_1

Fig 212

The projection on plane Π_5 of the perpendicular to the lines AB and CD, by which the shortest distance between these lines is determined, will give its true length, since this line is parallel to the plane Π_5 and line AB is perpendicular to Π_5. It follows that the right angle between this perpendicular and line when projecting CD onto plane Π_4 will also be seen as a right angle (see § 13).

Therefore having drawn from point $A_5 \equiv B_5$ a perpendicular to C_5D_5 we find the segment K_5L_5, giving the shortest distance between lines AB and CD. To determine the projection of line KL in the system of the Π_1-Π_2 planes of projection first find point L_4. Then find point L_1, and finally point L_2 by means of the respective projectors. The point K_4 will be found if a perpendicular is drawn from L_4 to A_4B_4. The right angle between K_4L_4 and A_4B_4 will not be distorted when projecting onto Π_4 since line AB is parallel to Π_4.

The projections K_1 and K_2 of point K are also found with the help of projectors.

Example 3. It is required to determine the true value of the angle formed between the planes of the triangles KMN and LMN (Fig. 212).

The true size of the dihedral angle is given by the angle formed by the straight line traces of the planes in a plane perpendicular to the edge or line of intersection of these given planes or by the orthographic projection of the dihedral angle onto a plane perpendicular to their line of intersection (Fig. 213).

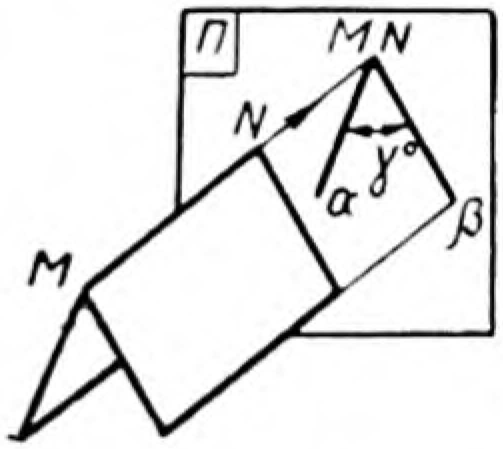


Fig. 213

In the case, shown in Fig. 212, the line MN of intersection of the given planes is an oblique line. So it is necessary to transform the projection of the angle so that the line of intersection is perpendicular to the new plane of projection.

First, project the triangles onto plane Π_4 which is perpendicular to plane Π_2 and parallel to the line of intersection MN. The ground line s_{24} is drawn parallel to the projection M_2N_2 of the line of intersection which remains unchanged. To obtain the projection of the triangles KMN and LMN on plane Π_4 from points M_2, N_2, K_2 and L_2 draw perpendiculars to line s_{24}. On the extension of these perpendiculars lay off from s_{24} the values of y for points M, N, K and L. These distances are the distances of the projections of M_1, N_1, K_1 and L_1 from ground line Ox. The line of intersection MN of the given planes then projects onto Π_4 in its true value ($M_4N_4 = MN$).

Next, the new plane of projection Π_5, perpendicular to plane Π_4 and to MN, is introduced. The ground line s_{45} of the system of planes of projection Π_4-Π_5 is drawn perpendicular to projection M_4N_4. The edge MN is projected onto plane Π_5 as a point $M_5 \equiv N_5$. When constructing projections K_5 and L_5 of points K and L it is borne in mind that the distances of points M_5, N_5, K_5 and L_5 from line s_{45} are equal to the distances of points M_2, N_2, K_2 and L_2 from ground line s_{24}, respectively. Points K_5 and L_5 are joined to point $M_5 \equiv N_5$. The plane angle $K_5M_5L_5$ will be the true dihedral angle formed by the given planes KMN and LMN.

When planes forming a dihedral are given by their traces the procedure for substituting projections in order to obtain the plane dihedral angle remains the same.

Example 4. It is required to determine the angle between line AB and plane α given by the triangle CDE (Fig. 214).

If the line and plane were situated relative to the principal planes of projection Π_1 and Π_2 in such a way that the line was parallel to Π_2 and the plane parallel to Π_1, then the angle γ

between them would be projected without distortion onto one of the principal planes of projection, in the case shown in Fig. 215, onto Π_2.

In the example shown in Fig. 214, the line AB and the given plane are both oblique. The drawing must be transformed so that, in the new system of planes of projection, the line is parallel to one plane of projection and the plane is parallel to the other plane of projection of the new system.

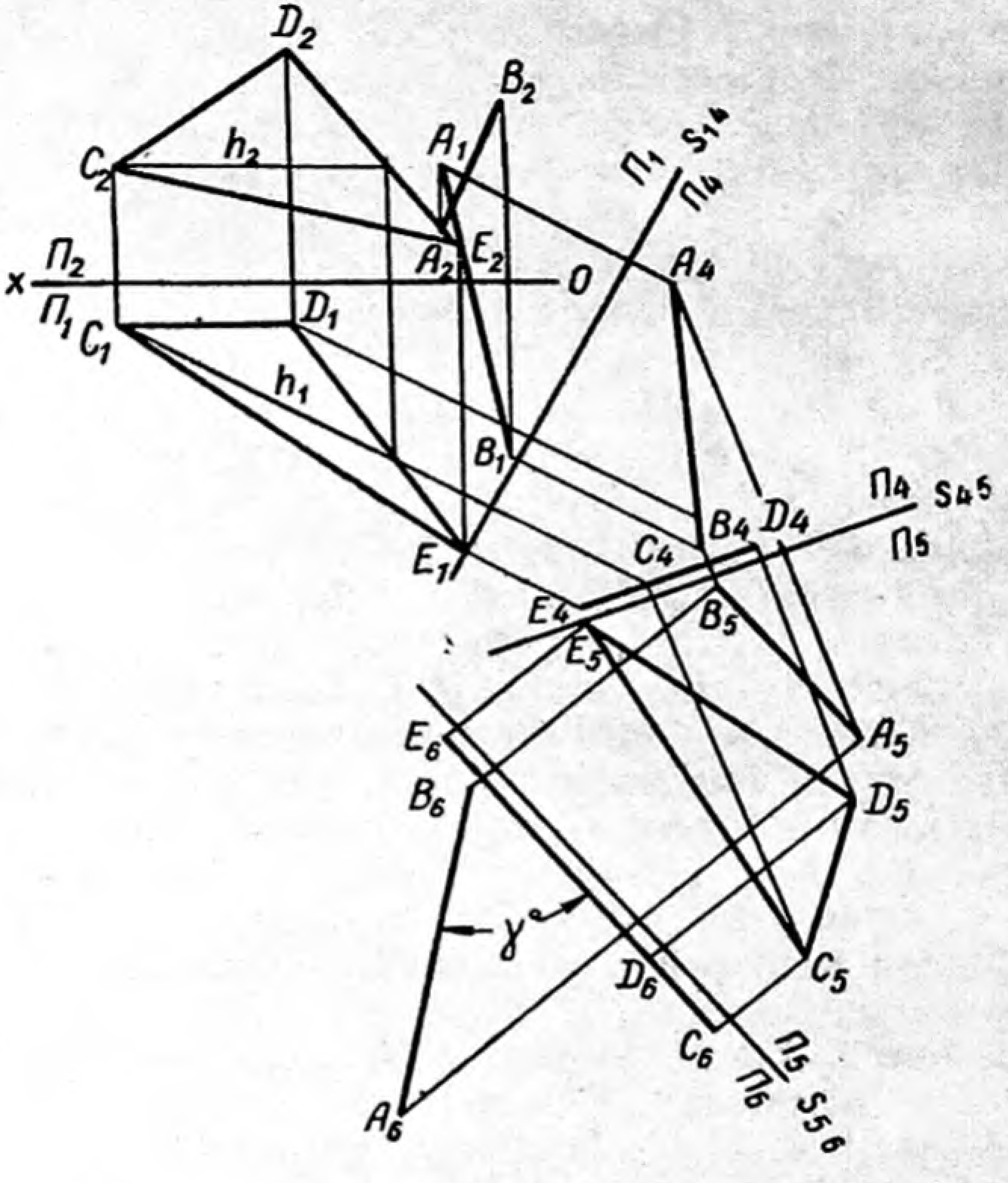

D_2
B_2
A_1
Π_1
S_{14}
Π_4
C_2
h_2
E_2
A_2
A_4
x
Π_2
Π_1
C_1
D_1
0
h_1
B_1
E_1
Π_4
S_{45}
Π_5
B_4
D_4
C_4
E_4
B_5
E_5
E_6
A_5
B_6
D_5
γ
D_6
C_5
C_6
Π_5
Π_6
S_{56}
A_6

Fig. 214

The first new plane Π_4 is drawn perpendicular to both plane Π_1 and plane α. The triangle CDE will then be projected onto plane Π_4 as line $C_4D_4E_4$, and the line AB as line A_4B_4.

The second new plane of projection, Π_5, should be perpendicular to Π_4 and parallel to plane α. Then the plane of triangle CDE will be projected onto Π_5 as a true value ($C_5D_5E_5=CDE$), and the line AB as A_5B_5.

Finally, the third new plane Π_6 is drawn perpendicular to plane Π_5 and parallel to line AB. The projection of line AB on plane Π_6 will be a true value ($A_6B_6=AB$). The projection of triangle CDE

will be line $C_6E_6D_6$ parallel to the ground line s_{56} of the Π_5-Π_6 system of planes of projection.

As a result of three successive substitutions of planes of projection, i. e., Π_2 by Π_4, Π_1 by Π_5, Π_4 by Π_6 we obtain the Π_5-Π_6 system in which the projections of the line and plane are those shown in Fig. 215. The angle γ contained between lines A_6B_6 and $C_6E_6D_6$ is the true size of the angle formed by line AB and the plane of triangle CDE (Fig. 214).

Example 5. It is required to construct the projection of a sphere the surface of which must contain points A, B, D, F, not situated in one plane (Fig. 216).

It can be seen that the given points A, B, D, F in reality are not contained in one plane. The auxiliary line DK, drawn so that its frontal projection D_2K_2 passes through projection F_2 of point F in space. However, this line does not pass through point F, since F_1 does not lie on D_1K_1. It follows that point F is not contained in the plane of triangle ABD.

The centre of the sphere passing through the four given points A, B, D and F may be found as follows (Fig. 217):

1) construct in plane ABD a circle with its centre in point M, passing through points A, B and D; 2) in plane ABF construct a circle with its centre in point N, passing through point A, B and F; 3) through points M and N draw lines respectively perpendicular to planes ABD and ABF. The intersection of these perpendiculars will determine the centre C of the sphere and the distance from point C to any of the given points A, B, D, F will be the radius of the sphere.

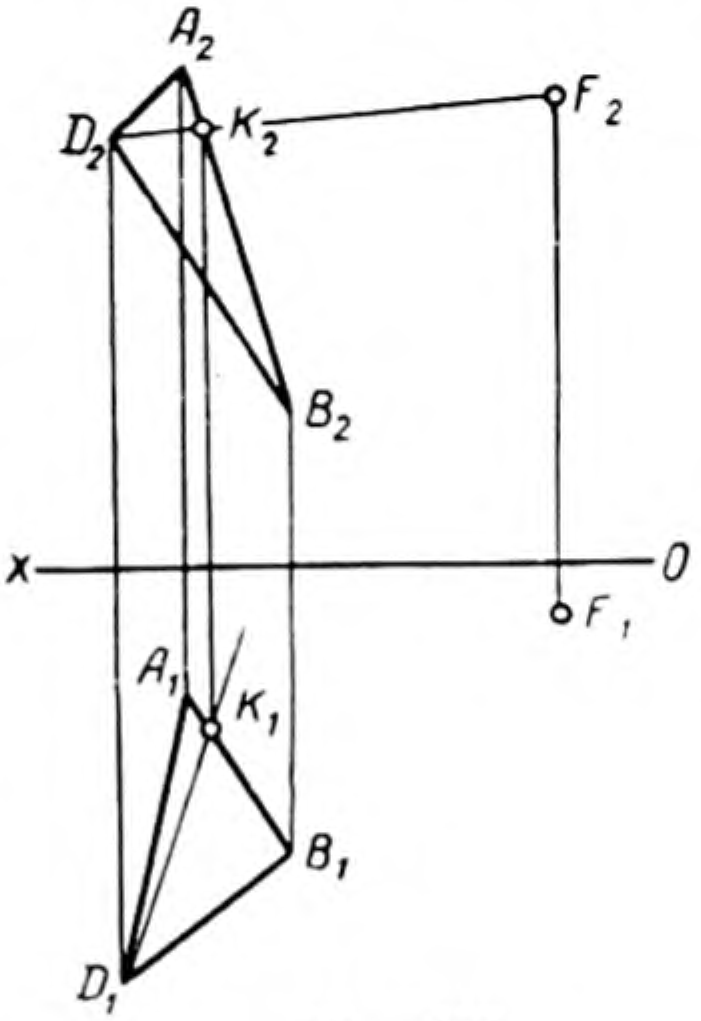


Fig. 216

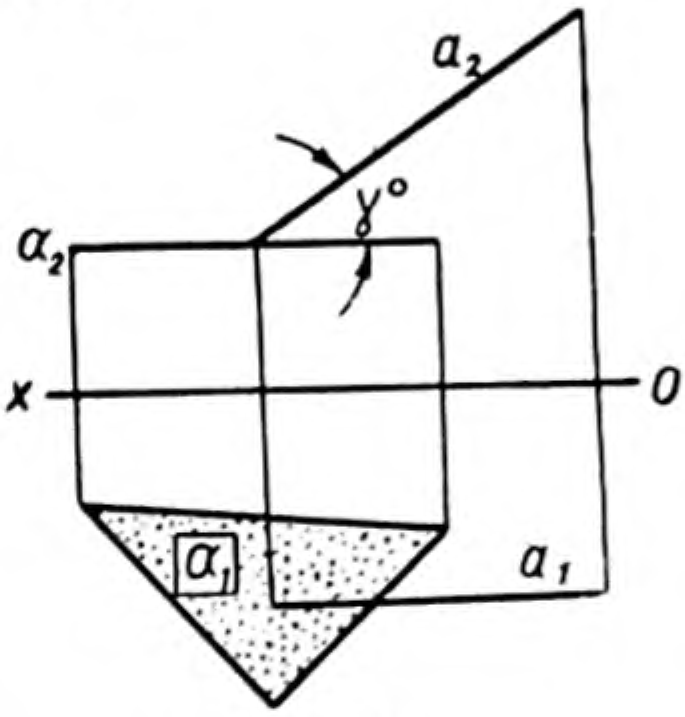


Fig. 215

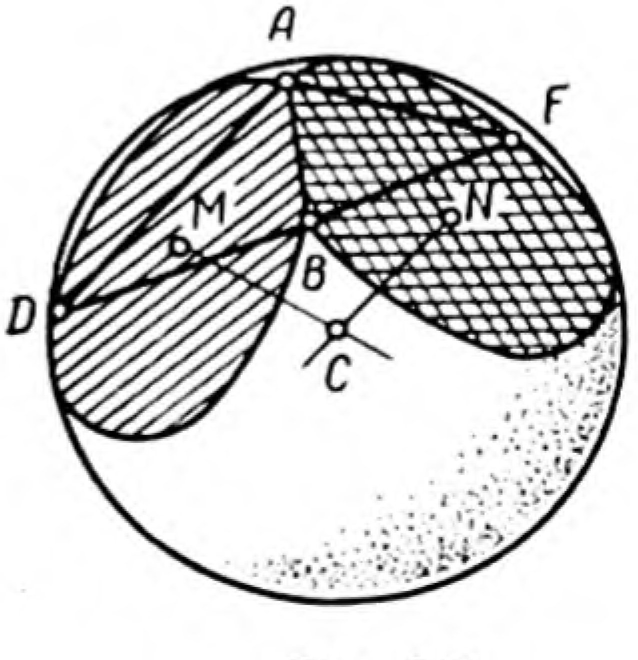


Fig. 217

Let us now construct the multiview drawing (Fig. 218). First, the Π_1-Π_2 system of planes of projection is substituted by the Π_2-Π_4 system in which the new plane of projection Π_4 is parallel to the line AB in which the planes ABD and ABF intersect, i. e., parallel to the edge of the dihedral angle. The ground line s_{24} is therefore drawn parallel to the projection A_2B_2 of this edge. The

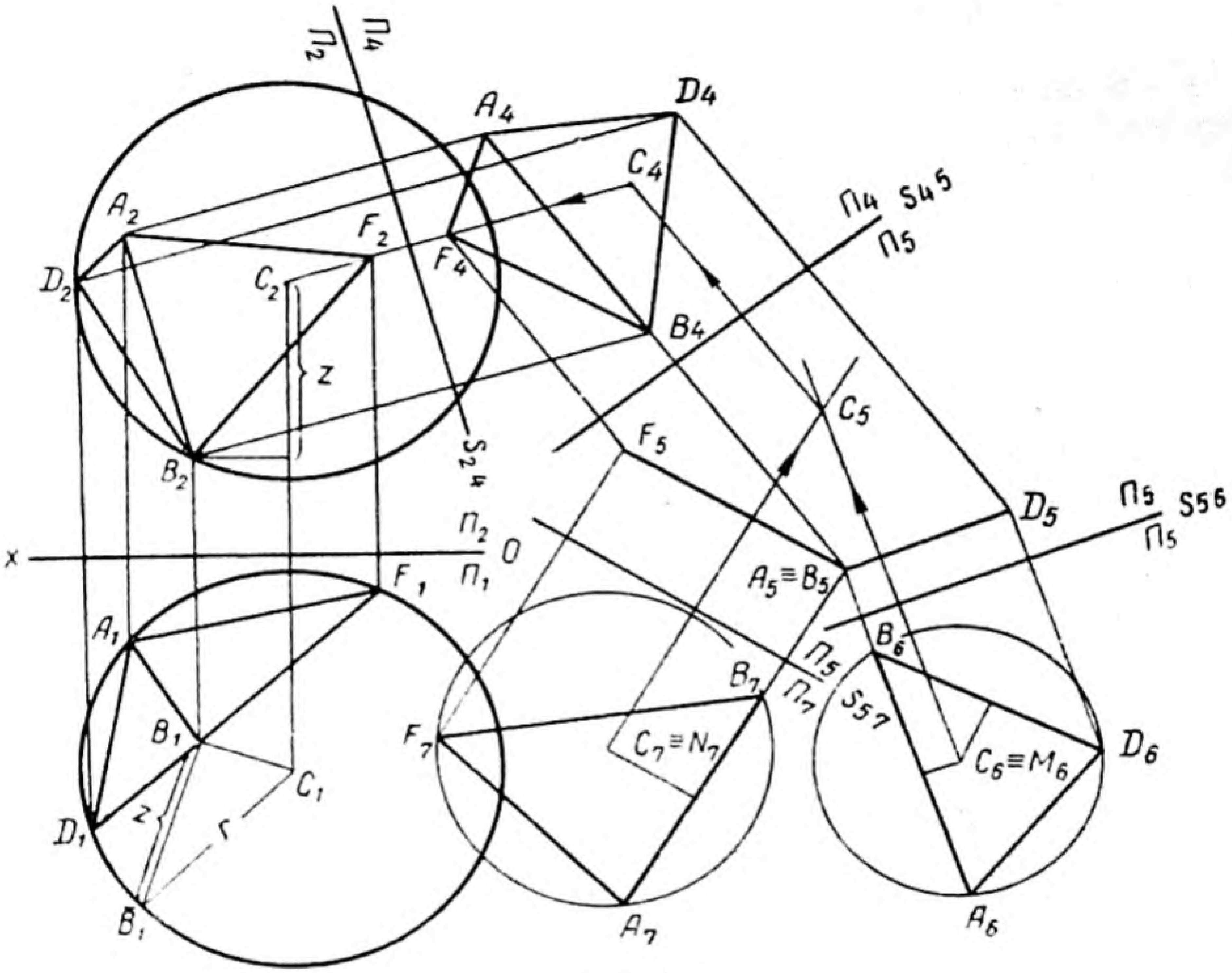


Fig. 218

projection of AB on plane Π_4 is a true value ($A_4B_4=AB$). Having found D_4 and F_4 we next find the projection of the dihedral planes on Π_4.

Further, we project the planes forming the angle onto plane Π_5 in the Π_4-Π_5 system. The plane Π_5 is drawn perpendicular to AB, i. e., the ground line s_{45} is perpendicular to A_4B_4. The edge AB is projected onto plane Π_5 as point $A_5 \equiv B_5$. The projection of triangle ABD is line A_5D_5, and that of triangle ABF line A_5F_5.

Angle $D_5A_5F_5$ gives the true dihedral angle of planes ABD and ABF.

The other two new planes of projection, Π_6 and Π_7, are to be perpendicular to plane Π_5. They are also to be parallel to the planes of triangles ABD and ABF, respectively. So ground line s_{56} is drawn parallel to A_5D_5 and ground line s_{57} parallel to A_5F_5. The triangle ABD is projected onto plane Π_6 in true size ($A_6B_6D_6=ABD$), and triangle ABD is also projected onto plane Π_7 in its true size ($A_7B_7F_7=ABF$).

Next, determine the centres M_6 and N_7 of the circles passing, respectively, through the vertices of the two triangles in these projections. To do this draw perpendiculars to the midpoints of the sides of the triangles. These circles so constructed are the projections of sections of the required sphere when it is cut by the planes of triangles ABD and ABF.

The projection C_5 of the centre C of the sphere on plane Π_5 is found by drawing a perpendicular to ground line s_{56} through point M_6 and a perpendicular to ground line s_{57} through point N_7 to meet in C_5.

By means of reverse projections find C_4, C_2 and C_1 which are the projections of point C on planes Π_4, Π_2 and Π_1. Point C_4 lies at a distance from line s_{45} equal to the distance of projection M_6 from line s_{56}, which is the same as the distance of N_7 from s_{57}. It should be recalled that points M_6 and N_7 coincide with projections C_6 and C_7 of centre C.

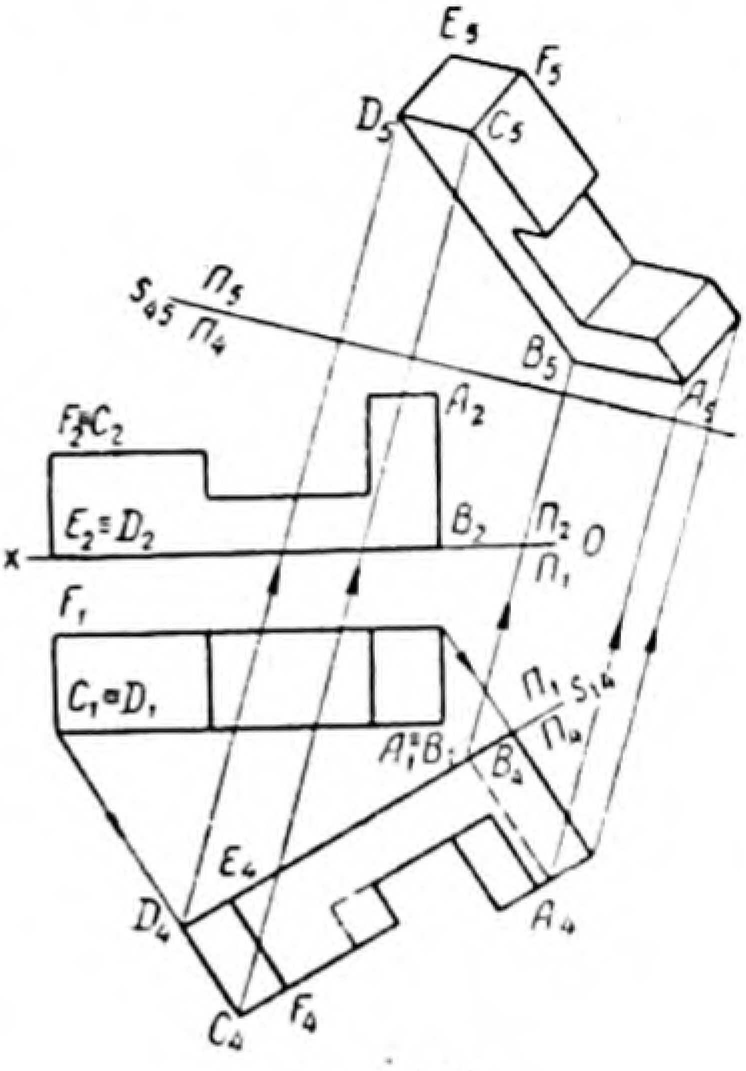


Fig. 219

Having found the principal projections C_1 and C_2 of the centre of the sphere join them by a straight line with the corresponding projections on Π_1 and Π_2 of any of the four given points, for example, with point B, and, by constructing a right triangle $B_1C_1\bar{B}_1$, determine the true length CB, which is equal to the radius r of the sphere ($C_1\bar{B}_1=CB$).

With a radius r describe circles from points C_1 and C_2. These are the horizontal and frontal projections of the sphere, the surface of which passes through the given points A, B, D and F.

Example 6. In Fig. 219 the transformation of the principal projections of a detail is carried out by successively substituting the planes of projection Π_2 and Π_1 by new planes, i. e., by Π_4 perpendicular to Π_1, and Π_5 perpendicular to Π_4. With the help of this construction the transition from an orthographic multiview drawing to the axonometric representation of the detail and its cut-out on plane Π_5 is achieved.

Problems and Exercises

1. Does a change in the distance of a new plane of projection from the projected object have any bearing on the final projection?

2. How should a new plane of projection be placed so that an oblique line is projected onto it in its true length?

3. In what position must a triangle be situated if its true size can be determined by substituting only one plane of projection?

4. What transformations must be carried out in order to project an oblique line onto a new plane of projection as a point?

5. How must a new plane of projection be placed in order that a plane figure is projected onto it as a straight line?

6. When can the true value of a dihedral angle be determined without distortion by substituting only one of the planes of projection?

7. When can the shortest distance between two non-intersecting lines be determined by introducing only one new plane of projection?

8. Determine the true length of an oblique line *AB* and its angles of inclination to planes Π_1 and Π_2 (Fig. 220).

9. Draw a perpendicular from a point *O* to the plane of triangle *ABC* (Fig. 221).

10. Determine the distance between the parallel planes α and β (Fig. 222).

11. Determine the distance from point *E* to the plane given by parallel lines *AB* and *CD* (Fig. 223).

12. Determine the true dimension of the quadrangle *ABCD* (Fig. 224).

13. Determine angles α and β formed by plane α with planes of projection Π_1 and Π_2 (Fig. 222).

Hint. Angle α may be found in the system Π_1-Π_4 if the new ground line s_{14} is drawn perpendicular to the traces k_1 and m_1. Angle β is found in the Π_2-Π_5 system with ground line s_{24} perpendicular to traces l_2 and n_2.

14. From point *K* draw a perpendicular to the line of intersection of the plane of triangle *ABC* with plane α given by traces *k* and *l* (Fig. 225).

15. Determine the angle formed by the plane of triangle *ABC* with the plane α (Fig. 225).

16. Draw a plane at the same distance from both the non-intersecting lines *AB* and *CD* (Fig. 226).

Hint. The construction is carried out as follows. Through point *B* draw a line parallel to *CD*. Then draw a level line in the plane of parallelism obtained. This level line enables us to find the new system of planes of projection in which the projection of the plane of parallelism is a straight line. The required plane will be constructed in this same system of planes of projection.

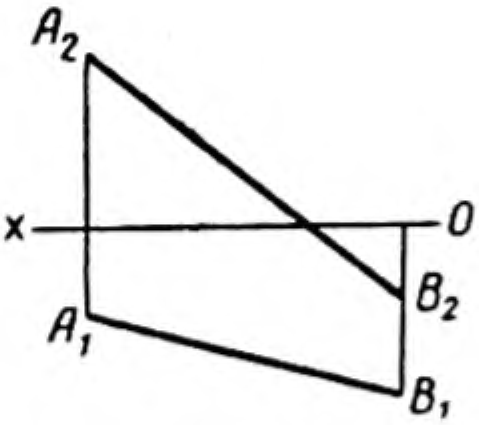

A2
x
0
B2
A1
B1

Fig. 220

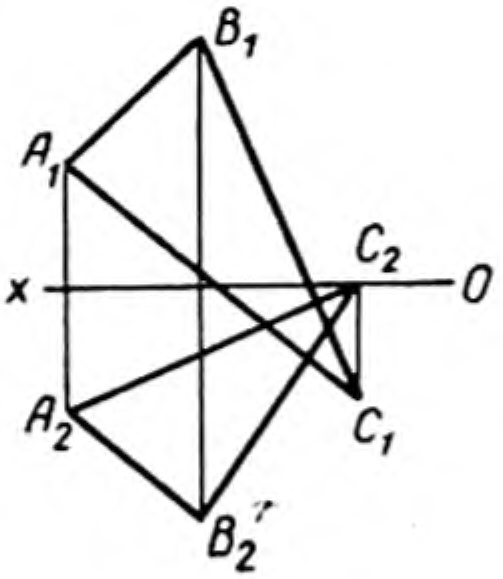

B1
A1
C2
x
0
A2
C1
B2

Fig. 221

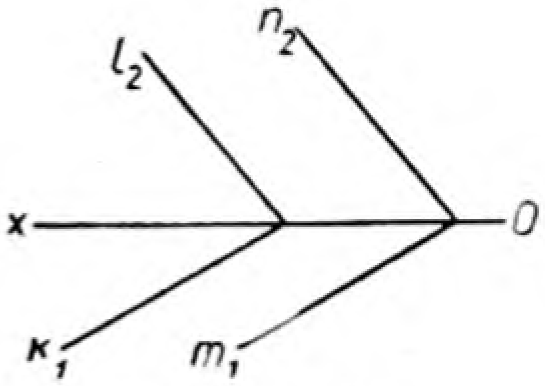

l2
n2
x
0
k1
m1

Fig. 222

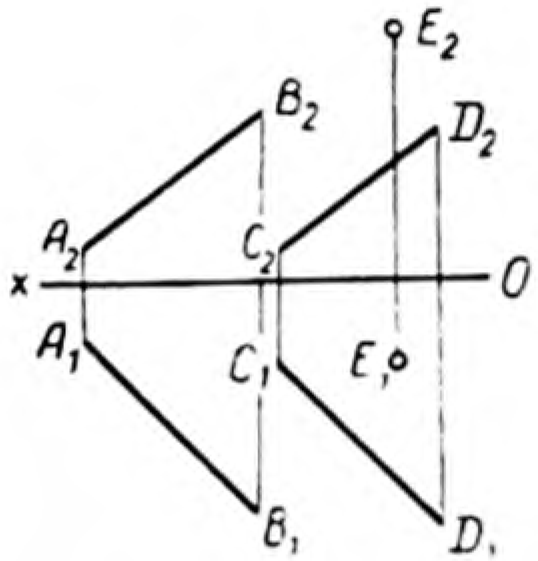

E2
B2
D2
A2
C2
x
0
A1
C1
E1
B1
D1

Fig. 223

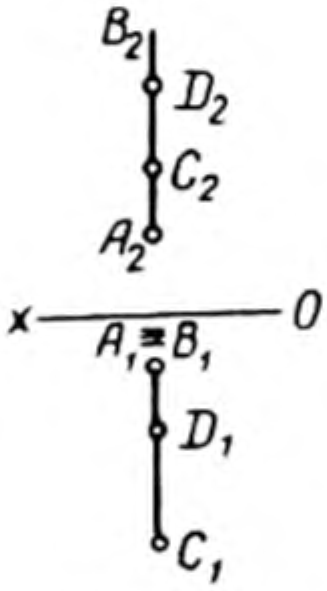

B2
D2
C2
A2
x
0
A1≡B1
D1
C1

Fig. 224

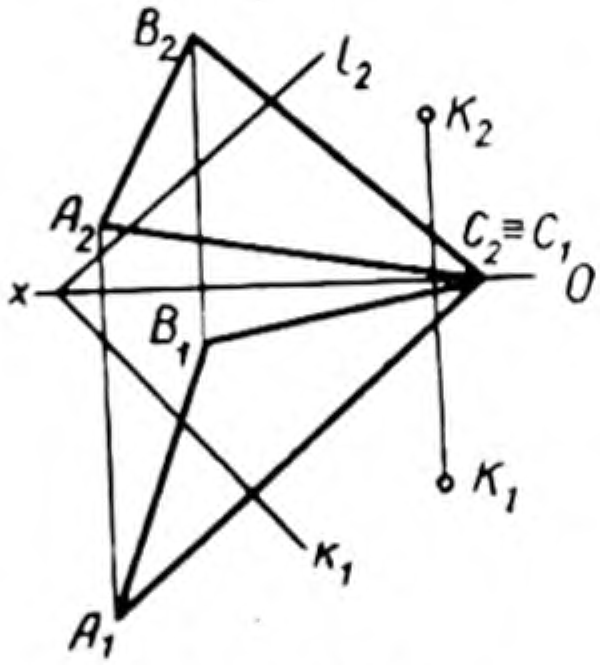

B2
l2
K2
A2
C2≡C1
x
0
B1
K1
k1
A1

Fig. 225

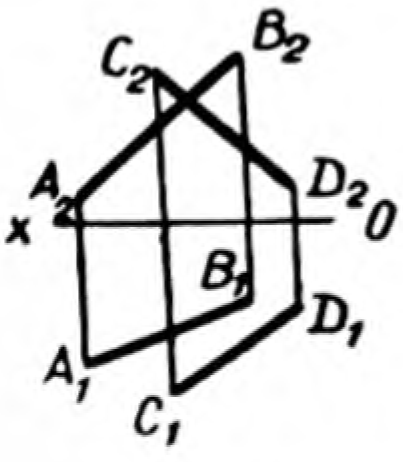

C2
B2
A2
D2
x
0
B1
D1
A1
C1

Fig. 226

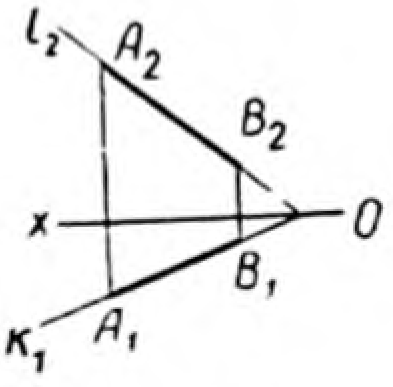

l2
A2
B2
x
0
B1
k1
A1

Fig. 227

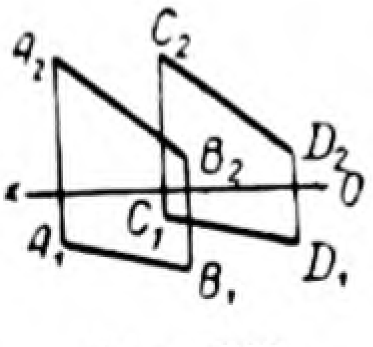

A2
C2
B2
D2
x
0
C1
A1
B1
D1

Fig. 228

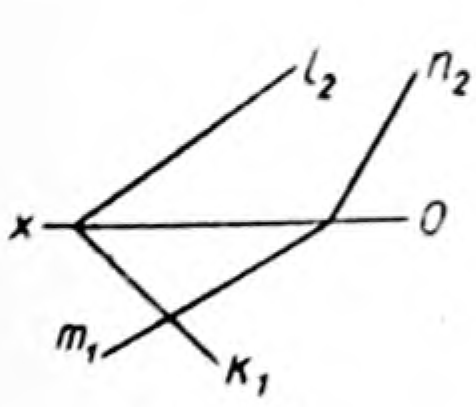

l2
n2
x
0
m1
k1

Fig. 229

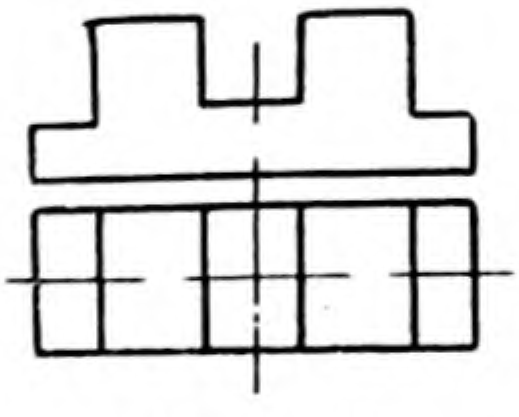


Fig. 230

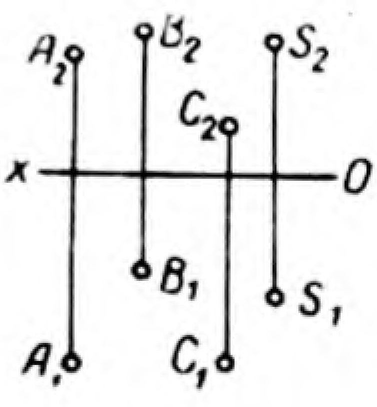

A2
B2
S2
C2
x
0
B1
S1
A1
C1

Fig. 231

17. Determine the angle between line *AB* and plane α given by the traces k and l (Fig. 227).

Hint. This problem may be solved as in example 4, § 24, (see Fig. 214).

18. Determine the distance between parallel lines *AB* and *CD* (Fig. 228).

Hint. Both line segments should first be projected onto a new plane in their true length. Then, by drawing a second supplementary plane perpendicular to the lines received, project them as points. The distance between these points is the required distance.

19. Construct a bisecting plane between planes α and β (Fig. 229).

Hint. This problem can be solved by changing two planes of projection (see Fig. 212 and related text). First, the line of intersection of the given planes should be determined.

20. Construct new projections of the three-dimensional detail (Fig. 230) on planes $\Pi_4 \perp \Pi_2$ and $\Pi_5 \perp \Pi_4$.

21. Four points not lying in one plane are shown in Fig. 231. Construct a cone with one of the points as the vertex. The other three points should lie on the lateral surface of the cone.

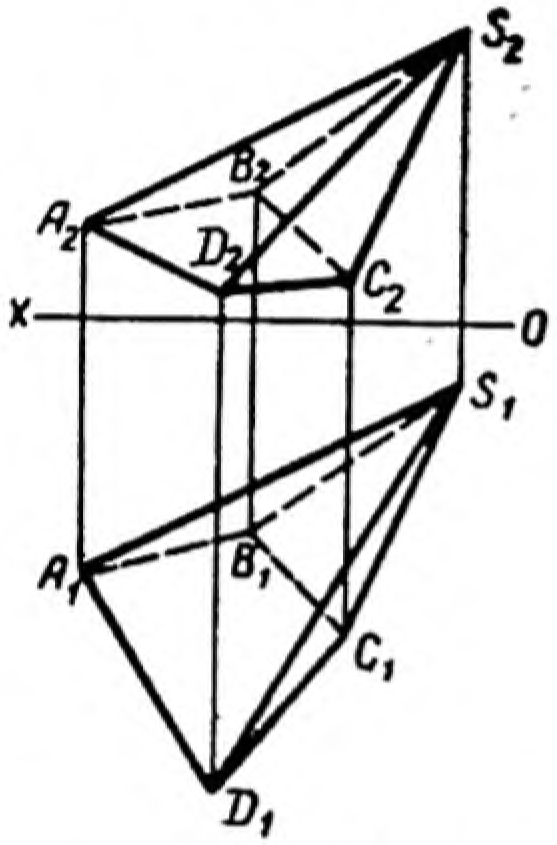


Fig. 232

22. Given a pyramid *SABCD* (Fig. 232). Determine: a) the angle between the base *ABCD* and face *SAB*, b) the altitude of the pyramid, c) the shortest distance between edges *AD* and *CB*.

23. Draw a circle through the vertices of any triangle *ABC*, the plane of which is oblique.

24. On ground line *Ox* find a point equidistant from traces *M* and *N* of a given line *AB*. (See Figs. 209, 211 and 212 and related text.)

§ 25. Method of Revolution

In accordance with this method the principal system of planes of projection Π_1-Π_2 remains unchanged while the lines and plane figures to be projected are rotated about corresponding axes to positions in which their projections in the multiview drawings become the true views of the lines, figures, etc.

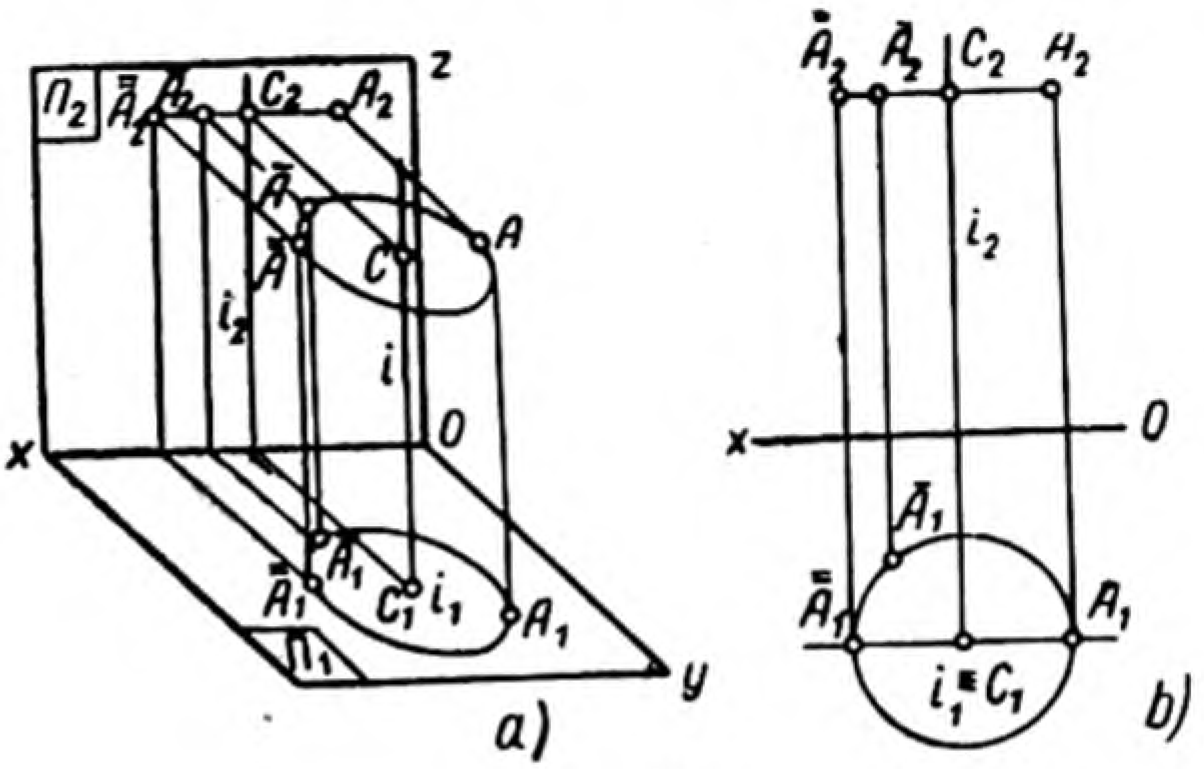
The axes of revolution are straight lines perpendicular to a plane of projection, parallel to it, or contained in a plane of projection.

Oblique lines are generally not used as axes of rotation since they complicate the construction.

If it becomes necessary in the process of construction to revolve any geometrical element about an oblique axis, the axis is first brought to some suitable position, after which the necessary rotation is carried out.

§ 26. Revolution about Axes Perpendicular to Planes of Projection

1. Let there be given point A and the axis of revolution i perpendicular to the horizontal plane of projection Π_1 (Fig. 233, *a*). When rotated about axis i point A will describe a circle in a plane perpendicular to axis i. The radius of this circle is equal to the distance of point A from axis i. The centre of the circle of revolution is therefore contained in the axis of rotation i. The points $\overline{A}$, $\overline{\overline{A}}$ are the various positions of point A on the circumference as it is rotated.



Π_2
$\overline{\overline{A}}_2$
$\overline{A}_2$
C_2
A_2
z
$\overline{\overline{A}}$
$\overline{A}$
C
A
i_2
i
x
0
$\overline{A}_1$
$\overline{\overline{A}}_1$
C_1
i_1
A_1
Π_1
y
a)
$\overline{\overline{A}}_2$
$\overline{A}_2$
C_2
A_2
i_2
x
0
$\overline{A}_1$
$\overline{\overline{A}}_1$
A_1
$i_1 \equiv C_1$
b)

Fig. 233

Since the plane of the circle of revolution of point A is parallel to plane Π_1, the circle projects on this plane without distortion, but on plane Π_2 it will project as a line, the length of which is equal to the diameter of the circle. Moreover, this line is parallel to ground line Ox.

In the multiview drawing (Fig. 233, *b*) the following elements are given: projections i_1 and i_2 of the axis of revolution i, the projection of the circle of revolution of point A and projections C_1 and C_2 of centre C of this circle. The projection C_1 coincides with point i_1 which is the horizontal projection of the axis of revolution i. If point A revolving about axis i has been moved into position $\overline{A}$, its horizontal projection occupies the position $\overline{A}_1$ on the horizontal projection of the circle of rotation. Its frontal projection is $\overline{A}_2$ on the frontal projection of the circle. The points $\overline{A}_1$ and $\overline{A}_2$ lie on the same perpendicular to the ground line Ox.

When the axis of revolution i is perpendicular to the frontal plane of projection Π_2 (Fig. 234), the circle of revolution of point

A is contained in a frontal plane parallel to Π_2 and so is projected onto plane Π_2 as a circle and as a line parallel to ground line Ox onto plane Π_1.

It follows that a circle of revolution of a point is projected without distortion onto that plane of projection to which the axis of revolution is perpendicular. On the other principal plane of projection the circle is projected as a line parallel to the ground line Ox.

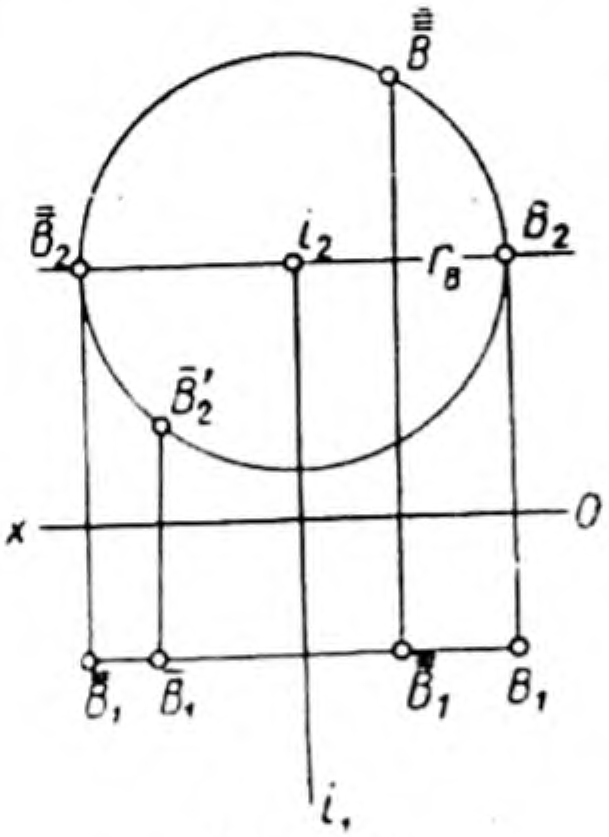


Fig. 234

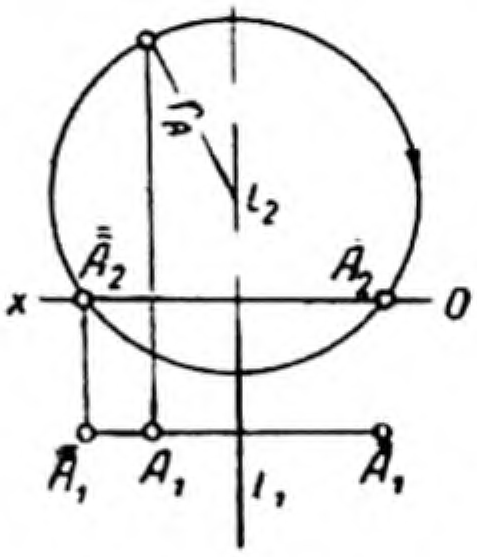


Fig. 235

2. Let us consider some examples.

Example 1. Revolve point A about the axis i perpendicular to plane Π_2 until it coincides with plane Π_1 (Fig. 235).

The circle of revolution of the point in this case is contained in the frontal plane, therefore it will be projected onto Π_2 without distortion and onto Π_1 as a line parallel to ground line Ox.

Join projection i_2 of the axis of revolution i, with which the frontal projection of the centre of the circle of revolution coincides, with point A_2, and with point i_2 as the centre describe a circle having its radius $r_A = A_2 i_2$. This circle cuts ground line Ox in points $\overline{A}_2$ and $\overline{\overline{A}}_2$.

These points are the frontal projections of the two possible positions $\overline{A}$ and $\overline{\overline{A}}$ of point A when it coincides with plane Π_1. The horizontal projections $\overline{A}_1$ and $\overline{\overline{A}}_1$ of points $\overline{A}$ and $\overline{\overline{A}}$ are obtained by drawing projectors from $\overline{A}_2$ and $\overline{\overline{A}}_2$ to intersection with line $\overline{A}_1\overline{\overline{A}}_1$, which is the projection of the circle of revolution on plane Π_1 and parallel to ground line Ox.

Example 2. By rotation about the axis i perpendicular to plane Π_1, make point A coincide with plane α (Fig. 236).

Through point A pass the horizontal plane β in which point A moves when revolved about axis i. Plane β will cut the given plane α along the horizontal NK.

Point A will coincide with plane α when the arc of the circle, which this point describes, cuts the horizontal NK. So, to obtain the horizontal projection $\overline{A}_1$ of point A in its position $\overline{A}$ in plane α it is necessary to describe an arc $A_1\overline{A}_1$ from the centre i_1 with a radius r_A to cut the horizontal projection N_1K_1 of the horizontal NK. The frontal projection $\overline{A}_2$ of the new position of point A in plane α is at the intersection of the projector passing through A_1 with the frontal projection N_2K_2 of the horizontal NK. This problem, like the previous one, has two solutions.

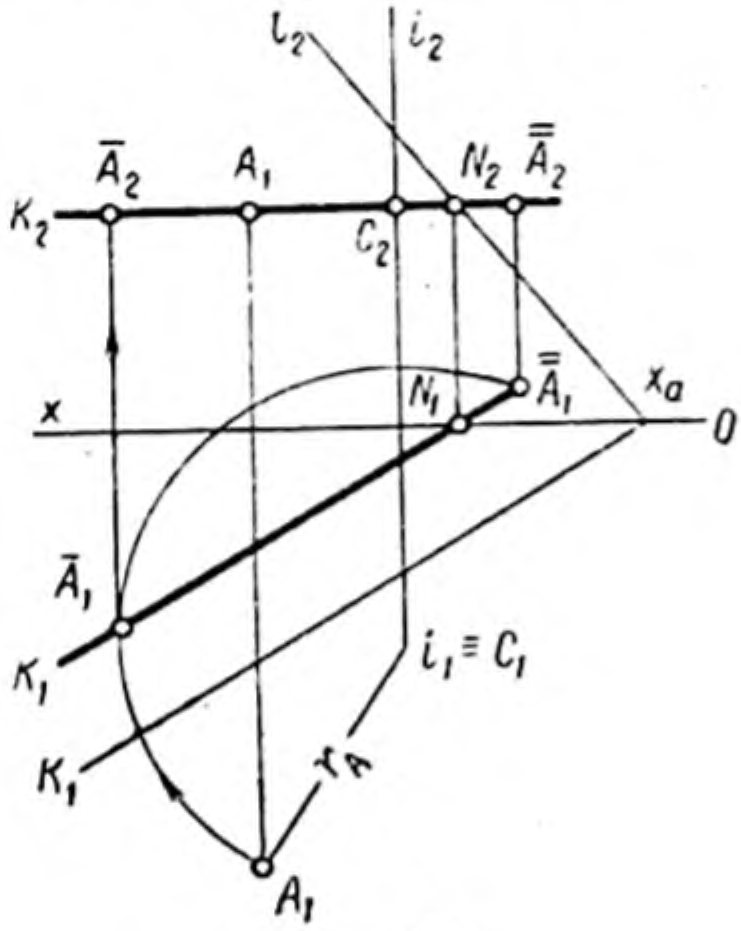


Fig. 236

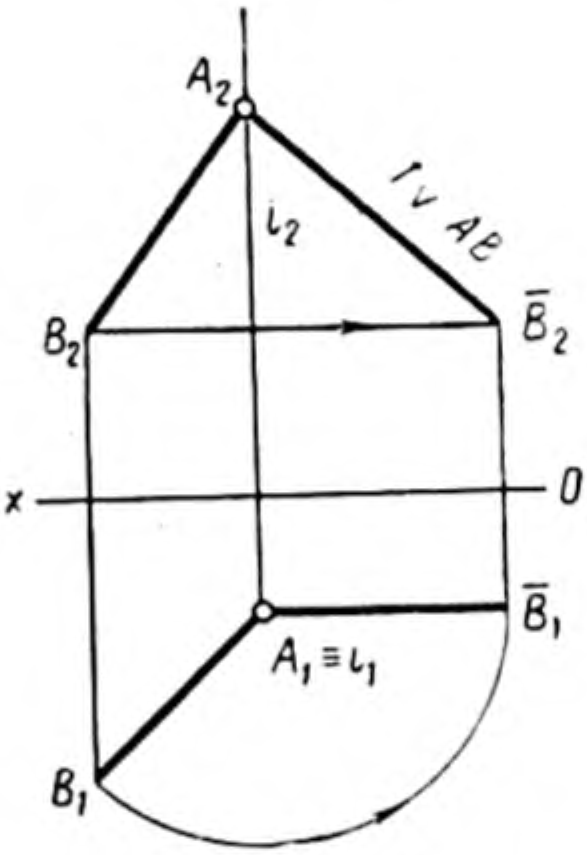


Fig. 237

3. The revolution of a straight line about an axis intersecting the line is reduced to the revolution of one of its points, because a point of intersection of the line with the axis of revolution remains stationary and, together with the point, which is moved, will determine the new position of the line.

In the drawing (Fig. 237) an oblique line AB is revolved about the axis i passing through point A perpendicular to plane Π_1 until it becomes parallel to the plane of projection Π_2. Then the projection $A_2\overline{B}_2$ on Π_2 gives the true length of the line AB.

Point A remains stationary. Its horizontal projection A_1 coincides with the horizontal projection of line i $(A_1 \equiv i_1)$. The projection A_1B_1 is equal in length to the radius of revolution of point B.

If line AB is to be parallel to plane Π_2, its horizontal projection A_1B_1 must be rotated until it is parallel to ground line Ox. This is done by moving point B_1 through arc $B_1\overline{B}_1$ to a new position $\overline{B}_1$. The frontal projection B_2 moves along line $B_2\overline{B}_2$ parallel to ground line Ox. Having drawn a projector through $\overline{B}_1$ point $\overline{B}_2$ is found at the intersection of the projector and line $B_2\overline{B}_2$. Point $\overline{B}_2$ is the frontal projection of point B in its new position $\overline{B}$. By joining the points A_2 and $\bar{B}_2$ we obtain the frontal projection $A_2\bar{B}_2$ of the given line after rotation about axis i. The length of $A_2\bar{B}_2$ is the true length of AB.

4. If the axis of revolution does not intersect a given straight line two points of the line are rotated through the same angle and in the same direction.

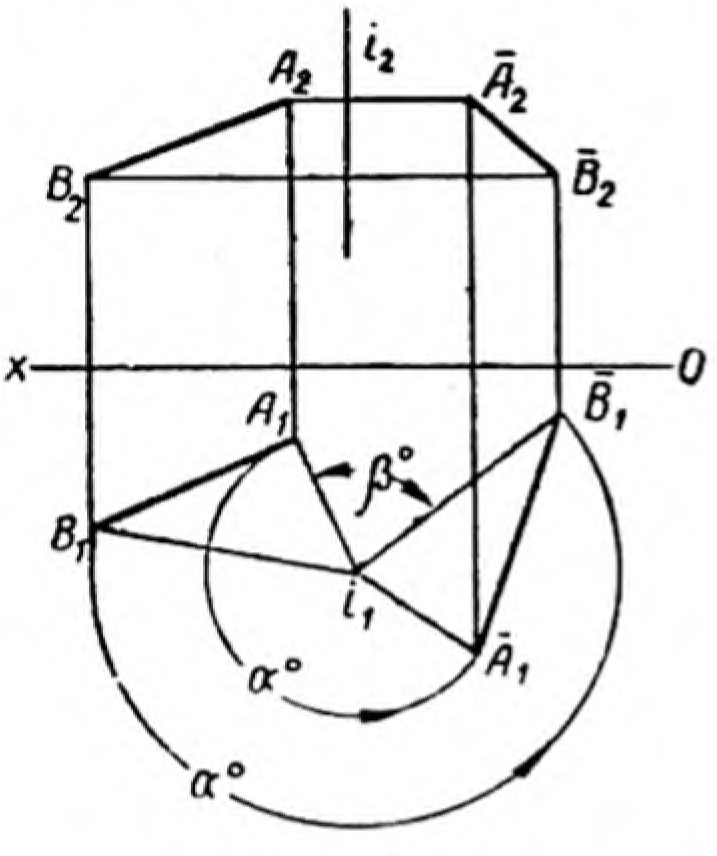


Fig. 238

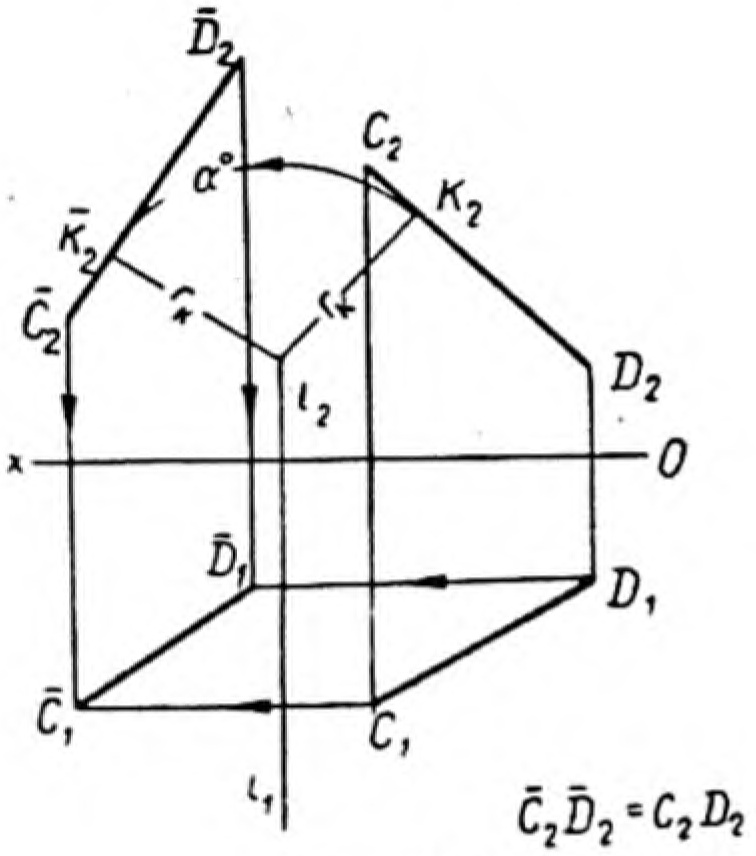


Fig. 239

Let line AB be rotated about axis i through an angle α in a counter-clockwise direction (Fig. 238). Having moved point A_1 through the arc $A_1\bar{A}_1$ with radius A_1i_1 to point $\bar{A}_1$, and point B_1 through the arc $B_1\bar{B}_1$ with radius B_1i_1 to point $\bar{B}_1$, both moving through one and the same angle α, we find the horizontal projection $\bar{A}_1\bar{B}_1$ of line AB after moving through angle α. The frontal projection $A_2\bar{B}_2$ is obtained by passing projectors through points $\bar{A}_1$ and $\bar{B}_1$. These projectors intersect with the lines drawn through points A_2 and B_2 parallel to ground line Ox.

In this construction the triangles $A_1B_1i_1$ and $\bar{A}_1\bar{B}_1i_1$ are identical, since two of their sides and the angle between them remain unchanged.

That is to say, $A_1i_1 = \bar{A}_1i_1$, $B_1i_1 = \bar{B}_1i_1$, $\measuredangle A_1i_1B_1 = \measuredangle \bar{A}_1i_1\bar{B}_1$, since each of these angles equals $360° - (\alpha° + \beta°)$.

Since these triangles $A_1B_1i_1$ and $\bar{A}_1\bar{B}_1i_1$ are equal, their third sides are equal, i. e., $A_1B_1 = \bar{A}_1\bar{B}_1$.

It follows that the projection of any line or figure on the plane, to which the axis of rotation is perpendicular, after rotation remains unchanged.

This deduction simplifies the construction of projections of given lines revolved about an axis perpendicular to the plane of projection and not intersecting the given line.

Such a simplified construction is shown in Fig. 239. The line CD, when revolved counter-clockwise through an angle α about axis i which is perpendicular to plane Π_2, takes the new position of the line $\bar{C}\bar{D}$. From point i_2 a perpendicular to the projection C_2D_2 is drawn. The base K_2 of this perpendicular is moved through an angle α to point $\bar{K}_2$. Point $\bar{K}_2$ is joined to point i_2 by a straight line. Then through point $\bar{K}_2$ a line at an angle 90° to $i_2\bar{K}_2$ is drawn. From point $\bar{K}_2$ on either side distances $\bar{K}_2\bar{C}_2$ and $\bar{K}_2\bar{D}_2$ are laid off respectively equal to the lengths K_2C_2 and K_2D_2. The line $\bar{C}_2\bar{D}_2$ then is the projection of CD on plane Π_2 in its new position $\bar{C}\bar{D}$.

The horizontal projections $\bar{C}_1$ and $\bar{D}_1$ of points $\bar{C}$ and $\bar{D}$ are found with the help of projectors drawn through points $\bar{C}_2$ and $\bar{D}_2$ to intersect lines drawn parallel to ground line Ox through $\bar{C}_1$ and $\bar{D}_1$. By joining up points $\bar{C}_1$ and $\bar{D}_1$ the horizontal projection of CD in its new position $\bar{C}\bar{D}$ is obtained.

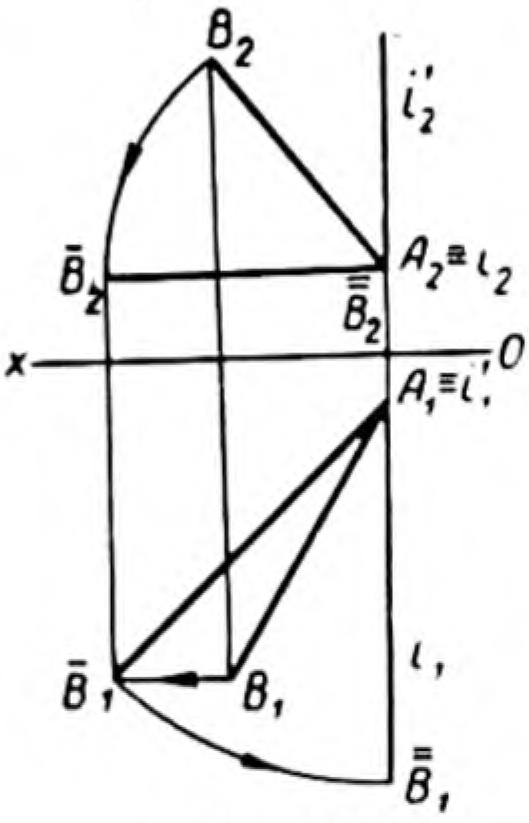


Fig. 240

5. Revolution about two axes. Fig. 240 illustrates the method of bringing an oblique line AB into a position, perpendicular to plane Π_2, by succesive revolutions of this segment about two axes respectively perpendicular to the planes of projection Π_1 and Π_2.

First, we revolve line AB about axis i perpendicular to plane Π_2; it also passes through point A. Line AB is brought into position $A\bar{B}$ parallel to plane Π_1. The frontal projection $A_2\bar{B}_2$ will be parallel to ground line Ox and the horizontal projection $A_1\bar{B}_1$ gives the true length of AB.

The second axis of rotation i' is drawn through point A perpendicular to plane Π_1. The line is rotated about this axis until it becomes perpendicular to plane Π_2. The horizontal projection $A_1\bar{\bar{B}}_1$ will then be perpendicular to ground line Ox and the frontal projection $A_2\bar{\bar{B}}_2$ becomes a point.

6. Let us consider some examples.

Example 1. It is required to construct a straight line AB inclined at angles α and β to planes Π_1 and Π_2.

If line KL, making an angle α with plane Π_1 (Fig. 241), revolves about an axis i, perpendicular to plane Π_1 and passing through point K, point L will move along a circle lying in plane Π_1 and the line KL in each new position ($K\bar{L}$; $K\bar{\bar{L}}$; $K\bar{\bar{\bar{L}}}$; $K\bar{\bar{\bar{\bar{L}}}}$ and so on) will form one and the same angle α with plane Π_1. The surface formed by rotation of the line is a conical surface.

If point K remains stationary and line KL is made to form an angle β with plane Π_2, after which it is revolved about the axis passing through point K and perpendicular to plane Π_2, then point L will describe another circle, this time in the frontal plane, and the movement of line KL will describe a new conical surface.

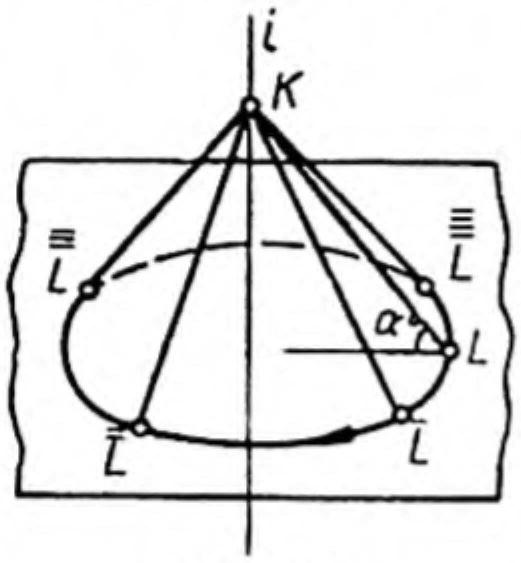

i
K
$\overline{\overline{L}}$
$\overline{\overline{\overline{L}}}$
α
L
$\overline{L}$
L

Fig. 241

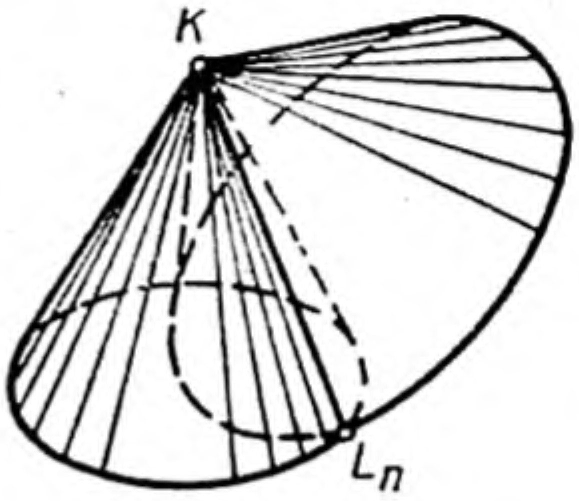


Fig. 242

All points of each of the two circles are at one and the same distance KL from point K. Therefore both circles will lie on the surface of a sphere, the centre of which coincides with point K and the radius of which is equal to the length of line KL.

Let the circles intersect in some point L_n (Fig. 242). Then line KL_n will be common to both the first and second cones and so form angle α with plane Π_1 and angle β with plane Π_2.

These considerations enable the following construction to be carried out in the multiview drawing (Fig. 243).

Draw the projections A_1B_1 and A_2B_2 of the frontal line AB at an angle α to plane Π_1. Through point A pass the axis i perpendicular to plane Π_1. Revolve the line AB about this axis. Point B will describe a circle with radius A_1B_1, since when rotated about axis i line AB forms a conical surface with its vertex at point A.

Further, leaving point A stationary, draw the horizontal $A\overline{B}$ equal in length to AB and making an angle β with plane Π_2. The horizontal projection $A_1\overline{B}$ of line $A\overline{B}$ is equal in length to AB and makes an angle β with ground line Ox.

Through point A draw the second axis of revolution perpendicular to plane Π_2. Revolve line $A\overline{B}$ about this axis. The point $\overline{B}$ in the frontal plane, parallel to Π_2, will describe a circle with radius $A_2\overline{B}_2$. The movement of line $A\overline{B}$ generates the second conical surface with its vertex at the same point A.

The circles of radii A_1B_1 and $A_2\overline{B}_2$ will intersect in points $\overline{\overline{B}}$ and $\overline{\overline{\overline{B}}}$. Joining up these points with point A by lines $A\overline{\overline{B}}$ and $A\overline{\overline{\overline{B}}}$ we obtain the two positions of line AB forming angles α and β with planes Π_1 and Π_2 as was required.

The lines, symmetrical to the lines constructed, $A\overline{\overline{B}}$ and $A\overline{\overline{\overline{B}}}$, with respect to the frontal plane passing through point A, will also form angles α and β with planes Π_1 and Π_2. These lines may be obtained if both nappes of one of the cones are constructed.

It follows that, as a rule, this problem has four solutions. If the circles do not intersect but touch, the problem has only one solution. The required line will then be a profile line. In general the problem may be solved only in the case when the sum of the angles α and β is less than 90°, which corresponds to the possible positions of a line in space in relation to planes Π_1 and Π_2.

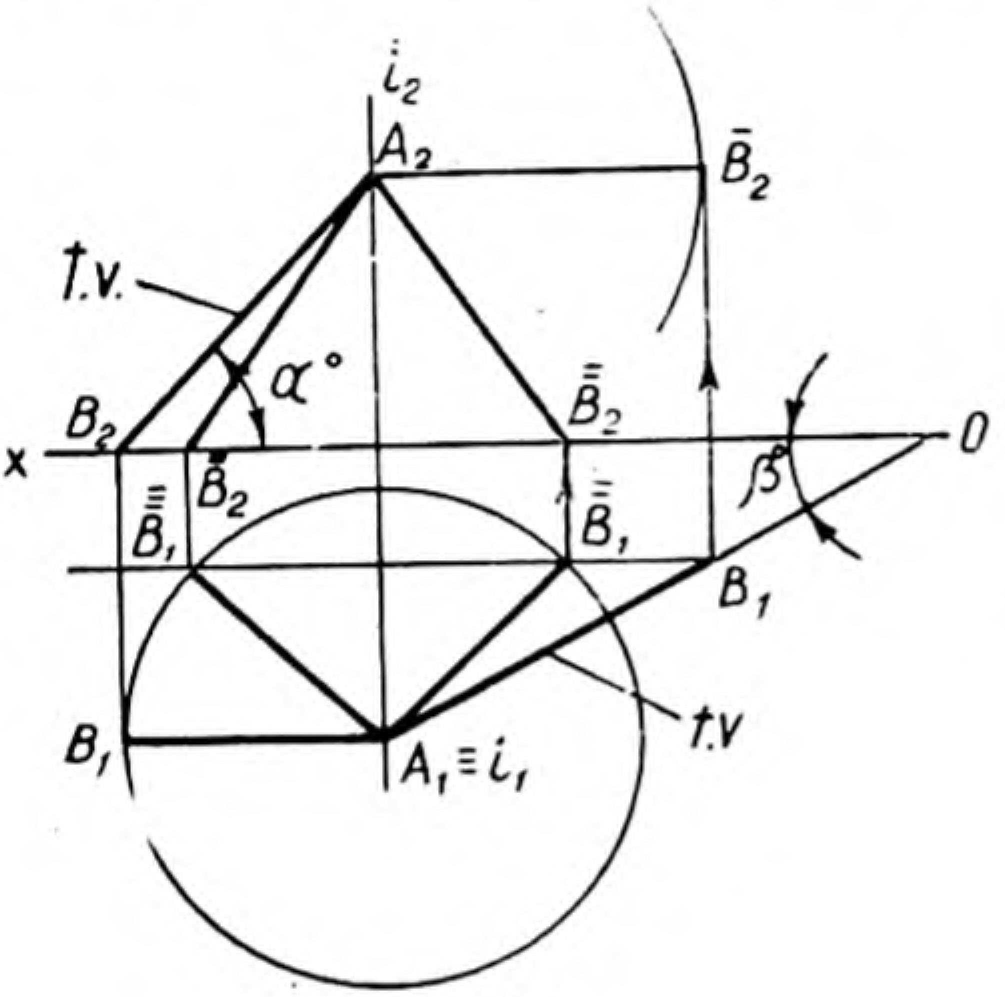


Fig. 243

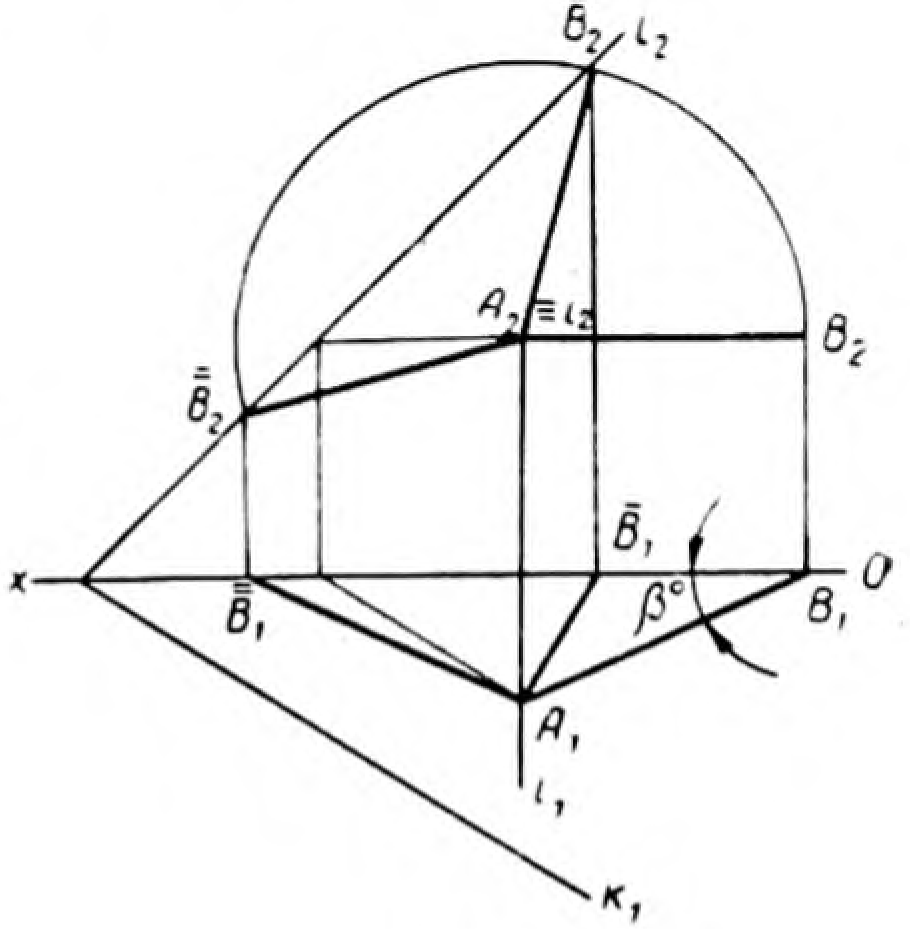


Fig. 244

Example 2. It is required to draw through point A a line in plane α at an angle α to plane Π_2 (Fig. 244).

The problem is solved by constructing a conical surface with its vertex in point A and the generatrix AB in its first position parallel to plane Π_1 and forming an angle α with plane Π_2, point B being contained in plane Π_2.

Through point A draw the axis of revolution i perpendicular to plane Π_2. Then revolve line AB about this axis. The point B will describe a circle in plane Π_2 and the line AB will generate a conical surface. The circle with a radius A_2B_2 intersects the frontal trace l_2 of plane α in points $\bar{B}$ and $\bar{\bar{B}}$. Joining these points with point A we find in this case two possible solutions. The lines $A\bar{B}$ and $A\bar{\bar{B}}$ are situated in plane α and make an angle α with plane Π_2.

If the circle which has been constructed merely touches trace l_2, the problem has one solution and in that case line AB and plane α have one and the same angle of inclination α to the plane of projection Π_2.

When the circle and the trace l_2 have no common points it becomes impossible to draw in plane α a line inclined at angle α to plane Π_2.

Example 3. By rotation about axis i which is perpendicular to plane Π_1, make the given line AB coincide with plane α (Fig. 245).

Having found the piercing point K of line AB with plane α through it draw the axis i perpendicular to plane Π_1. Rotate about this axis one of the extreme points of AB, for example point A, until it coincides with plane α. This operation is carried out with the help of the horizontal AN drawn in plane α through point A (see Fig. 236).

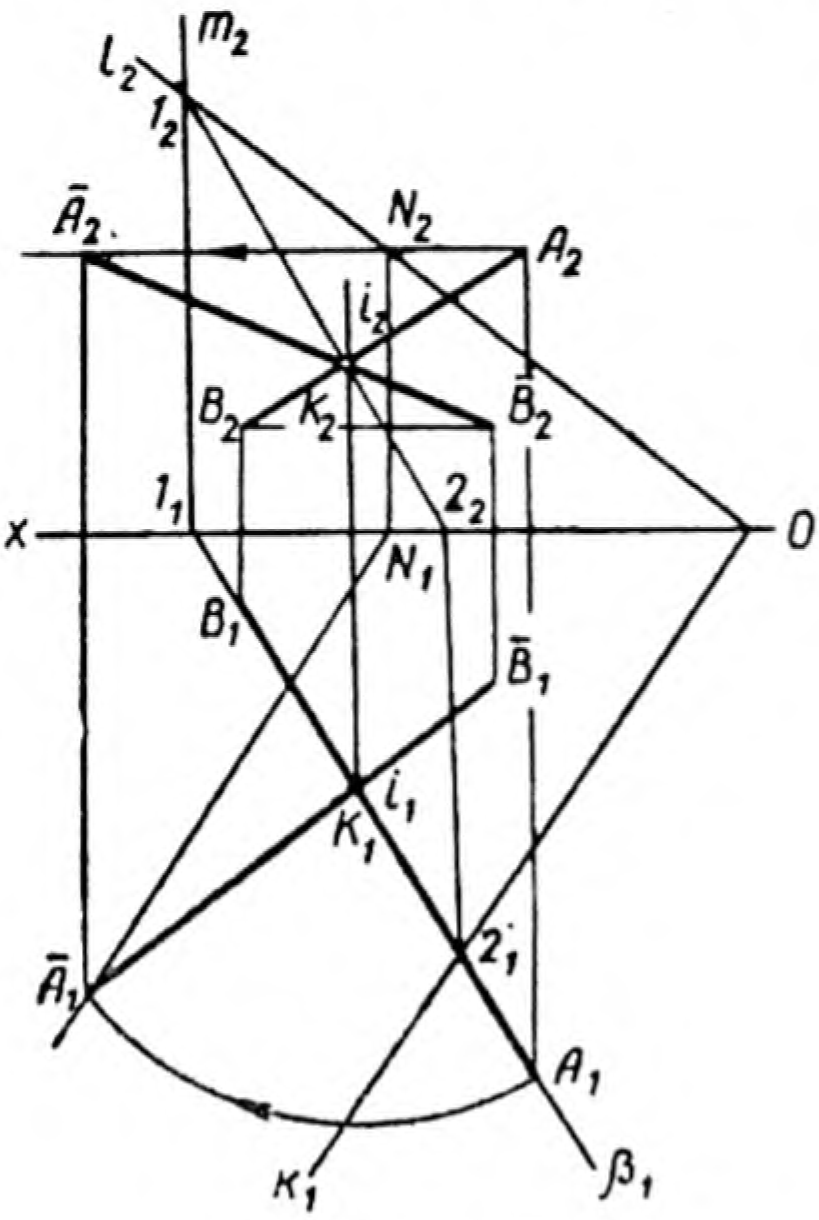


Fig. 245

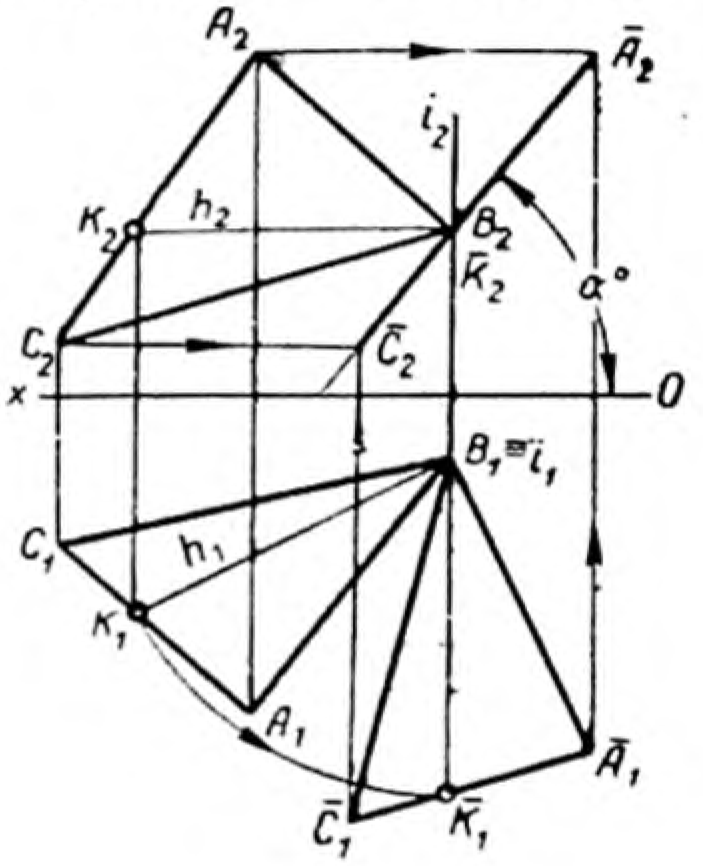


Fig. 246

Having found the position $\bar{A}$ of point A, when it coincides with plane α, join point K with point $\bar{A}$. Line $\bar{A}K$ is that part of line AB which has been rotated to coincide with plane α.

The position $\bar{B}$ of the point B of the line, when it has been made to coincide with α, may be constructed by extending the projection $\bar{A}_2K_2$ beyond point K_2 to intersect a line drawn through point B_2 parallel to ground line Ox. The point $\bar{B}_2$ is the frontal projection of the point $\bar{B}$ after rotation. Then with the help of a projector find the point $\bar{B}_1$ on the extension of projection $\bar{A}_1K_1$.

7. The revolution of a plane about an axis perpendicular to the plane of projection is performed by rotating the points and lines by which the plane is given through one and the same angle and in the same direction.

It is required to revolve the plane of triangle ABC into a position perpendicular to plane of projection Π_2 (Fig. 246).

All horizontals of a horizontal-projecting plane are projected onto plane Π_2 as points since these lines in space are perpendicular to plane Π_2 (see Fig. 110). Therefore in the triangle ABC draw any horizontal, say BK, and revolve it about the axis i perpendicular to plane Π_1 and passing through point B until it takes up the position $B\overline{K}$ and is perpendicular to plane Π_2, namely $B_1\overline{K}_1 \perp Ox$, and the projection $B_2\overline{K}_2$ becomes a point. The horizontal projection of the triangle, when rotated through the same angle as the projection B_1K_1 of the horizontal was moved, will occupy the position $\overline{A}_1B_1\overline{C}_1$. The triangle $\overline{A}_1B_1\overline{C}_1$ is equal to triangle $A_1B_1C_1$.

To obtain the frontal projections $\overline{A}_2$ and $\overline{C}_2$ of points $\overline{A}$ and $\overline{C}$ through A_2 and C_2 draw lines parallel to Ox, and intersect them by projectors passing through points $\overline{A}_1$ and $\overline{C}_1$. By joining points $\overline{A}_2$, B_2 and $\overline{C}_2$ line $\overline{A}_2B_2\overline{C}_2$ is obtained. This is the frontal trace-projection of the given triangle plane when turned until it is perpendicular to the frontal plane of projection Π_2.

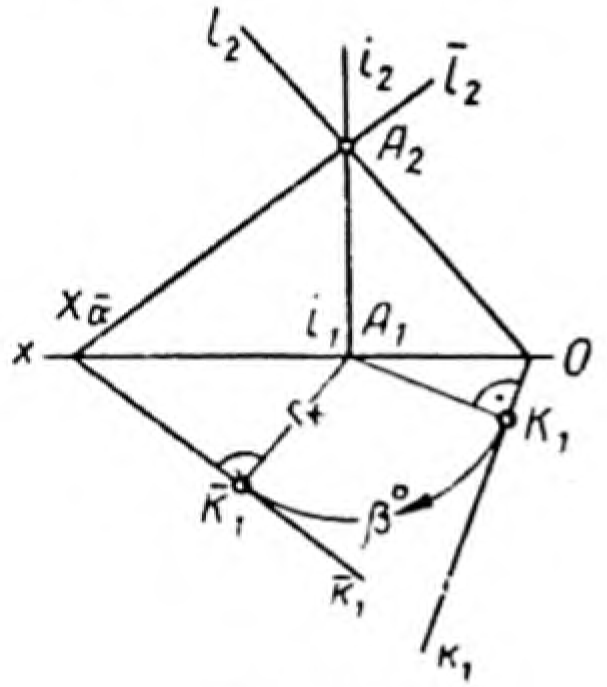
The angle α formed by the trace-projection $\overline{A}_2B_2\overline{C}_2$ and ground line Ox determines the true value of inclination of triangle ABC to the horizontal plane of projection Π_1.

8. When the *plane is given by traces* it can be rotated by revolving one of its traces and a point or line belonging to the plane.

In Fig. 247 the plane α assigned by traces k_1 and l_2 is turned about axis i perpendicular to plane Π_1 clockwise through angle β. The revolution of the horizontal trace into position $\overline{k}_1$ is carried out with the help of the perpendicular i_1K_1 drawn from i_1 to k_1. After moving point K_1 into the position $\overline{K}_1$ and joining it up with point i_1 of line $i_1\overline{K}_1$ draw through point $\overline{K}_1$ the perpendicular $\overline{k}_1$. This line, which is the horizontal trace of plane α after rotation, intersects ground line Ox in point $X_{\overline{\alpha}}$, the point of convergence of traces of plane α in its new position $\overline{\alpha}$.

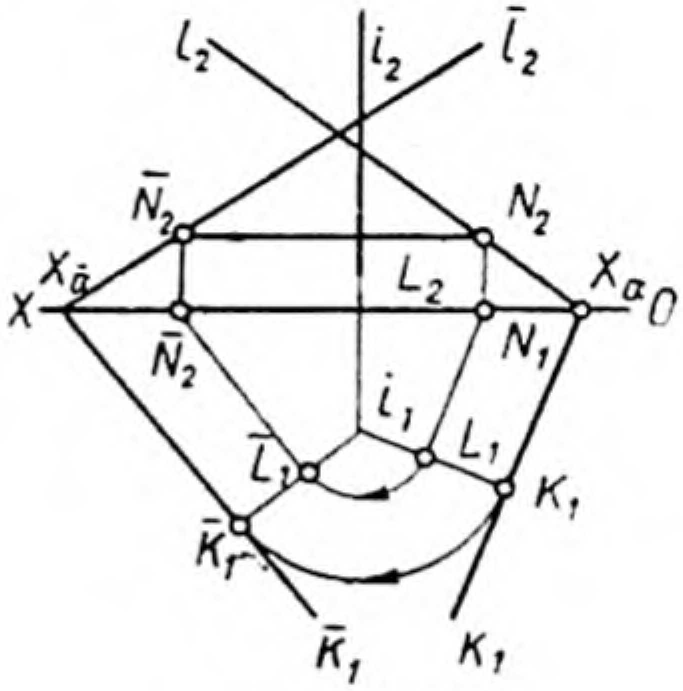
The axis of rotation i, as can be seen from the drawing, is contained in plane Π_2 and intersects trace l_2 in point A. This point A is located on the axis of rotation i and so remains stationary. By joining the frontal projection A_2 of point A with point $X_{\overline{\alpha}}$ we find the new frontal trace $\overline{l}_2$ of plane α after it has been revolved about axis i through angle β.

In Fig. 248 the axis i about which plane α is revolved into the position $\overline{\alpha}$, is perpendicular to plane Π_1 but is not contained in plane Π_2. The position of the new frontal trace $\overline{l}_2$ is determined by the point of convergence $X_{\bar{\alpha}}$ obtained after turning trace k_1 through the given angle and by the new frontal trace $\bar{N}$ of the auxiliary horizontal LN, drawn at any suitable altitude in plane α. This horizontal is revolved into position $\overline{LN}$ with the help of point L in the same way as the trace k_1 was turned using point K_1. The new frontal trace $\bar{l}_2$ does not pass through the point in which the trace l_2 intersects projection i_2 of the axis of revolution i.



l_2
i_2
$\bar{l}_2$
A_2
$X_{\bar{\alpha}}$
x
i_1
A_1
O
K_1
$\bar{K}_1$
$\beta°$
$\bar{k}_1$
k_1

Fig. 247



l_2
i_2
$\bar{l}_2$
$\bar{N}_2$
N_2
$X_{\bar{\alpha}}$
X
L_2
X_{α}
O
$\bar{N}_2$
N_1
i_1
$\bar{L}_1$
L_1
K_1
$\bar{K}_1$
$\bar{k}_1$
k_1

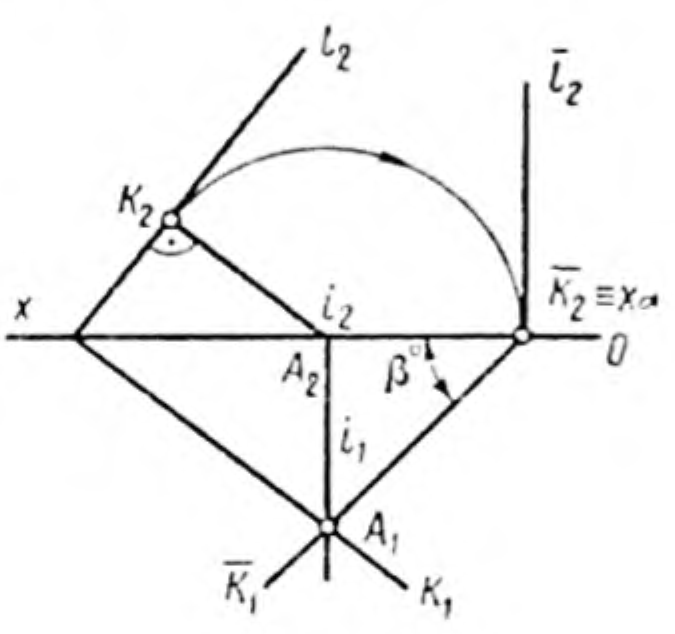
Fig. 248

9. The following examples of the transformation of projections based on the principle of the revolution of planes are given.

Example 1. It is required to determine the inclination of an oblique plane α to the frontal plane of projection Π_2 (Fig. 249).

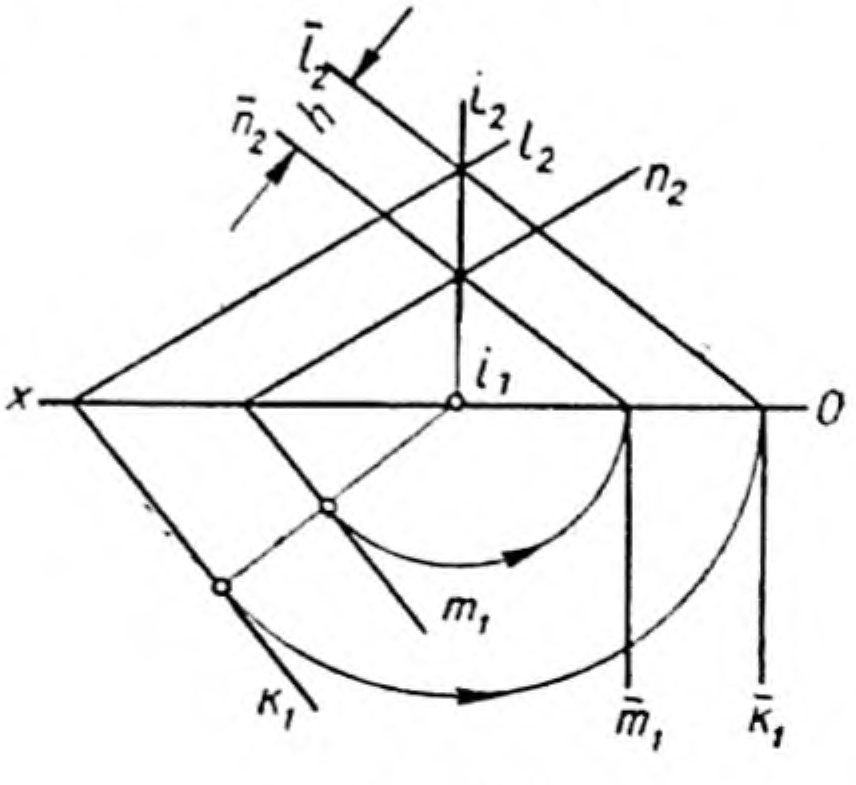
Plane α can be converted into a horizontal-projecting plane by revolving it about axis i perpendicular to plane Π_2 and contained in plane Π_1. To construct the frontal trace $\bar{l}_2$ from point i_2 draw a perpendicular i_2K_2 to l_2 and revolve it till it coincides with ground line Ox. Through point $\bar{K}_2 \equiv X_{\bar{\alpha}}$ draw the trace $\bar{l}_2$ perpendicular to Ox. Through points $X_{\bar{\alpha}}$ and A_1 draw the horizontal trace $\bar{k}_1$.

After revolution the plane α is perpendicular to plane Π_1;



l_2
$\bar{l}_2$
K_2
x
i_2
$\bar{K}_2 \equiv X_{\bar{\alpha}}$
O
A_2
$\beta°$
i_1
A_1
$\bar{k}_1$
k_1

Fig. 249



$\bar{l}_2$
$\bar{n}_2$
h
i_2
l_2
n_2
x
i_1
O
m_1
k_1
$\bar{m}_1$
$\bar{k}_1$

Fig. 250

therefore the angle β between trace $\bar{k}_1$ and ground line Ox is the required angle of inclination of plane α to Π_2.

Example 2. It is required to determine the distance between the parallel oblique planes α and β (Fig. 250).

Convert both planes into frontal-projecting planes by revolving them about the axis i which is perpendicular to plane Π_1 and lying in plane Π_2 (see example 1). The planes α and β after the revolution are perpendicular to plane Π_2. The distance h between their new frontal traces $\bar{l}_2$ and $\bar{n}_2$ is the required distance.

Example 3. It is required to revolve plane α about the axis i perpendicular to plane Π_1 so that it passes through point A (Fig. 251).

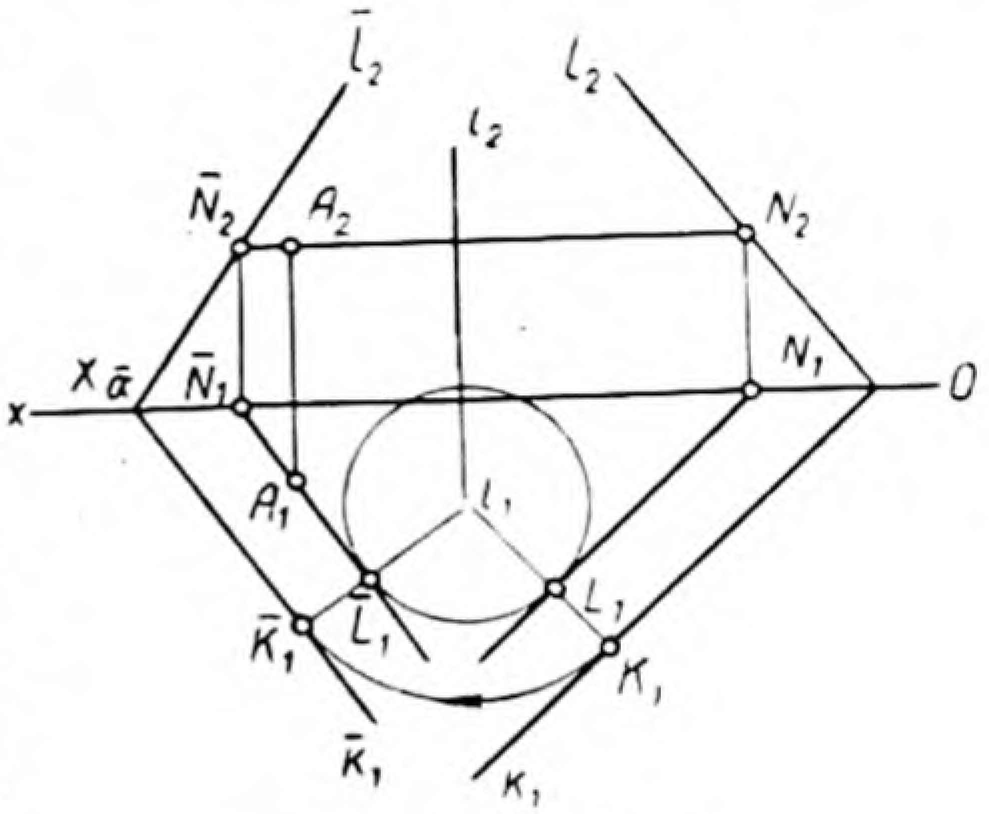


Fig. 251

In plane α, at the altitude of point A, draw the horizontal NL and revolve it about axis i until it passes through point A. Point L_1 which is the foot of the perpendicular drawn from i_1 to N_1L_1 will describe a circle when the horizontal is rotated. By drawing a tangent to this circle through point A_1 the new position $\overline{N}_1\overline{L}_1$ of the horizontal projection of the horizontal NL of plane α is found.

Point K_1, which is the foot of the perpendicular drawn from i_1 to trace k_1 when plane α is revolved, will also describe a circle. A tangent drawn to this circle parallel to the projection $\overline{N}_1\overline{L}_1$ of the horizontal $\overline{N}\overline{L}$ gives the horizontal trace $\bar{k}_1$ of plane α in its new position $\bar{\alpha}$.

The frontal trace $\bar{l}_2$ will pass through $X_{\bar{\alpha}}$, i. e., the point of convergence of the traces, and through the frontal trace $\overline{N}_2$ of the horizontal $\overline{N}\overline{L}$.

This problem has two solutions, because through point A_1 two tangents to the circle with radius i_1L_1 can be drawn.

Example 4. It is required to find the true size of a triangle ABC contained in an oblique plane (Fig. 252).

This problem is solved by successive revolutions of triangle ABC about two axes respectively perpendicular to the planes of projection Π_1 and Π_2.

By means of the first revolution about axis i perpendicular to plane Π_2, passing through vertex A, triangle ABC is brought into a position $A\bar{B}\bar{C}$ so that it becomes perpendicular to plane Π_1 (see Fig. 246). The projection of axis i is not shown in Fig. 252. The frontal AK is drawn in the plane of the triangle and then rotated about axis i until it becomes perpendicular to plane Π_1.

The triangle $A\overline{B}\overline{C}$ is brought into position $\overline{A}\overline{\overline{B}}\overline{C}$ parallel to plane Π_2 by revolving it about axis i' perpendicular to plane Π_1. This axis passes through point $\overline{C}$. The projection of axis i' is not shown in the drawing. The horizontal projection $\overline{A}\overline{\overline{B}}_1\overline{C}_1$ is parallel to ground line Ox, and the frontal projection $\overline{A}_2\overline{\overline{B}}_2\overline{C}_2$ is a true view of the given triangle ABC.

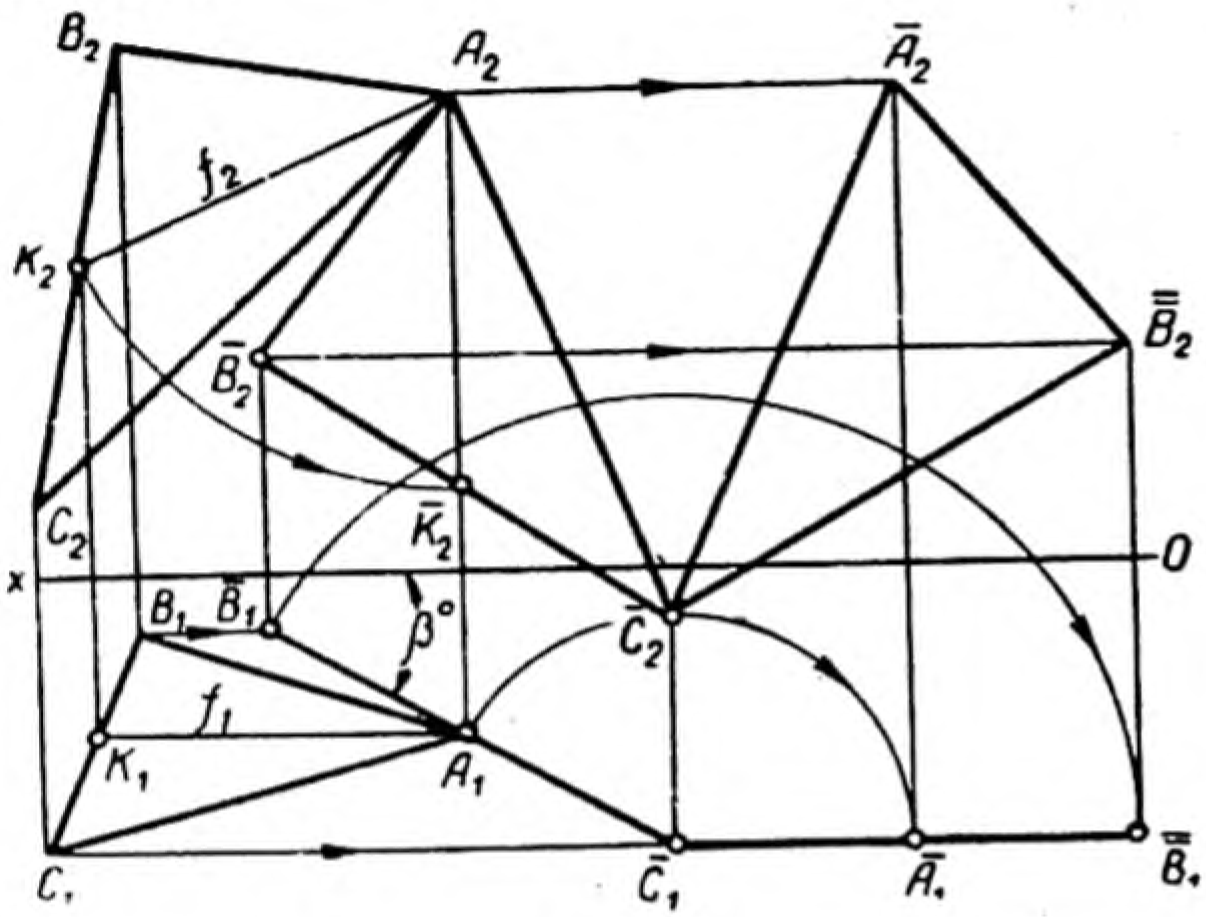


Fig. 252

The first revolution about the axis i is sufficient to determine the angle of inclination β of triangle ABC plane to the frontal plane of projection Π_2.

Problems and Exercises

1. In what plane does a point move when it is revolved about an axis a) perpendicular to plane Π_1, b) perpendicular to plane Π_2?

2. To which plane of projection should the axis of revolution be perpendicular in order to revolve an oblique line a) into a horizontal position, b) into a frontal position?

3. What revolutions about axes perpendicular to the planes of projection are required to bring an oblique line into a vertical position?

4. How can an oblique plane, given as a triangle, be rotated so as to be perpendicular a) to plane Π_1, b) to plane Π_2?

5. The same as (4) when the plane is given by its traces.

6. Revolve a point A through 60° in a clockwise direction about an axis i perpendicular to plane Π_1 (Fig. 253).

7. Make point A coincide with the plane of triangle BCD by rotating it about an axis i perpendicular to plane Π_1 (Fig. 254).

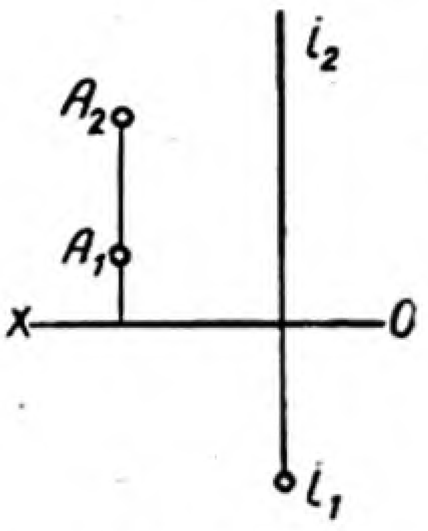

A₂
A₁
l₂
x
0
l₁

Fig. 253

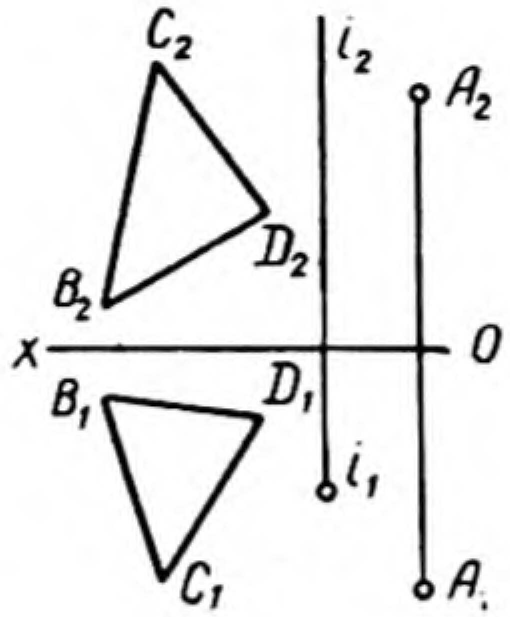

C₂
l₂
A₂
D₂
B₂
x
0
D₁
B₁
l₁
C₁
A.

Fig. 254

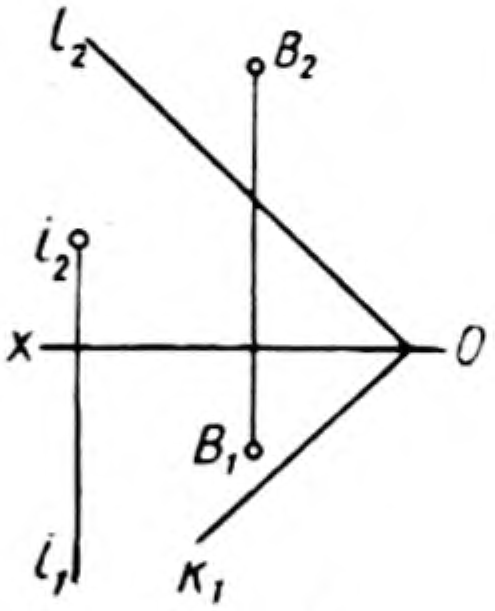

l₂
B₂
l₂
x
0
B₁
l₁
K₁

Fig. 255

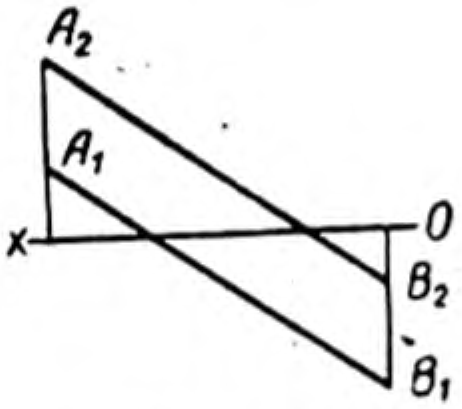

A₂
A₁
x
0
B₂
B₁

Fig. 256

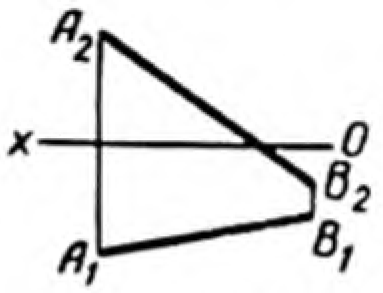

A₂
x
0
B₂
A₁
B₁

Fig. 257

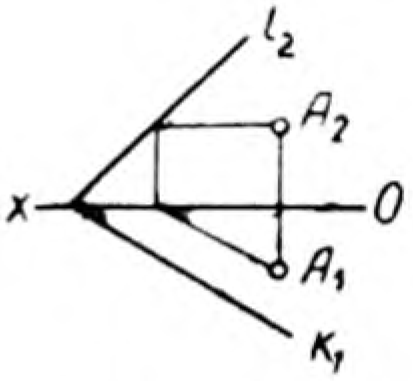

l₂
A₂
x
0
A₁
K₁

Fig. 258

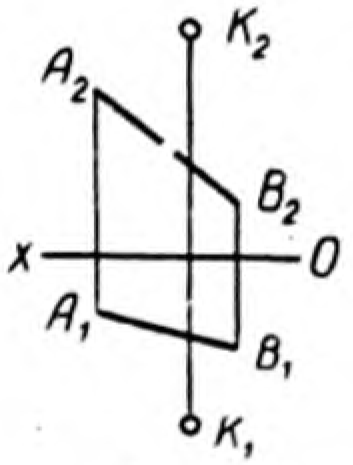

K₂
A₂
B₂
x
0
A₁
B₁
K₁

Fig. 259

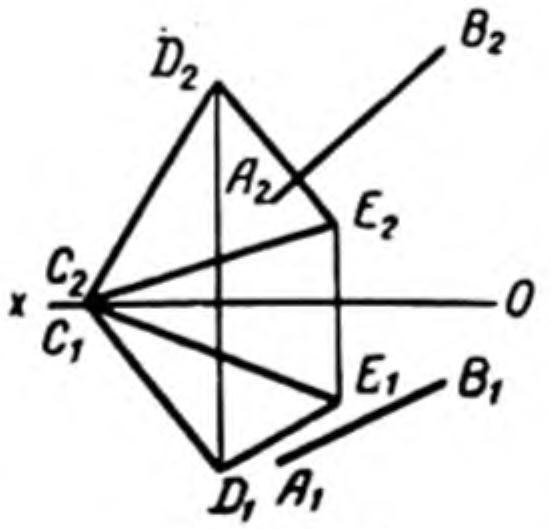

D₂
B₂
A₂
E₂
C₂
x
0
C₁
E₁
B₁
D₁
A₁

Fig. 260

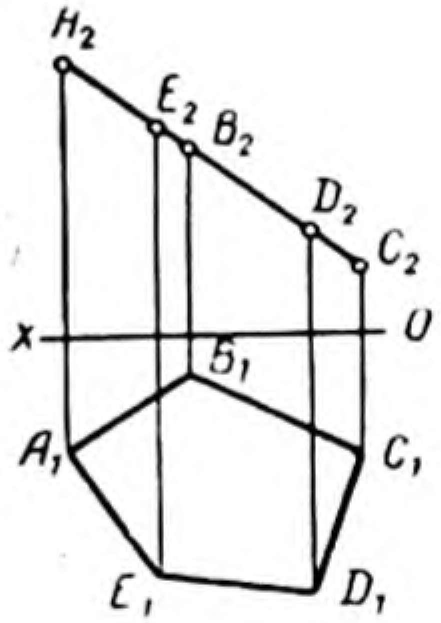

H₂
E₂
B₂
D₂
C₂
x
0
B₁
A₁
C₁
E₁
D₁

Fig. 261

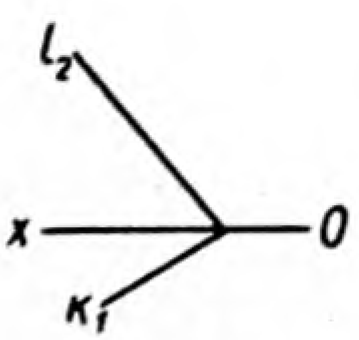


Fig. 262

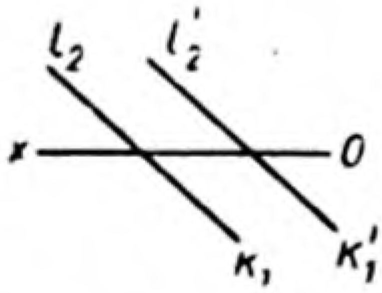

l₂
l'₂
x
0
K₁
K'₁

Fig. 263

Hint. In plane BCD draw an auxiliary horizontal at the altitude of point A.

8. Bring point B into a position so that it lies at 10 mm from plane α (**Fig.** 255) using the method of rotation about an axis i perpendicular to Π_2.

Hint. First, a plane β should be drawn parallel to plane α at a distance of 10 mm from it.

9. Determine the angle β formed by line AB and plane Π_2 (Fig. 256).

10. Rotate line AB until it becomes perpendicular to plane Π_1 (Fig. 257).

Hint. First, line AB should be converted into a frontal by turning it about an axis perpendicular to plane Π_1. Then, it should be revolved about an axis perpendicular to plane Π_2.

11. Draw a line AB in plane α to form an agle of 45° with Π_1. Point A is given (Fig. 258). (See Fig. 244 and text.)

12. By means of rotation determine the distance from point K to line AB (Fig. 259).

Hint. Two successive revolutions of the line and the point about axes perpendicular to the planes of projection should be carried out so as to make the line perpendicular to one of the planes of projection. The line AB will then be projected as a point. The distance between this point and the new projection of point K will be the required distance.

13. Revolve line AB until it lies in the plane of triangle CDE (Fig. 260). (See Fig. 245 and text.)

14. Revolve line AB until it is parallel to the plane of triangle CDE (Fig. 260).

Hint. This problem may be solved in the same way as the previous one. However, the line AB should be made to coincide with a plane drawn parallel to the given plane CDE.

15. Revolve the oblique line AB about an axis i perpendicular to plane Π_1 so that its projection A_1B_1 becomes parallel to the projection K_1L_1 of a line KL contained in plane Π_1.

16. Construct a line AB so that the angles it makes with planes Π_1 and Π_2 are $\alpha=45°$ and $\beta=30°$, respectively. (See Fig. 243 and text.)

17. Determine the true size of the pentagon $ABCDE$ (Fig. 261).

18. Rotate the plane α so that it becomes a frontal projecting plane (Fig. 262).

Hint. Revolve the plane about an axis perpendicular to Π_1 (compare with Fig. 249).

19. Determine the distance between the parallel planes α and β (Fig. 263).

20. Revolve a triangle ABC which is part of an oblique plane so that it becomes parallel to plane Π_2.

§ 27. Revolution about Axes Parallel to Planes of Projection

1. An oblique line AB is shown in Fig. 264. Let a line i be drawn parallel to plane Π_1 intersecting line AB in point K. Assuming line i to be the axis of revolution let line AB be rotated until it becomes parallel to plane Π_1.

After the line has assumed its new position $\overline{A}\,\overline{B}$ its frontal projection $\overline{A}_2\overline{B}_2$ will merge with the frontal projection i_2 of the axis of revolution i and its horizontal projection $\overline{A}_1\overline{B}_1$ will give the true length of AB.

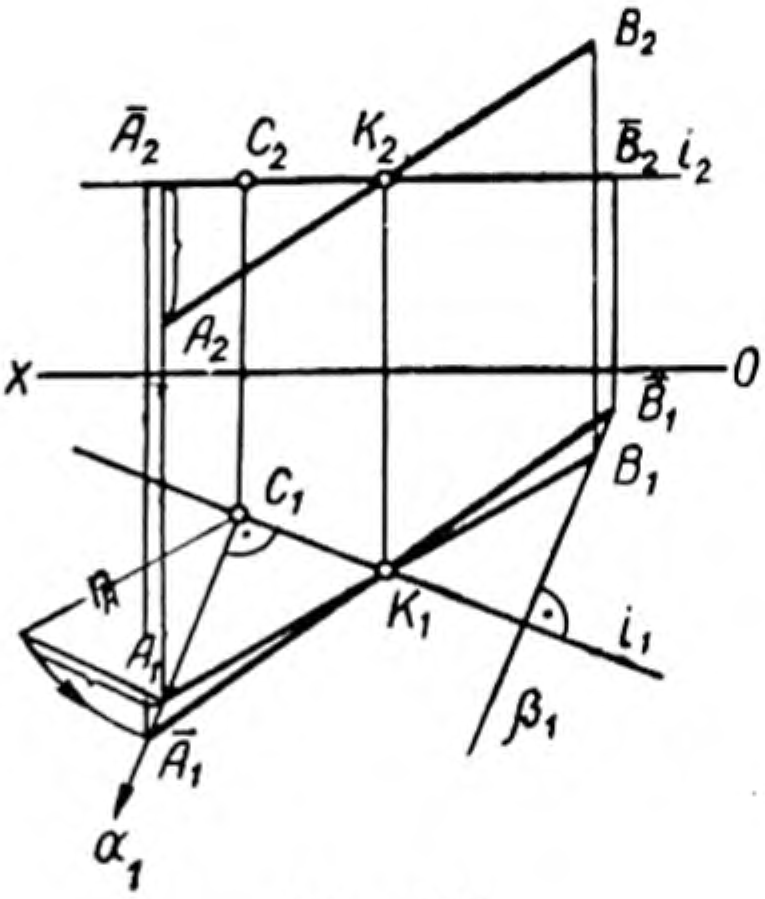


Fig. 264

The construction of the horizontal projection $\overline{A}_1\overline{B}_1$ after the line has been turned is carried out as follows. The points A and B are revolved about axis i and they move in horizontal-projecting planes α and β perpendicular to the axis of revolution i. It follows, that the projection $\overline{A}_1$ and $\overline{B}_1$ of the extremes of line AB in its new position $\overline{A}\,\overline{B}$ will lie on the traces α_1 and β_1 perpendicular to the projection A_1B_1. When line $\overline{A}\,\overline{B}$ is in a horizontal position the radii of revolution of points A and B will be projected onto plane Π_1 in their true length. The true length of the radius r_A for point A is found with the help of a right triangle. The radius r_A is laid off from point C_1, which is the horizontal projection of the centre of revolution C for point A. Point $\overline{A}_1$ lies on trace α_1.

Point $\overline{A}_1$ is joined by a line with the projection K_1 of the stationary point K in which axis i intersects line AB. In this way the horizontal projection of line AB after rotation about axis i is obtained. The horizontal projection $\overline{B}_1$ of point $\overline{B}$ lies at the intersection of projection A_1K_1 with trace β_1. The projection $\overline{A}_1\overline{B}_1$ gives the true length of line AB.

2. The construction of the true view of a plane figure by revolution about an axis parallel to a plane of projection is shown in the multiview drawing (Fig. 265). By rotation about axis i, which is parallel to plane Π_1, the triangle ABC is brought to a position parallel to plane Π_1, on which it is then projected in its true size ($\triangle A_1\overline{B}_1\overline{C}_1 = \triangle ABC$).

The frontal projection $A_2\overline{B}_2\overline{C}_2$ of triangle ABC after revolution merges with the frontal projection of axis i.

In order to construct triangle $A_1\overline{B}_1\overline{C}_1$ a perpendicular to the projection i_1 of the axis of revolution is drawn. Then the true dimension of radius r_B is determined by turning point B. To do this the right triangle method is employed. The distance r_B is marked off on the perpendicular. Point $\overline{B}_1$ obtained in this way is the projection of the vertex $\overline{B}$ of the original triangle when it becomes parallel to plane Π_1.

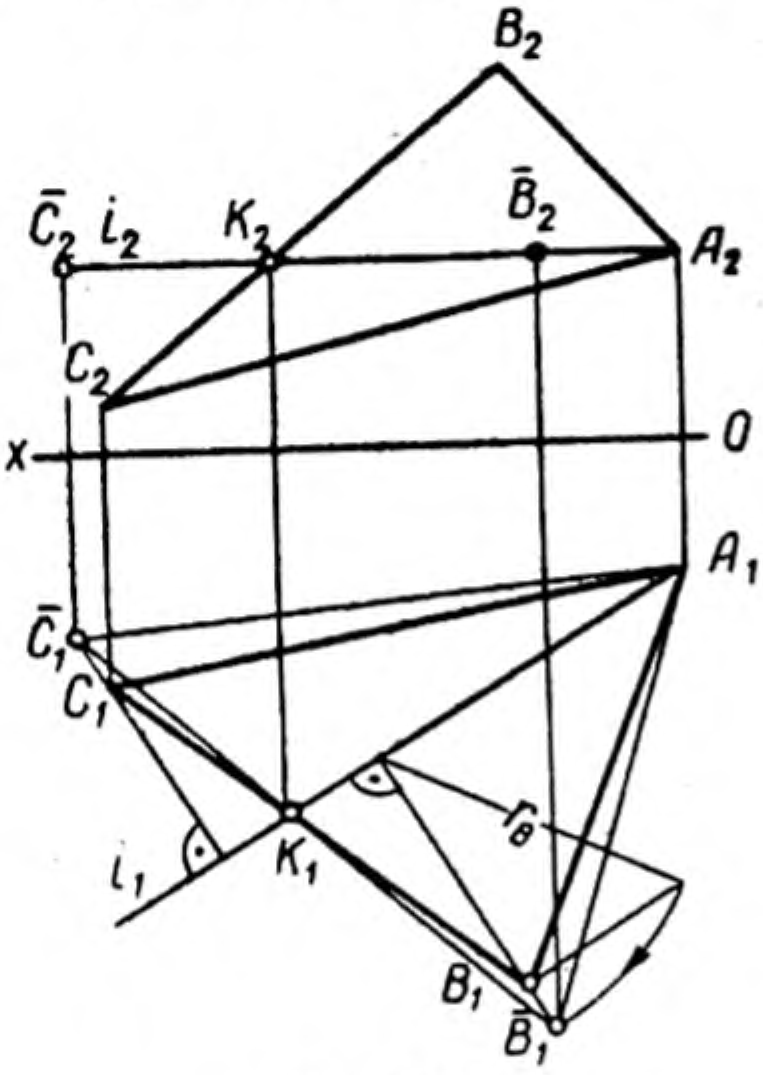


Fig. 265

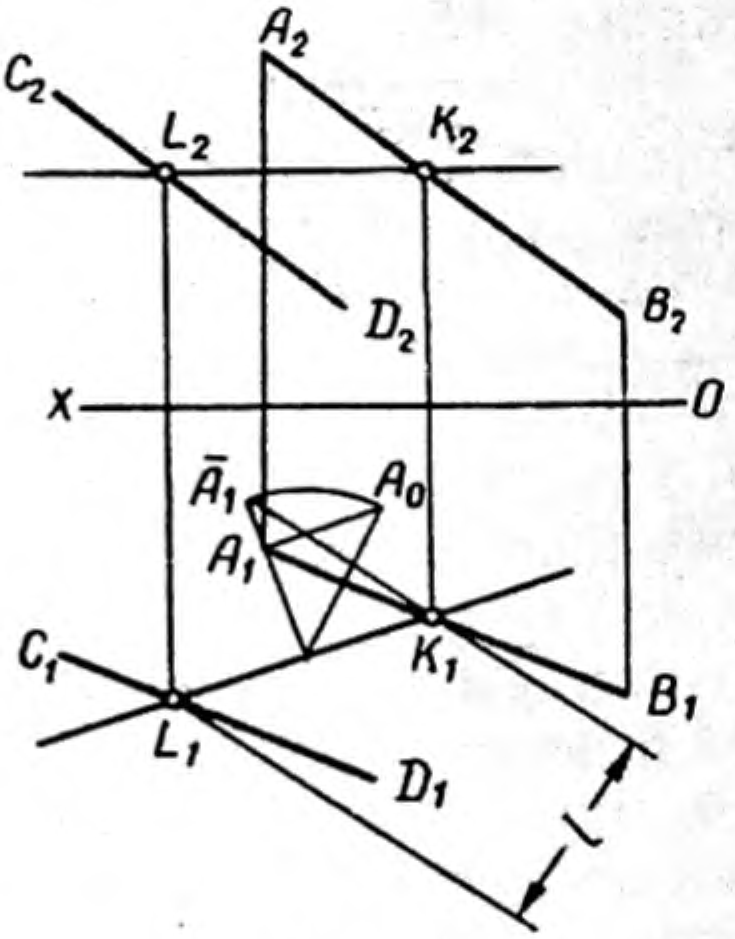


Fig. 266

Point $\overline{C}_1$, which is the horizontal projection of vertex C of the triangle ABC when it is parallel to plane Π_1, is found by drawing a line through points $\overline{B}_1$ and K_1 to intersect the perpendicular drawn from C_1 to i_1.

The vertex A of the triangle remains stationary since it is a point on the axis of revolution. Its projection A_1 is joined with projections $\overline{B}_1$ and $\overline{C}_1$ by straight lines. This gives the horizontal projection $A_1\overline{B}_1\overline{C}_1$ of triangle $A\overline{B}\overline{C}$ when it is parallel to plane Π_1.

Example. Determine the distance between the parallel lines AB and CD (Fig. 266).

Draw the horizontal KL of the plane determined by the lines AB and CD. About this horizontal lines AB and CD should be rotated so that they become parallel to plane Π_1.

Projection $\overline{A}_1$ of point A is found by determining the radius r_A. A straight line is drawn through $\overline{A}_1$ and K_1. This is the horizontal projection of line AB after rotation. Then, through projection L_1 of the stationary point L a second line is drawn parallel to line $\overline{A}_1K_1$. The distance l between these lines gives the true distance at which AB lies from CD.

§ 28. The Method of Coincidence

1. The method of coincidence is a particular case of the method of revolution. It is used to bring geometrical elements, points, lines and planes into coincidence with one of the planes of projection by means of revolution about an axis contained in that plane of projection.

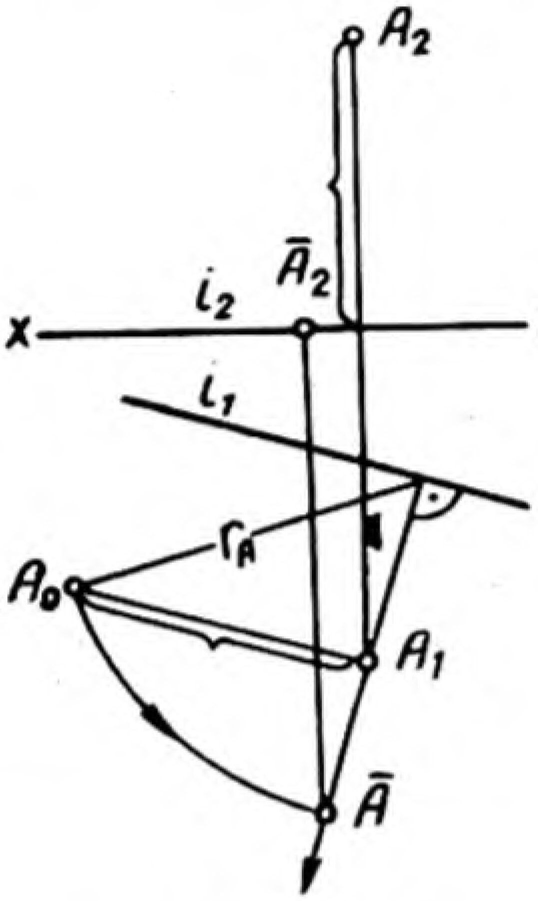


Fig. 267

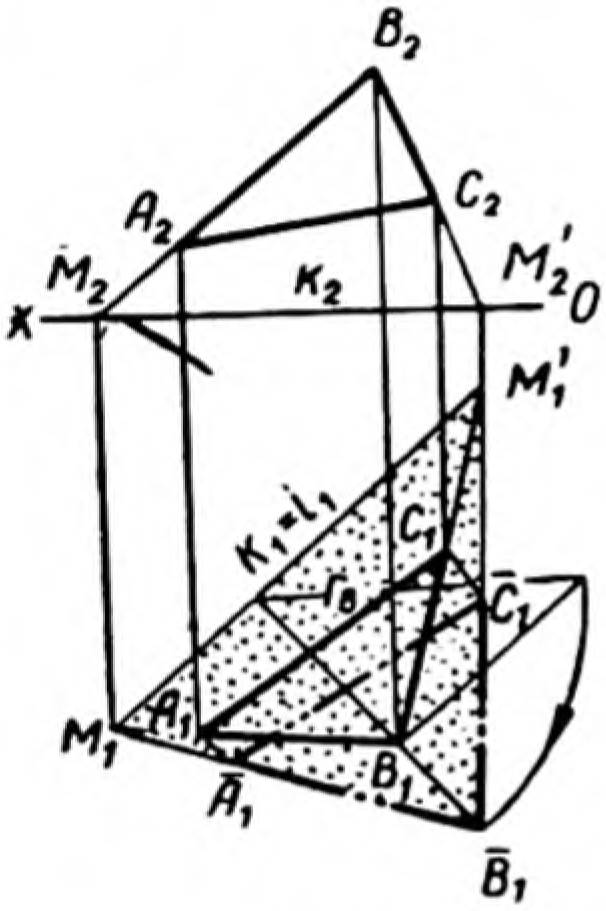


Fig. 268

In the drawing (Fig. 267) it is seen that point A is brought into coincidence with the horizontal plane of projection Π_1 by revolving it about axis i contained in that plane. Point $\bar{A}$ is the position of point A in Π_1 and it lies on a perpendicular drawn from A_1 to i_1. Its distance from axis i is equal to the radius of revolution r_A of point A. The radius r_A is obtained by the right-triangle method. The frontal projection $\bar{A}_2$ of A after it has been made to coincide with Π_1 is located on the ground line Ox.

2. The method of bringing a plane given as the triangle ABC into coincidence with plane Π_1 is shown in Fig. 268. The horizontal trace of this plane serves as the axis of revolution. By rotation about the axis k triangle MBM' is made to lie in plane Π_1. When carrying out this construction only the vertex B has to be brought into coincidence with Π_1, since the vertices M and M' are themselves contained in plane Π_1. Having found point $\bar{B}_1$ join it by straight lines to M_1 and M_1'. This gives triangle $M_1\bar{B}_1M_1'$ which coincides with Π_1. Triangle $\bar{A}\bar{B}\bar{C}$ is part of triangle $M_1\bar{B}_1M_1'$. To find this triangle it is sufficient to draw from points A_1 and C_1 lines $A_1\bar{A}_1$ and $C_1\bar{C}_1$ perpendicular to the projection k_1 of the axis of rotation. These lines intersect the sides $M_1\bar{B}_1$ and $M_1'\bar{B}_1$ of triangle $M_1\bar{B}_1M_1'$

which has been brought into coincidence wih Π_1; points $\overline{A}_1$ and $\overline{C}_1$ obtained are then joined by the straight line $\overline{A}_1\overline{C}_1$.

3. The construction required in order to bring planes given by traces into coincidence with the plane of projection Π_1 is shown in Figs. 269 and 270.

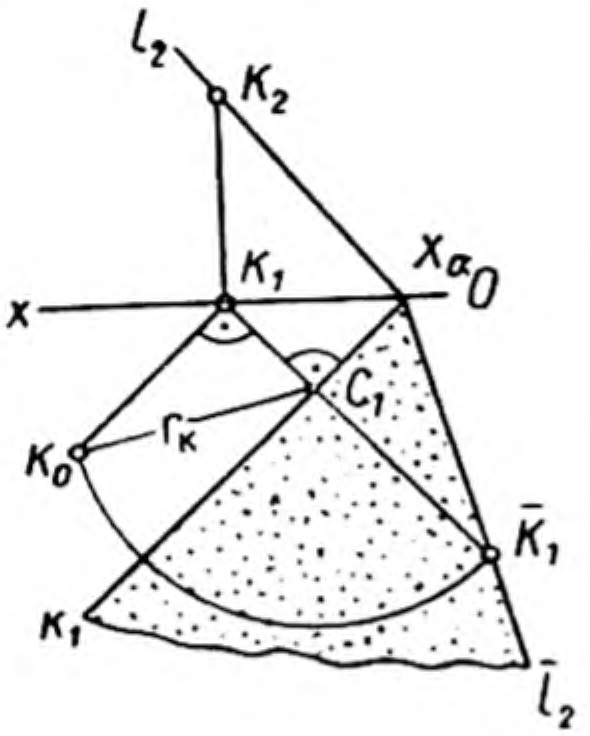


Fig. 269

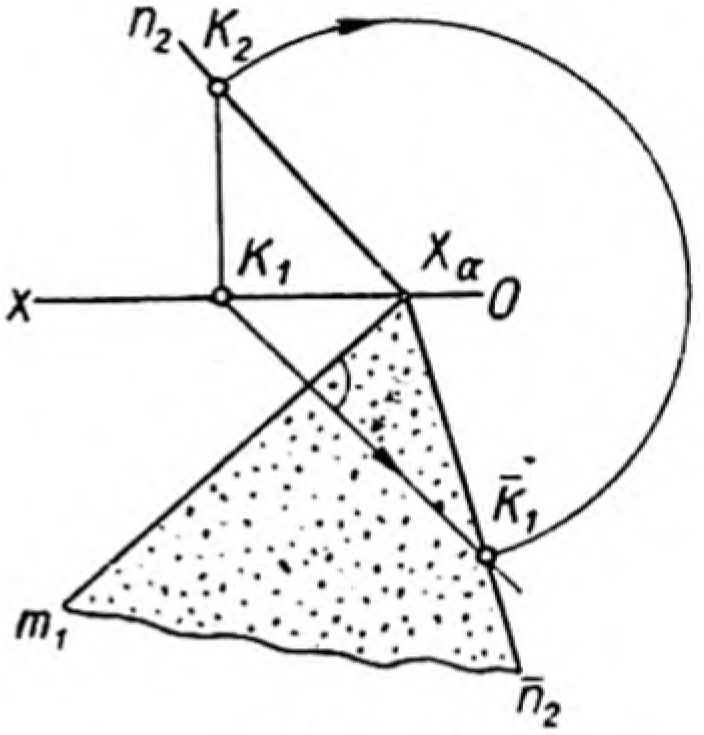


Fig. 270

In Fig. 269 the plane α is made to coincide with plane Π_1 by revolving it about the horizontal trace k_1.

To construct the frontal trace l_2 of plane α, which coincides with plane Π_1, it is sufficient to make one of its points coincide with Π_1, for example point K. The other point through which trace $\overline{l}_2$ will pass is X_α, the point of convergence of the traces k_1 and l_2. The projection K_1 will move when the point K is revolved and moves along line $K_1\overline{K}_1$ perpendicular to the axis of revolution k_1.

The position of point K when made to coincide with plane Π_1 is $\overline{K}_1$. This point lies at a distance from C_1 equal to the radius of revolution r_K of point K. The radius itself is found by the right triangle method.

Point $\overline{K}_1$ is then joined with point X_α of line $\overline{l}_2$ which is the frontal trace of plane α when it coincides with Π_1.

In Fig. 270 the point $\overline{K}_1$ on the frontal trace of plane α after being brought into coincidence with Π_1 has been found by another method. The distances from point X_α to points K_2 and $\overline{K}_1$ are equal, since they represent true values of one and the same segment. Therefore from point K_1 a perpendicular to the horizontal trace k_1 of plane α is drawn. This perpendicular is then cut in point $\overline{K}_1$ by an arc of radius $r = X_\alpha K_2$ with point X_α as its centre.

Such a simplified construction is possible only when the traces of the given planes intersect on the ground line Ox within the limits of the drawing.

4. In order to bring the frontal trace l_2 of plane α which is parallel to ground line Ox into coincidence with plane Π_1 (Fig. 271), a point A is taken on trace l_2 and brought into coincidence with plane Π_1 by the right triangle method. In the new position of the plane trace $\overline{l}_2$ is parallel to trace k_1 and passes through point $\overline{A}_2$ which lies in plane Π_1.

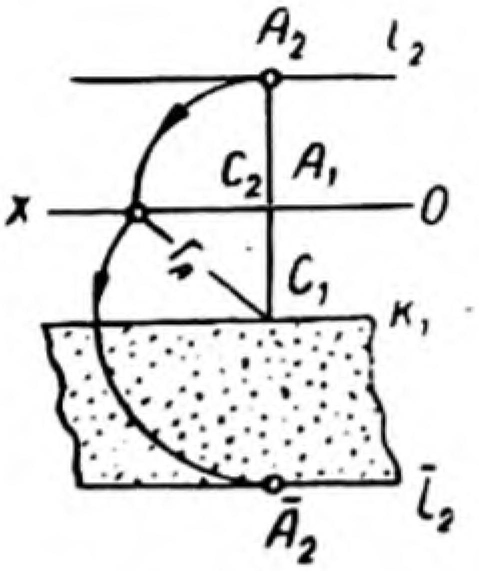


Fig. 271

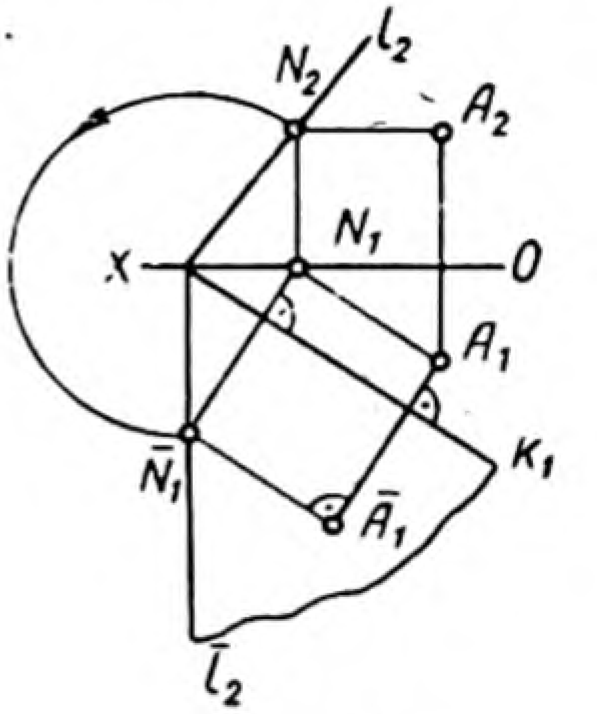


Fig. 272

5. A point contained in a given plane may be brought into coincidence with a plane of projection with the help of an auxiliary line drawn in the given plane through the point. A horizontal or frontal is generally chosen as the auxiliary line.

The construction required to bring point A contained in plane α into coincidence with plane Π_1 is shown in Fig. 272.

Through point A in plane α the horizontal AN is drawn. The frontal trace $N \equiv N_2$ of this horizontal is made to coincide with plane Π_1. Through point $\overline{N}_1$ and point of convergence of traces X_α the trace $\overline{l}_2$ of plane α, when it coincides with Π_1, is drawn. The horizontal $\overline{N}_1\overline{A}_1$ is drawn through point $\overline{N}_1$ parallel to k_1 This line is coincident with Π_1. Point $\overline{A}_1$ lies on this line at its intersection with the perpendicular drawn from A_1 to k_1 or, what is the same thing, to $\overline{N}_1\overline{A}_1$.

This construction could be carried out by using a frontal lying in plane α, as shown in Fig. 273.

The same number of graphical operations is required to determine point $\overline{A}_1$ by the methods, shown in Figs. 272 and 273. Nevertheless when bringing several points into coincidence with plane Π_1 it is advisable to use a frontal as the auxiliary line, because after the first point has been determined in its coincided position, the rest of the construction requires less graphical work and is simpler than when a horizontal is used (see Fig. 273).

It also follows that, when bringing a number of points of a given plane into coincidence with the frontal plane of projection Π_2, it is better to use a horizontal of the plane as the auxiliary.

6. The method of coincidence is very convenient when constructing a figure of given dimensions and form in an oblique plane. The given plane may be brought into coincidence with one of the planes of projection Π_1 or Π_2. The required figure is then constructed as a true view coinciding with the plane of projection. The given plane with the figure on it is returned to its original position. This gives the projections of the figure on Π_1 and Π_2. Such an operation is called the *restoration* of points to the plane.

It is required to construct in plane α a regular hexagon with sides equal in length to line h (Fig. 274).

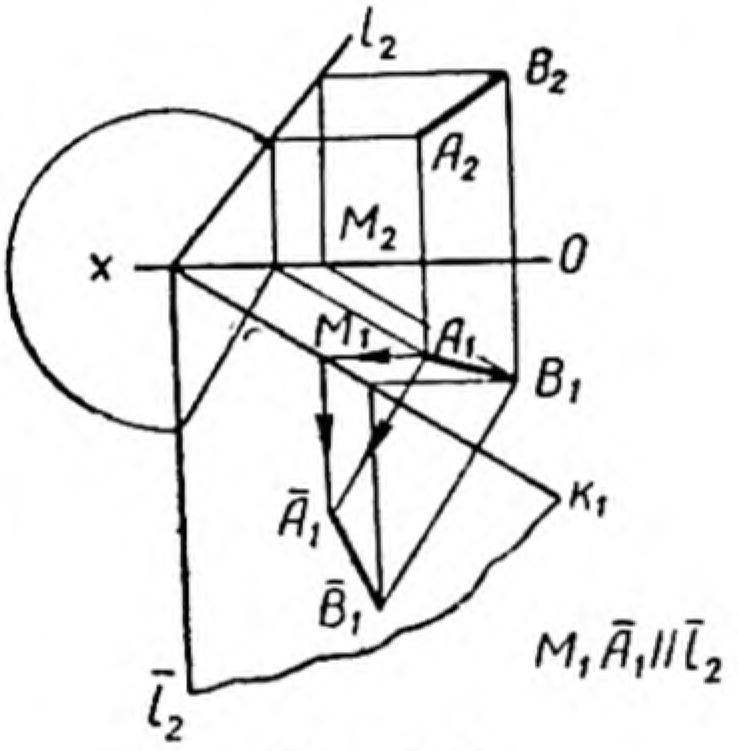


Fig. 273

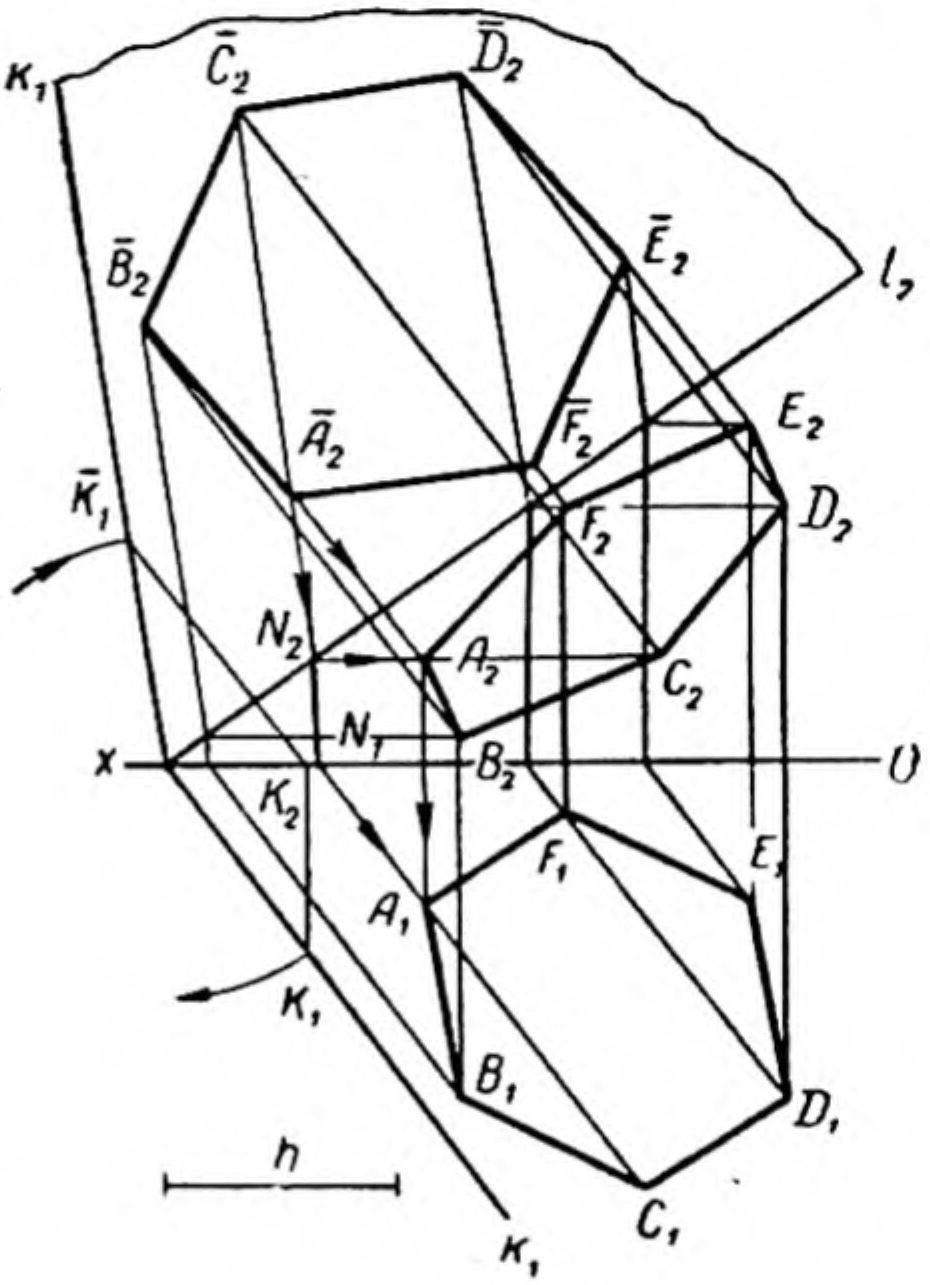


Fig. 274

After making plane α coincide with Π_2 with the help of point K taken on trace k_1 on plane α in its new position construct the regular hexagon $\bar{A}_2\bar{B}_2\bar{C}_2\bar{D}_2\bar{E}_2\bar{F}_2$ with its sides equal in length to h.

After this, with the help of horizontals in the coincided plane $\bar{\alpha}$ drawn parallel to the horizontal trace $\bar{k}_1$ of the plane and passing through the vertices of the hexagon we construct the frontal projection $A_2B_2C_2D_2E_2F_2$ and the horizontal projection $A_1B_1C_1D_1E_1F_1$ of the figure. The construction of the frontal and horizontal projections of one vertex, say point A, is as follows. Through point $\bar{A}_2$ parallel to trace $\bar{k}_1$ draw the horizontal $\bar{A}_2N_2$ coincided with Π_2. Through N_2 draw its frontal projection N_2A_2 parallel to ground line Ox. The projection A_2 of point A is found by passing a perpendicular from $\bar{A}_2$ to trace l_2 and extending it to intersect the frontal projection of the horizontal. Once N_2 has been found, the projection N_1 may be determined. Then through N_1 the horizontal projection of the horizontal is drawn and on it, with the help of a projector, the horizontal projection A_1 of point A is determined. The other five vertices of the hexagon $ABCDEF$ are found in a similar way.

The horizontal and frontal projections of a hexagon situated in plane α are found by joining the corresponding points by straight lines.

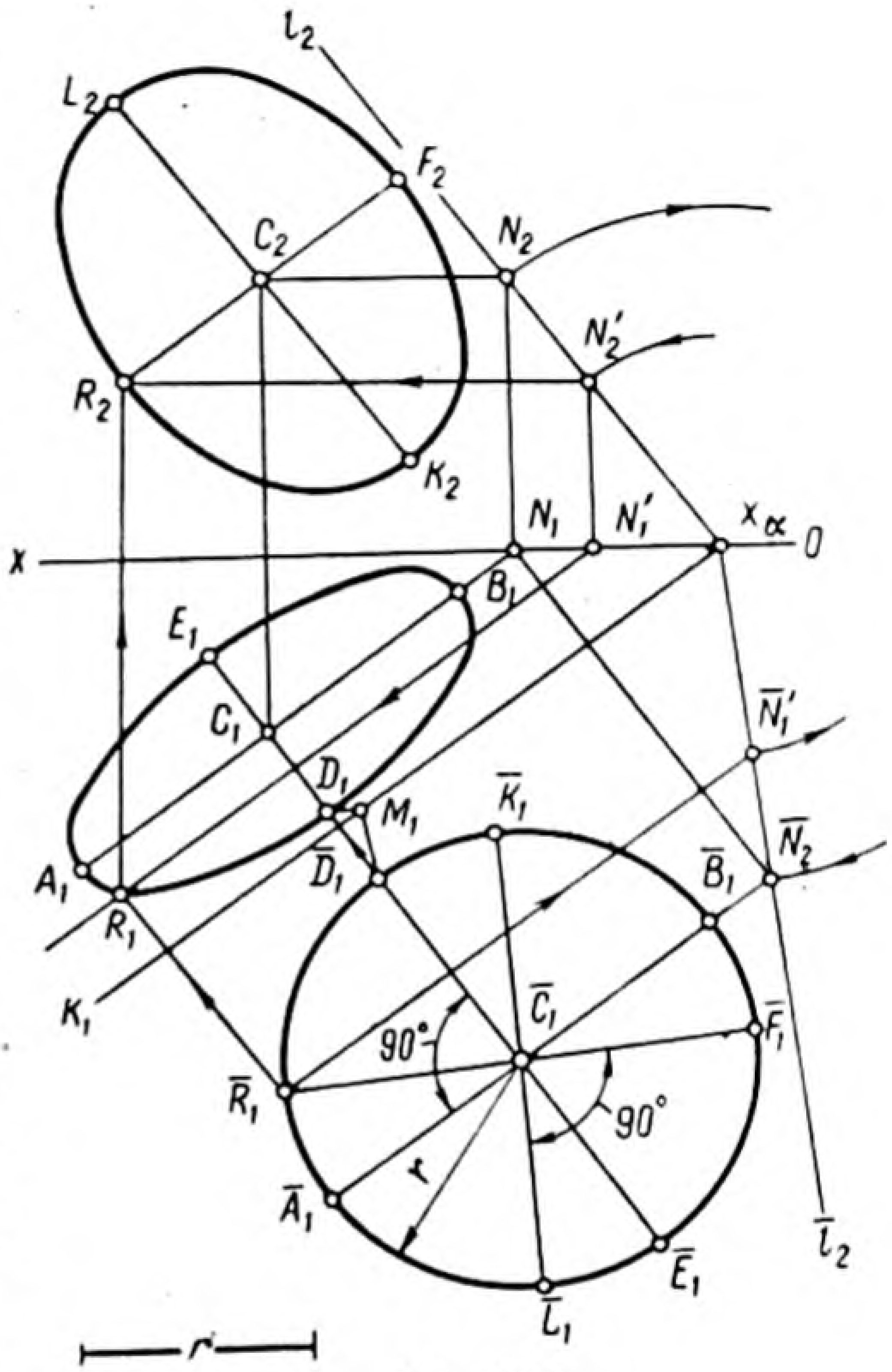


Fig. 275

7. Construct in plane α a circle of a radius $r = 17$ mm with its centre at point C (Fig. 275).

A horizontal CN is drawn in plane α through point C. The frontal trace N of this horizontal is then made to coincide with plane Π_1. After that through points X_α and $\overline{N}_2$ the frontal trace $\overline{l}_2$ of plane α coinciding with Π_2. is drawn. Through $\overline{N}_2$ parallel to k_1 a horizontal in coincidence with Π_1 is drawn. Next, from point C_1 a perpendicular is dropped to trace k_1 of plane α to intersect this coincided horizontal in point $\overline{C}_1$. From point $\overline{C}_1$ as the centre, a circle of radius r is described. Two pairs of mutually perpendicular diameters, $\overline{A}_1\overline{B}_1 \perp \overline{D}_1\overline{E}_1$ and $\overline{K}_1\overline{L}_1 \perp \overline{R}_1\overline{F}_1$, are drawn, diameter $\overline{A}_1\overline{B}_1$ being parallel to trace k_1 and diameter $\overline{K}_1\overline{L}_1$ parallel to trace $\overline{l}_2$.

Since diameter AB of the circle is parallel to plane Π_1, it will be projected onto Π_1 in its true length. Therefore A_1B_1 ($A_1B_1 = \overline{A}_1\overline{B}_1$) is the major axis of the ellipse which is the projection of the circle on Π_1. The diameter KL parallel to plane Π_2 determines the position and length of the major axis of the ellipse which is the frontal projection of the circle. The minor axes of these ellipses are perpendicular to the major axes and are determined by the projections of the diameters DE and RF.

After restoration of points of the circle from the coincided position to plane α points A_1 and B_1 will lie on the horizontal projection of the horizontal CN, each at a distance equal to radius

r from point C_1. Points K_2 and L_2 will lie on the frontal projection of the frontal drawn through point C_2, also at a distance equal to radius r from point C_2.

Projections A_1B_1 and K_2L_2 are equal in length to diameter of the circle. The projections of the diameters DE and RF on planes Π_1 and Π_2 will be perpendicular to the corresponding projections of diameters AB and KL, namely, $D_1E_1 \perp A_1B_1$ and $R_2F_2 \perp K_2L_2$.

To find the projection D_1E_1, which is the minor axis of the ellipse in plane Π_1, it is sufficient to construct the horizontal projection of only one extreme point such as D of the diameter DE. The length of the minor axis of the ellipse in plane Π_2 may be determined by restoring to plane α one of the ends of the diameter RF, for example point R, from the coincided position.

Point D_1 may be found with the help of frontal DM, $\overline{D}_1M_1$ being parallel to $\overline{l}_2$. Point E_1 is obtained by marking distance C_1D_1 on the minor axis D_1E_1 but on the opposite side of major axis A_1B_1.

The point R_2 in plane Π_2 is constructed with the help of the horizontal RN. Point F_2 lies on the opposite side of the major axis of ellipse K_2L_2 at the same distance from point C_2 as the point R_2.

Once the axes have been found the ellipses, which are the complete horizontal and frontal projections of a circle of radius r contained in plane α, may be drawn.

§ 29. Revolution with Parallel Transfer of Projections

Sometimes the views obtained by revolving projections about axes perpendicular to planes of projection are found to lie one on top of the other. This can be avoided by applying another method (see Fig. 276). In this drawing by means of two successive revolutions about axes respectively perpendicular to planes of projection Π_1 and Π_2, the plane of triangle ABC is made to lie parallel to the horizontal plane of projection Π_1.

The triangle is brought into the position ABC perpendicular to plane Π_2 by the first revolution. The construction is carried out with the help of the horizontal CF which is rotated about an axis perpendicular to plane Π_1 until it is perpendicular to the frontal plane of projection Π_2.

It is convenient to shift the horizontal projection $\overline{A}_1\overline{B}_1\overline{C}_1$ to the right side of the drawing where there is space. This corresponds to rotating it about an axis perpendicular to the plane of projection Π_1 and changes neither its form nor size.

The frontal projection $\overline{A}_2\overline{B}_2\overline{C}_2$ is then obtained in the usual way, i. e., from points $\overline{A}_1$, $\overline{B}_1$, $\overline{C}_1$ projectors are drawn perpendicular to ground line Ox to intersect the corresponding lines drawn through points A_2, B_2, C_2 parallel to ground line Ox. The points A_2, B_2 and $\bar{C}_2$ are then joined by the straight line $\bar{A}_2\bar{B}_2\bar{C}_2$.

By means of a second revolution about an axis perpendicular to plane Π_2 the triangle $\bar{A}\bar{B}\bar{C}$ is brought to the position $\bar{\bar{A}}\bar{\bar{B}}\bar{\bar{C}}$ parallel to plane Π_1. The frontal projection $\bar{\bar{A}}_2\bar{\bar{B}}_2\bar{\bar{C}}_2$ is then parallel to ground line Ox.

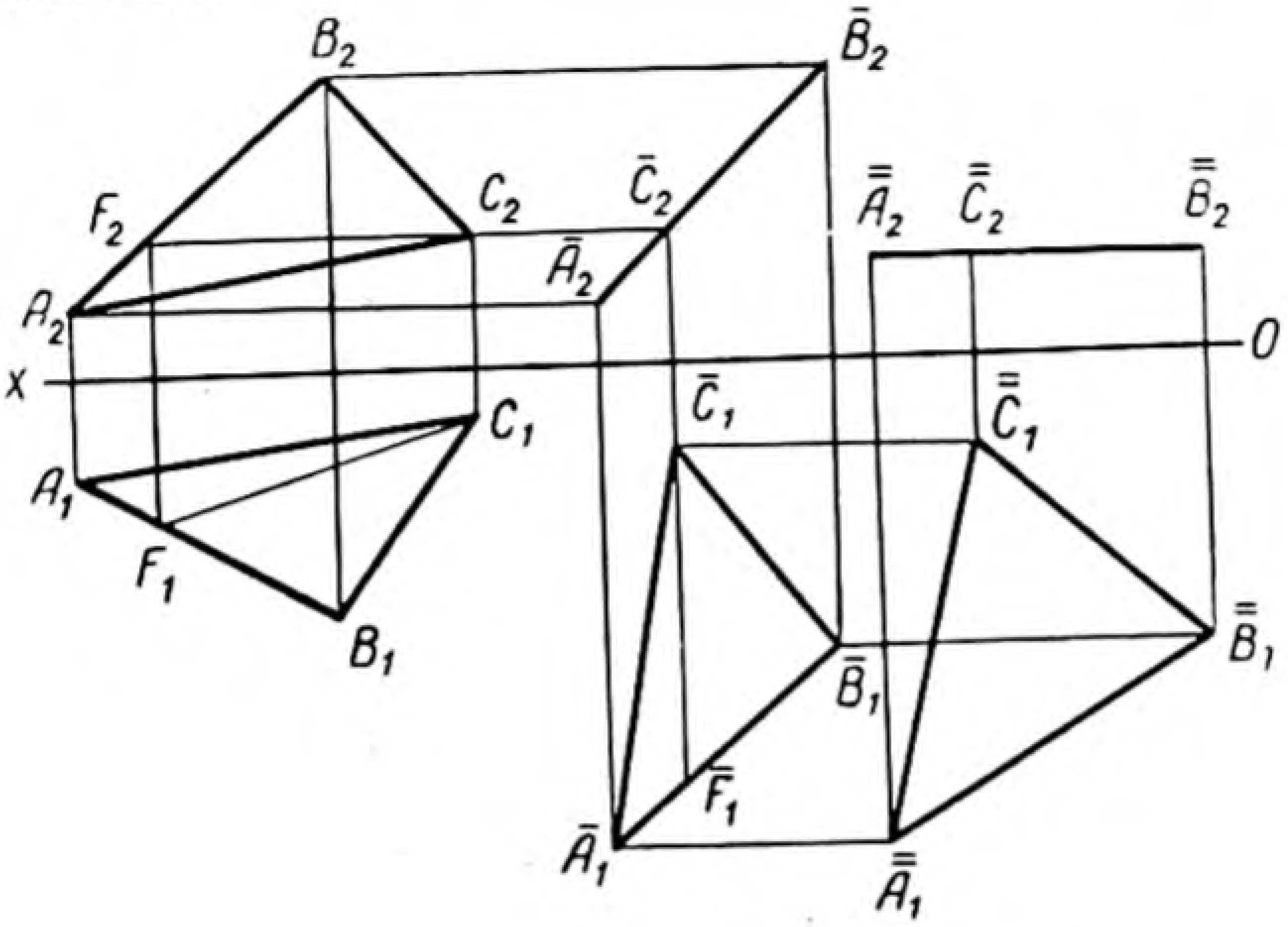


Fig. 276

This projection $\bar{\bar{A}}_2\bar{\bar{C}}_2\bar{\bar{B}}_2$ is shifted by a parallel transfer to a convenient position at the right of the drawing. Projectors are drawn through points $\bar{\bar{A}}_2$, $\bar{\bar{B}}_2$ and $\bar{\bar{C}}_2$ and through points $\bar{A}_1$, $\bar{B}_1$, $\bar{C}_1$ lines parallel to ground line Ox are passed. Their intersections give points $\bar{\bar{A}}_1$, $\bar{\bar{B}}_1$ and $\bar{\bar{C}}_1$. Triangle $\bar{\bar{A}}_1\bar{\bar{B}}_1\bar{\bar{C}}_1$ is a true view of triangle ABC.

§ 30. Combined Transformations of Projections

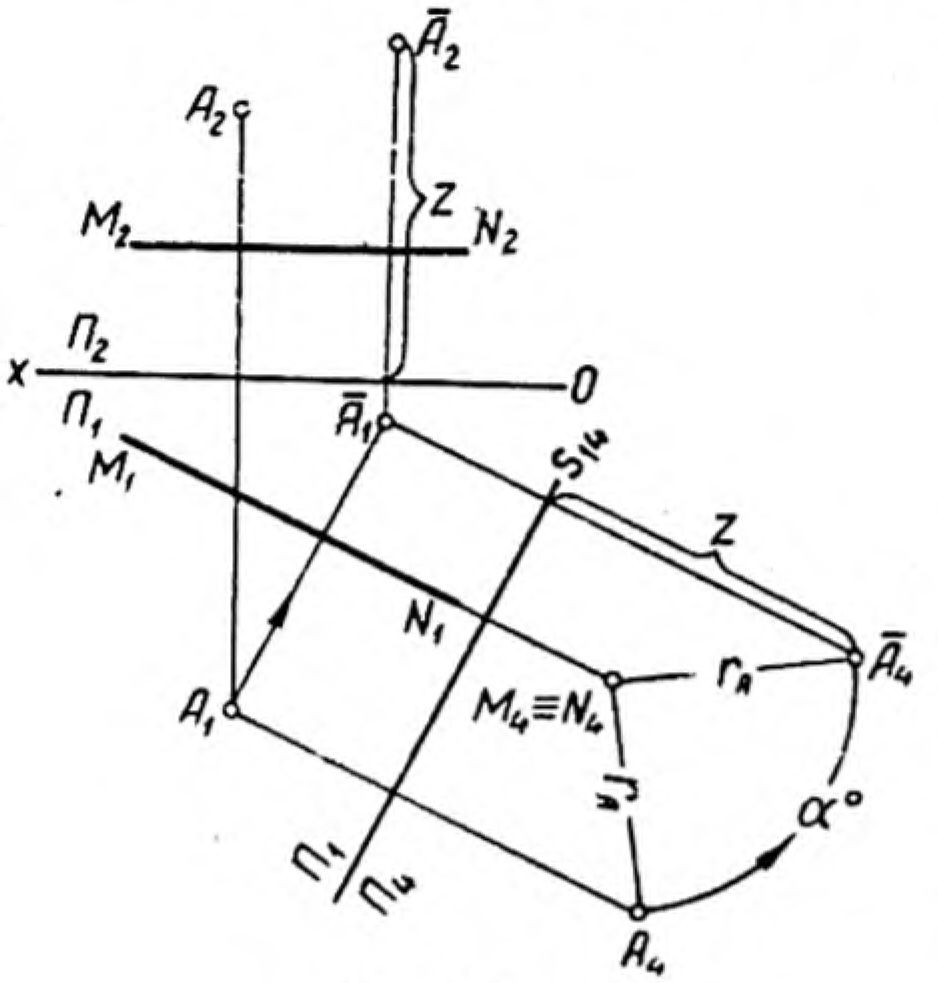
So far we have examined the revolution of various geometrical elements about axes perpendicular or parallel to one of the planes of projection, the rotation being made about axes either parallel to plane Π_1 or plane Π_2. Moreover, this rotation was carried out not through any definite angle but only up to the one particular position when the figure being drawn becomes parallel to a plane of projection.

Sometimes it becomes necessary to revolve geometrical elements through a definite angle about oblique axes or axes parallel to one of the planes of projections.

In such cases it is common practice to resort to combined transformations. The axis of revolution in that case is first positioned perpendicular to one of the planes of projection. If we revolve a geometrical element through a given angle about an axis in such a position we find the projection of the element in the system of principal planes of projection Π_1-Π_2.

Let us consider some examples.

Example 1. It is required to revolve point A about axis MN parallel to plane Π_1 through angle $\alpha = 90°$ in a counter-clockwise direction (Fig. 277).



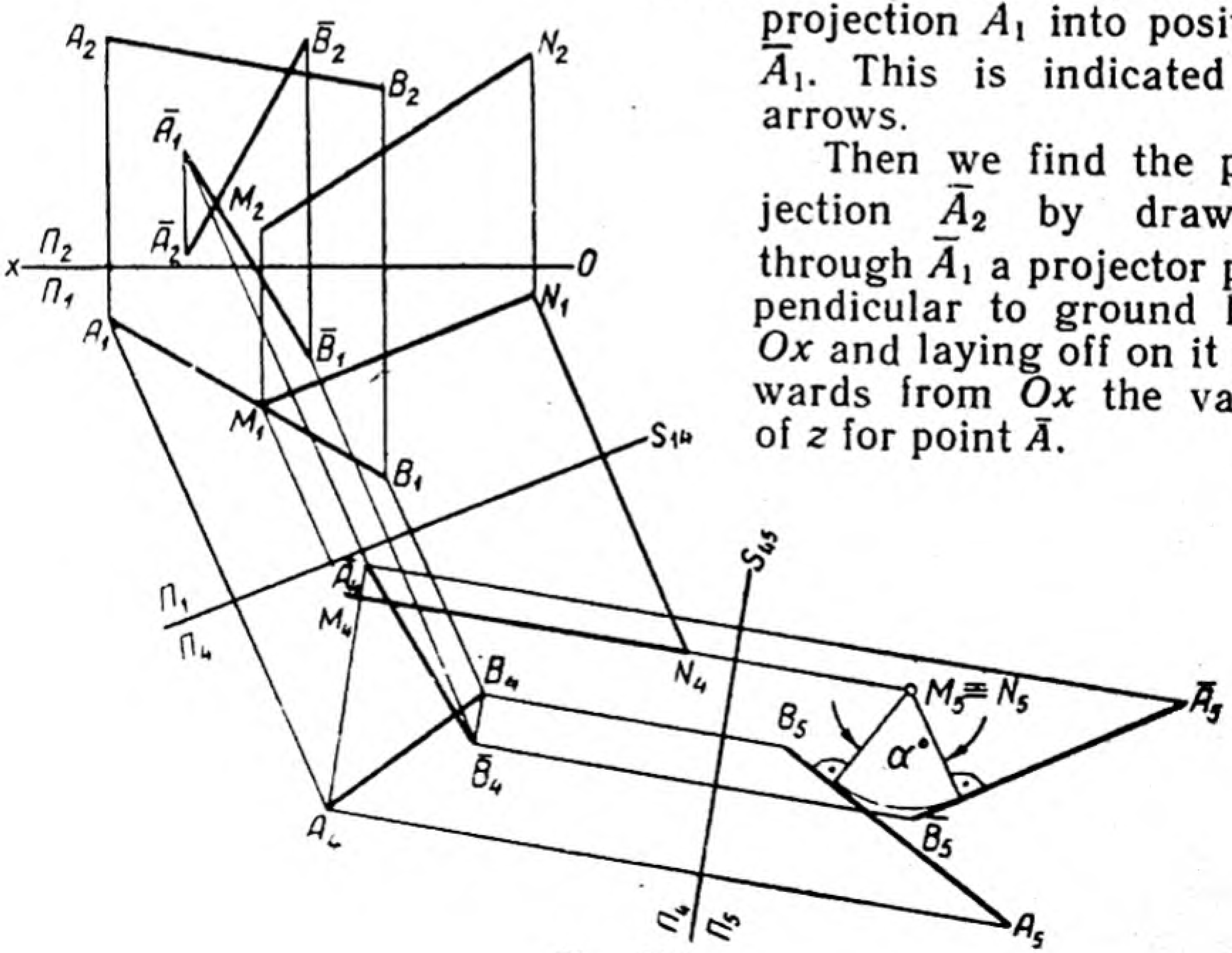
$\bar{A}_2$
A_2
Z
M_2
N_2
Π_2
x
0
Π_1
$\bar{A}_1$
M_1
S_{14}
Z
N_1
$\bar{A}_4$
r_A
A_1
$M_4 \equiv N_4$
r_A
$\alpha°$
Π_1
Π_4
A_4

Fig. 277

First, the plane of projection Π_2 for Π_4 is changed so that the axis of revolution is projected onto plane Π_4 as a point. To ensure this plane Π_4 is drawn perpendicular to both line MN and plane Π_1. This is possible since MN is parallel to Π_1. The axis MN is projected onto plane Π_4 as point $M_4 \equiv N_4$, and point A as point A_4.

We then revolve point A through the given angle α in the system of planes of projection Π_1-Π_4. The projection A_4 will move into position $\bar{A}_4$ and the projection A_1 into position $\bar{A}_1$. This is indicated by arrows.

Then we find the projection $\bar{A}_2$ by drawing through $\bar{A}_1$ a projector perpendicular to ground line Ox and laying off on it upwards from Ox the value of z for point $\bar{A}$.



A_2
$\bar{B}_2$
N_2
B_2
$\bar{A}_1$
M_2
$\bar{A}_2$
x
Π_2
Π_1
0
A_1
N_1
$\bar{B}_1$
M_1
S_{14}
B_1
$\bar{A}_4$
M_4
Π_1
Π_4
S_{45}
B_4
N_4
$M_5 \equiv N_5$
$\bar{A}_5$
B_5
$\alpha°$
$\bar{B}_4$
$\bar{B}_5$
A_4
Π_4
Π_5
A_5

Fig. 278

The constructed projections $\bar{A}_1$ and $\bar{A}_2$ are the projections of the given point A after it has been transferred to the required position $\bar{A}$.

Example 2. Given two oblique lines AB and MN, it is required to revolve line AB about line MN through an angle $\alpha=60°$, clockwise (Fig. 278).

By successive substitution of the planes of projection Π_1 and Π_2 by the planes Π_4 and Π_5 we transform the projection of the axis of revolution MN into a point contained in the system of planes of projection Π_4-Π_5. Line AB should also be constructed in this system of planes of projection.

Having done that we revolve line AB about axis $MN \perp \Pi_5$ through the angle α. Lines $\bar{A}_5\bar{B}_5$ and $\bar{A}_4\bar{B}_4$ are projections on Π_5 and Π_4 of line AB when it is turned into the required position $\bar{A}\bar{B}$. The projections $\bar{A}_1\bar{B}_1$ and $\bar{A}_1\bar{B}_2$ of line $\bar{A}\bar{B}$ in the given system of planes of projection Π_1-Π_2 are found by means of reverse projections.

§ 31. The Use of Related Conformity in Transforming Projections

1. It is possible to establish a definite relationship between the position of a geometrical element when it has been brought into coincidence with a plane of projection and the projection of that element on the given plane of projection.

For example, in Fig. 275 the horizontal projection C_1 corresponds to point $\bar{C}_1$ which coincides with plane Π_1. Projection A_1B_1 corresponds to the coincided diameter $\bar{A}_1\bar{B}_1$. The ellipse $A_1B_1D_1E_1$ corresponds to the circle with centre $\bar{C}_1$. Line D_1M_1 corresponds to line $\bar{D}_1\bar{M}_1$ and so on.

This relationship is called *related conformity* and is studied in the course of projective geometry, in which it is proved that between the geometrical element itself and each of its orthographic projections and also between the orthographic projections of the element on planes Π_1 and Π_2, a related conformity may always be established.

The following invariants exist in related conformity (see Fig. 275):

1) One and only one point of a plane corresponds to each point of another plane. For example, only point C_1 corresponds to point $\bar{C}_1$.

2) To each line contained in a plane there is only one corresponding line in another plane. For example, only line D_1M_1 corresponds to line $\bar{D}_1\bar{M}_1$.

3) Each pair of related lines intersects on a line called the *axis*

of alliance or the *axis of conformity*. Points contained in this axis are called *duplicate points* since they correspond to themselves.

4) The duplicate point of two related lines parallel to the axis of conformity lies at infinity (for example, $\overline{A}_1\overline{B}_1$ and A_1B_1).

5) For a point contained in a line there is one corresponding point contained in any other related line. For example, point C_1 on line A_1B_1 corresponds to point $\overline{C}_1$ on line $\overline{A}_1\overline{B}_1$.

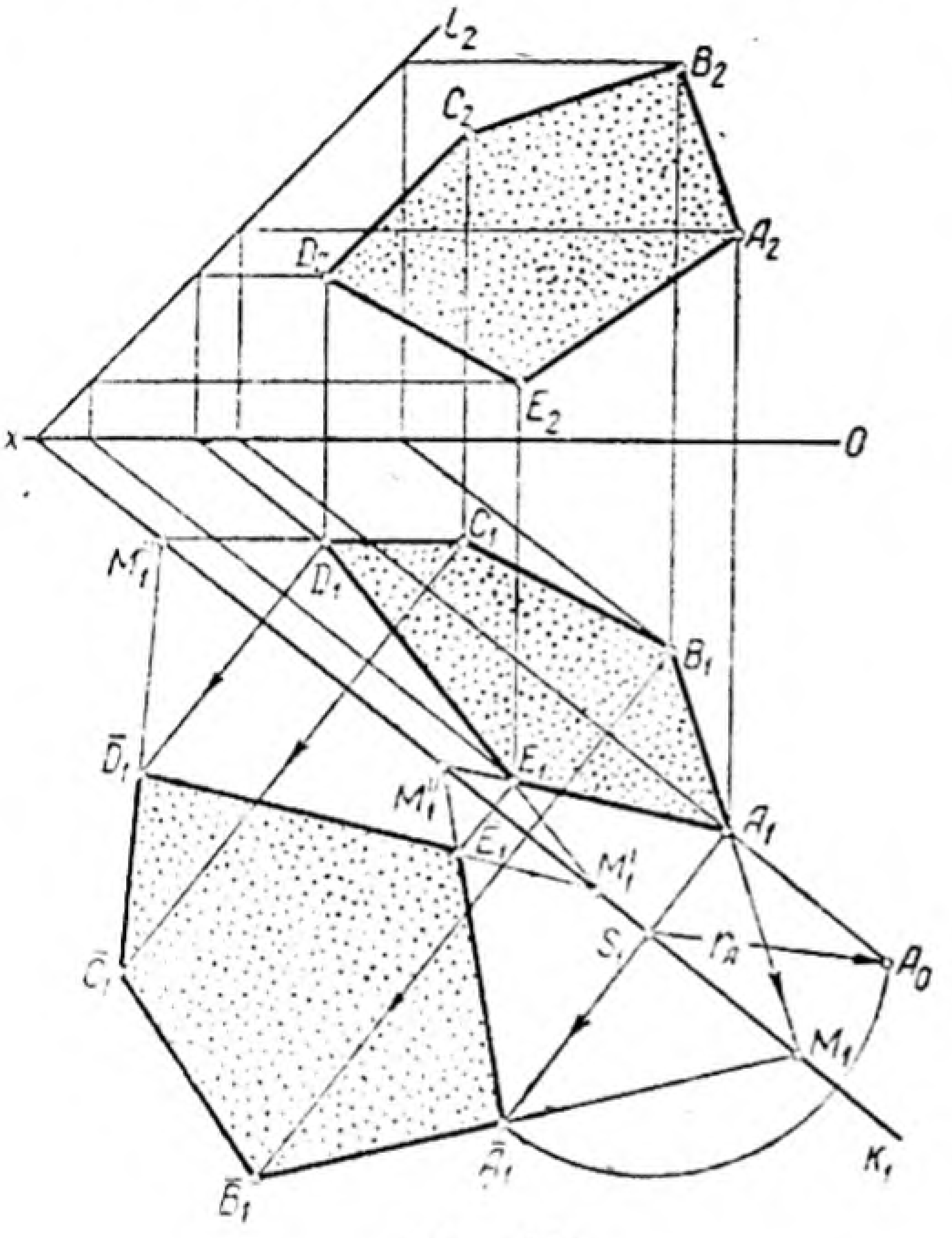


Fig. 279

6) Two parallel lines in one plane correspond to two parallel lines in another plane.

7) If a part of a line in one plane is divided into some ratio by a point, the part of a line related to it in another plane is divided in the same ratio by the point corresponding to the first point.

2. The use of related conformity for the transformation of projections allows a considerable decrease in the number of auxiliary lines in the drawing as compared with the more usual constructions.

For example, it is required to find the true size of the pentagon *ABCDE* contained in plane α (Fig. 279). When constructing this pentagon, after it has been made to coincide with plane Π_1, we shall use the properties of related conformity.

Vertex A of the pentagon is made to coincide with plane Π_1 by revolution about trace k_1 of plane α. This is done in the usual way, i. e., with the help of right triangle $S_1A_1A_0$, the true size of the radius of revolution r_A of point A is determined. This length is laid off from point S_1 on the perpendicular to k_1. We then obtain $\overline{A}_1$, that is the position of point A when it has been made to coincide with plane Π_1.

Having assumed trace k_1 as the axis of conformity of the figure $A_1B_1C_1D_1E_1$ to be transformed into the related figure $\overline{A}_1\overline{B}_1\overline{C}_1\overline{D}_1\overline{E}_1$ in coincidence with plane Π_1 and assuming that point $\overline{A}_1$ corresponds to point A_1, the subsequent construction is carried out by using related conformities.

Extend line A_1B_1 to intersect the axis of conformity k_1 in the duplicate point M_1. Join M_1 with point $\bar{A}_1$ by line $\bar{A}_1M_1$ which is related to line A_1M_1. Draw a perpendicular from B_1 to k_1, i. e., in the direction of conformity of $B_1\bar{B}_1$. The intersection of this perpendicular with line A_1M_1 gives point $\bar{B}_1$ related to point B_1.

Point $\bar{E}_1$ related to point E_1 is constructed in a similar manner with the help of line A_1M_1'' and the line $\bar{A}_1\bar{M}_1''$ related to it. Point $\bar{E}_1$ is found at the intersection of a perpendicular drawn form E_1 to k_1 with line $\bar{A}_1M_1''$.

By similar constructions points $\bar{D}_1$ and $\bar{C}_1$ are also found. The pentagon $\bar{A}_1\bar{B}_1\bar{C}_1\bar{D}_1\bar{E}_1$ constructed in this way is related to pentagon $A_1B_1C_1D_1E_1$ and is the pentagon $ABCDE$ when it is brought into coincidence with plane Π_1.

Problems and Exercises

1. In what type of projecting plane does a point move when it is revolved about a) a horizontal, b) a frontal?

2. Describe the procedure for constructing a multiview drawing of a point made to coincide with the frontal plane of projection by revolving it about a line contained in plane Π_2.

3. What line should be taken as the axis to revolve a plane figure: a) into plane Π_1, b) into plane Π_2?

4. Why is a horizontal of a plane located parallel to the horizontal trace of plane, and its frontal — parallel to its frontal trace after the plane has been rotated to lie in a plane of projection?

5. Given the position of a point rotated into a plane of projection, from an oblique plane, how can the projection of this point in the given oblique plane be determined?

6. Rotate vertex B of triangle ABC about the frontal AK until it lies at the same distance from plane Π_2 at which the frontal lies (see Fig. 252). (See § 27.)

7. Rotate point A about the line MN located in Π_2 into the frontal plane of projection (Fig. 280). (See § 27.)

8. Make the frontal-projecting plane α coincide with plane Π_2.

9. Make the plane α given by point A and ground line Ox coincide with plane Π_1.

10. In plane α draw a line CD at 20 mm from AB and parallel to AB which is contained in plane α (Fig. 281).

Hint. This problem may be solved by the method of coincidence (see § 28).

11. The trace l_2 of plane α is given. The angle between traces l_2 and k_1 in space is 75°. Construct in this plane an isosceles

triangle ABC with a base $AB=40$ mm and altitude $H=50$ mm (Fig. 282). (See Fig. 274 and related text.)

12. Construct a regular hexagonal pyramid with its base in a given plane α. The side of the base of the pyramid is 30 mm, the altitude — 50 mm. The centre of the base lies at point C (Fig. 283).

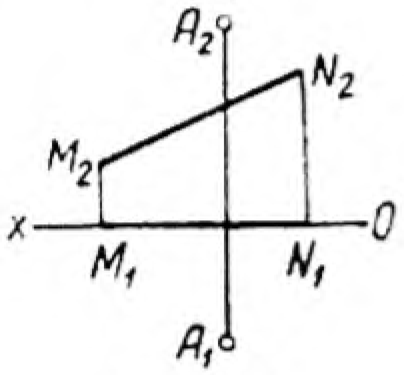


Fig. 280

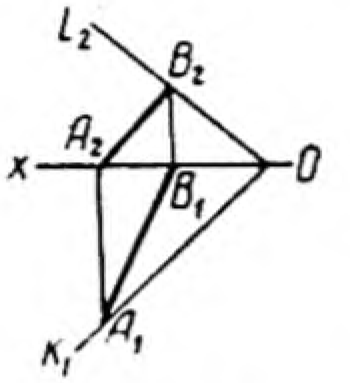


Fig. 281

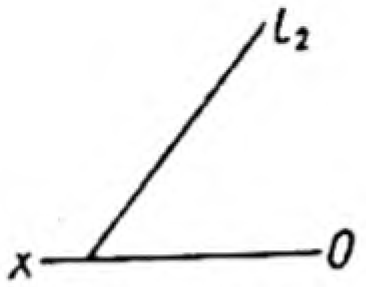


Fig. 282

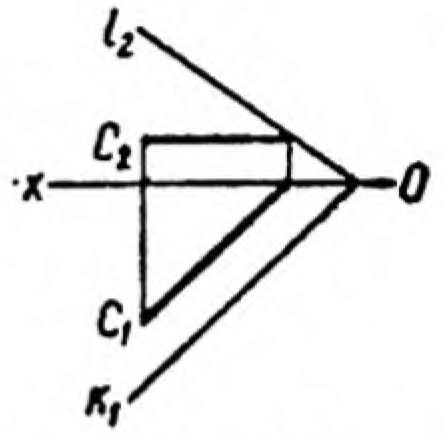


Fig. 283

Hint. When determining the projection of the vertex of the pyramid it is necessary to find the true length of any limited part of the line of altitude of the pyramid.

13. Find the true size of triangle ABC which lies in a plane parallel to ground line Ox and inclined to planes of projection Π_1 and Π_2.

14. On an oblique plane construct a cylinder the diameter of which is 40 mm and the altitude — 70 mm.

Hint. The solution is similar to that given in the hint to question 12.

Chapter VII

REPRESENTATIONS OF GEOMETRICAL BODIES AND THE DEVELOPMENTS OF SURFACES

§ 32. Multiview Drawings of Geometrical Bodies and Projections of Points and Lines on their Surfaces

1. Three projections of various geometrical bodies are shown in Fig. 284. These bodies are a tetragonal parallelepiped (Fig. 284, *a*), triangular and hexagonal prisms (Fig. 284, *b*, *c*), triangular and tetragonal pyramids (Fig. 284, *d*, *e*), the frustum of a tetragonal pyramid (Fig. 284, *f*), a cylinder (Fig. 284, *g*), a cone (Fig. 284, *h*), a sphere (Fig. 284, *i*), and an annular torus or anchor ring (Fig. 284, *j*).

All these bodies are depicted in the most convenient position determining their projections. That is to say, their axes, edges and faces are projected onto certain of the planes of projection in their true size and onto others as points and lines. Thus, for instance, all the vertical edges and the front face of the right parallelepiped (Fig. 284, *a*) are projected onto plane Π_2 in their true size and on plane Π_1 as points $A_1 \equiv A_1'$, $B_1 \equiv B_1'$, $C_1 \equiv C_1'$, $D_1 \equiv D_1'$, while the face is projected as line $A_1D_1 \equiv A_1'D_1'$.

When working out and reading the multiview drawings of geometrical bodies special attention should be paid to the conformity of the dimensions repeated in pairs on each of the three pairs of planes of projection, i. e., on Π_1 and Π_2 (parallel to ground line Ox), on Π_2 and Π_3 (parallel to line Oz) and on Π_1 and Π_3 (parallel to line Oy). For example, in Fig. 284, *c* the dimension h, taken in plane Π_1 parallel to ground line Oy_1, is repeated in plane Π_3 as the same dimension h parallel to ground line Oy_3. The altitude of the frontal and profile projections of the bodies, i. e., the dimensions parallel to line Oz in planes Π_2 and Π_3 are the same and so on.

The construction of multiview drawings of prisms, pyramids, cylinders and cones is usually begun on those planes of projection onto which their bases project in their true size, that is, as true view.

It should be taken into account that in a system of three planes

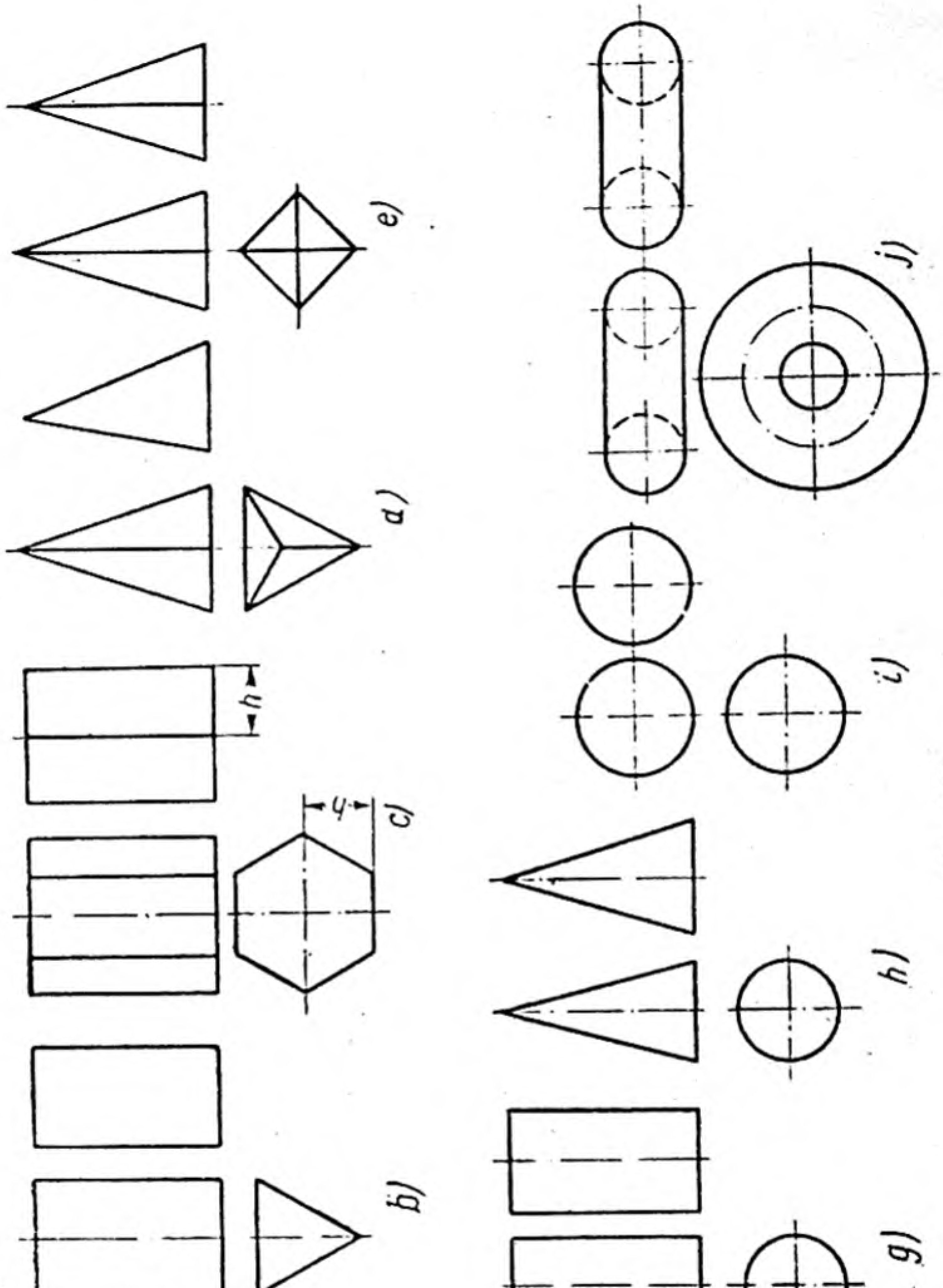

$A_2 \equiv B_2$
$C_2 \equiv D_2$
$A_2' \equiv B_2'$
$C_2' \equiv D_2'$
$B_1 \equiv B_1'$
$C_1 \equiv C_1'$
$A_1 \equiv A_1'$
$D_1 \equiv D_1'$
a)
b)
c)
h
h
d)
e)
f)
g)
h)
i)
j)

Fig. 284

of projection Π_1-Π_2-Π_3 a cylinder is projected onto two of those planes as equal rectangles, and the cone — as isosceles triangles. The sphere on all three planes is projected as a circle whose diameter is equal to the diameter of the sphere.

2. The construction of projections of points contained in the surface of a geometrical body, as a rule, is carried out with the help of lines drawn through these points on the surface of the body.

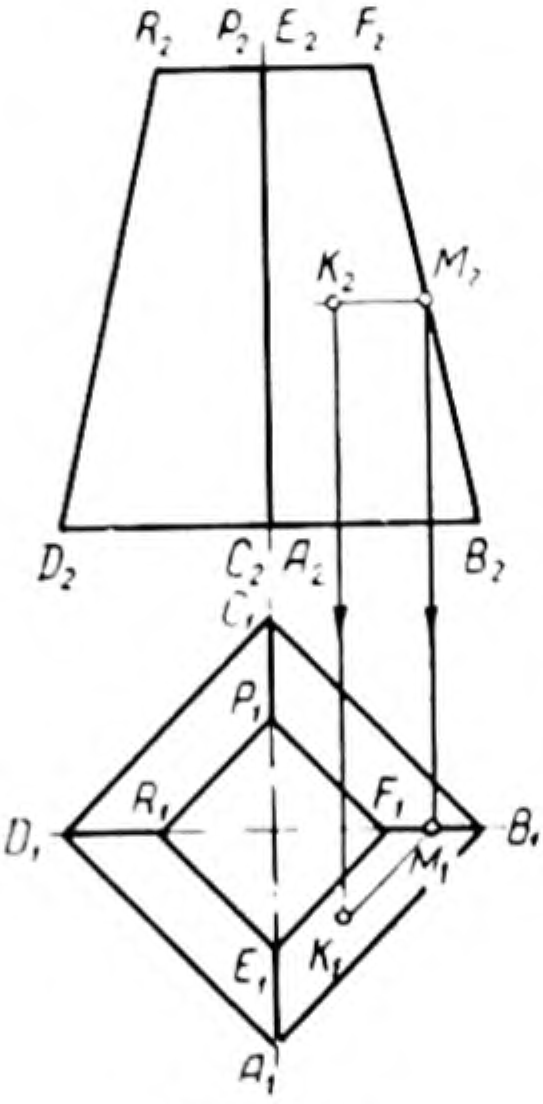


Fig. 285

It is possible to choose auxiliary lines in any direction. However, for simplicity these lines should be chosen so as to lie parallel to one of the edges of the given body.

For instance, it is required to construct at an altitude of 15 mm a point K on the right-hand front lateral face of the tetragonal regular pyramid frustum (Fig. 285).

This construction is carried out with the help of horizontal MK drawn on the face in question. For this purpose at an altitude of 15 mm from the ground line Ox we draw the frontal projection M_2K_2 of the horizontal, the point M_2 being contained in the frontal projection of the right-hand lateral edge BF. By drawing a projector from point M_2 to intersect line B_1F_1 we obtain point M_1 through which we draw the projection M_1K_1 of the horizontal parallel to the projection of edge A_1B_1, since A_1B_1 is the horizontal trace of the plane of the right-hand front face of the pyramid. By drawing a projector through point K_2 to intersect M_1K_1 we find the horizontal projection K_1 of point K contained in the surface of a pyramid and lying at a distance of 15 mm from plane Π_1. It is evident that any point contained in the horizontal MK will satisfy the conditions laid down.

3. The construction of points contained in the surface of a cone is carried out with the help of a circle or the generatrix, or generating line of the cone. Point A on the surface of the cone, Fig. 286, is constructed with the help of a circle of radius r ($r = O_2D_2 = O_1D_1$) and the point B, with the help of the generatrix SC. The sequence of the constructions in Fig. 286 is indicated by arrows. In this case the frontal projections A_2 and B_2 of points A and B, contained in the surface of the cone, were given.

Hint. The radius of the auxiliary circle, with the help of which point A_1 is constructed, is not equal in length to line O_2A_2, which is frequently and incorrectly assumed by students to be the true length of the required radius. The radius is equal to the length of O_2D_2. Line O_2A_2 is the frontal projection of the radius of the given circle, inclined to plane Π_2. It is clear, when its projection S_1A_1 on plane Π_1 is drawn, that OA is projected onto plane Π_2 as a foreshortened line O_2A_2.

4. In Fig. 287 a frontal projection A_2 of point A has been constructed. This point lies on the surface of a sphere and the horizontal projection A_1 of the point is given. An auxiliary circle has been drawn on the surface of the sphere through point A. This circle has a radius r which in the projection on Π_1 is equal to the distance between points C_1 and A_1, i. e., this is the true length of the radius. The frontal projection of the auxiliary circle is parallel to ground line Ox and passes through point K_2 determined after point K_1 has been determined, as shown by an arrow. By drawing a projector from point A_1 to intersect the frontal projection of the auxiliary circle the frontal projection A_2 of point A was found.

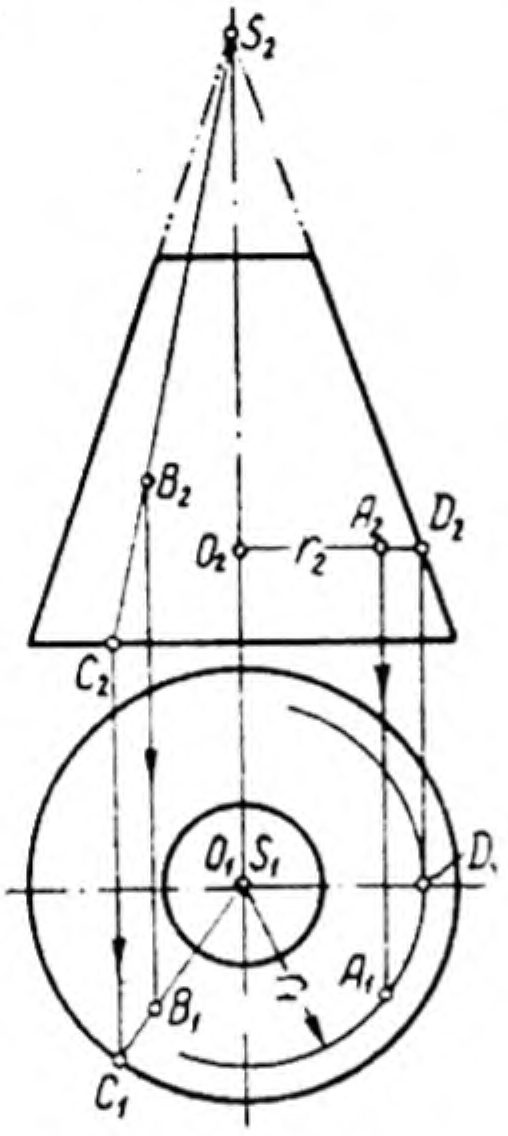


Fig. 286

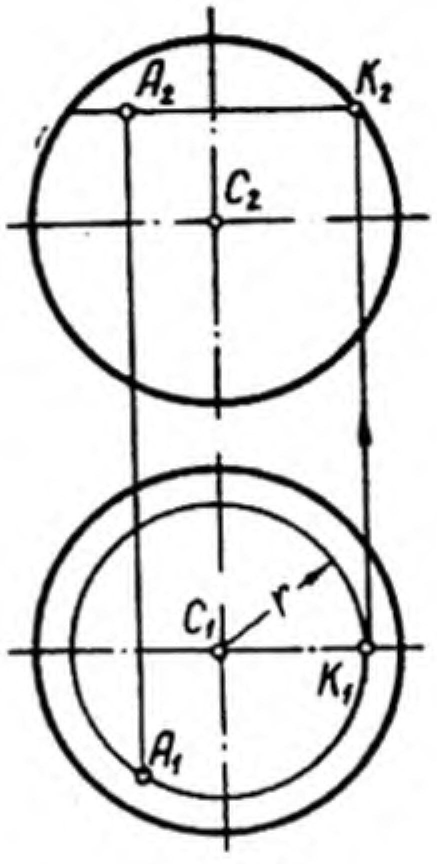


Fig. 287

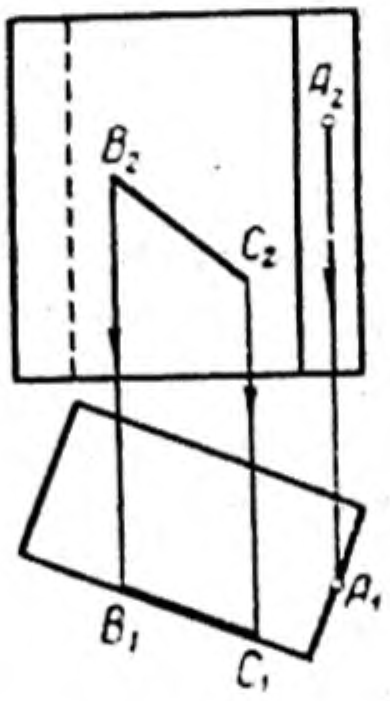


Fig. 288

5. If a face of a body is perpendicular to the plane of projection, the construction of points on that face is carried out directly, without auxiliary lines. For instance, in order to construct the horizontal projection A_1 of point A contained in the lateral face of a right tetragonal prism and given by the frontal projection A_2 (Fig. 288), it is sufficient to draw a projector through point A_2 to intersect the projection of the face on Π_1 in point A_1.

The horizontal projection of line BC contained in another lateral face of the same prism and its given frontal projection B_2C_2 are found in a similar manner.

6. A triangular inclined pyramid frustum standing on plane Π_1 is shown in Fig. 289. A part of this pyramid is cut out. The projection of this portion on plane Π_2 is bounded by the lines $1_2 3_2 5_2$ and $2_2 4_2 6_2$. By using the frontal projection let us construct the horizontal projection of the outline of the notch. Having found the horizontal projection 2_1 with the help of projector $2_1 2_2$, through point 2_1 we draw lines $2_1 4_1$ and $2_1 6_1$ respectively parallel to projections B_1D_1 and B_1F_1 of the edges of the base of the pyramid. At the intersection of line $2_1 4_1$ with the projection of the lateral edge C_1D_1 point 4_1 will be obtained. By drawing a line parallel to the projection D_1F_1 of the third edge of the base of the pyramid through this point 4_1, we find the projection $4_1 5_1$ of another line making up the outline of the notch. We find point 5_1 on the projection $4_1 5_1$ and point 6_1 on the projection $2_1 6_1$ at the intersection of these lines with a projector drawn through point $5_2 \equiv 6_2$. The horizontal projections of the other vertices of the notch, i. e., points 1_1 and 3_1, contained respectively in the projections A_1B_1 and C_1D_1 of the lateral edges, are also found with the help of projectors, as shown by arrows. Having joined the points 1_1 and 3_1 by straight lines and with points 6_1 and 5_1, respectively, we finish the construction of the horizonal projection of the notch left by the part cut out of the given geometrical body.

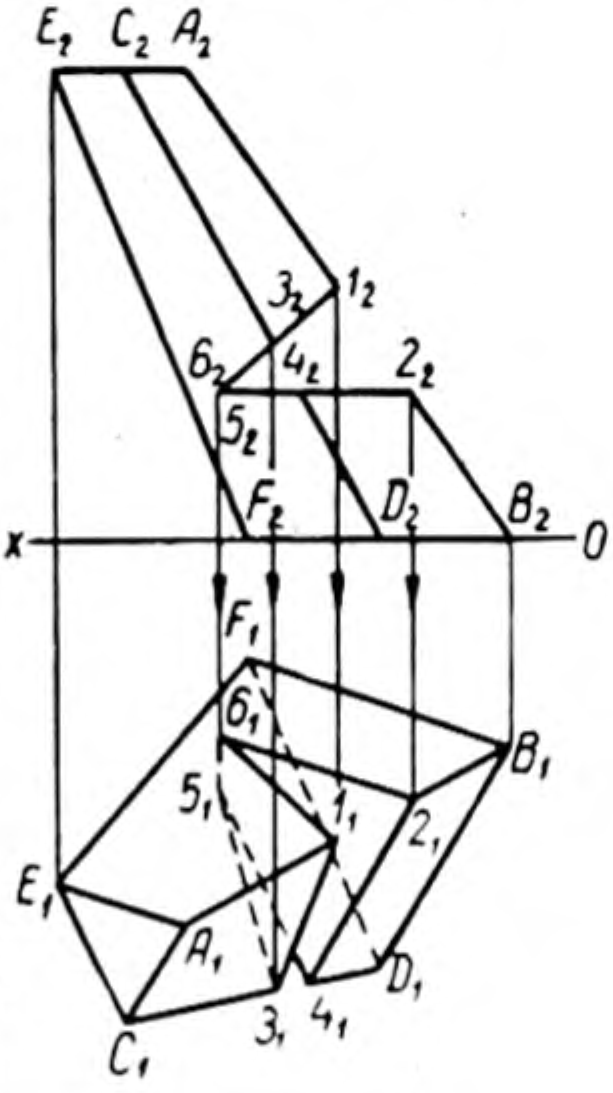


Fig. 289

§ 33. Development of the Surfaces of Geometrical Bodies

1. *The development of the surface of a body* is the figure obtained when the surface is made to coincide with a plane*.

The development of the surface of a polyhedron consists in determining the true views of each of its faces in succession in the plane of the drawing. This is done in the same order as the faces are located in space. When developing oblique prisms and pyramids it is first necessary to determine the true size of each face with the help of one of the methods of transforming projections.

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* The term "the development of surfaces", particularly in automobile body designing offices, is frequently understood to mean the construction of the drawings of complicated double curve surfaces.

The development of the surface of right prisms, all the angles of the lateral faces being right angles, is carried out directly without any transformation of projections.

For example, in order to develop the lateral surface of a hexagonal right regular prism (Fig. 290) it is sufficient to draw a line and step off on it six parts *AB*, *BC*, . . . , *FA*, each of which is

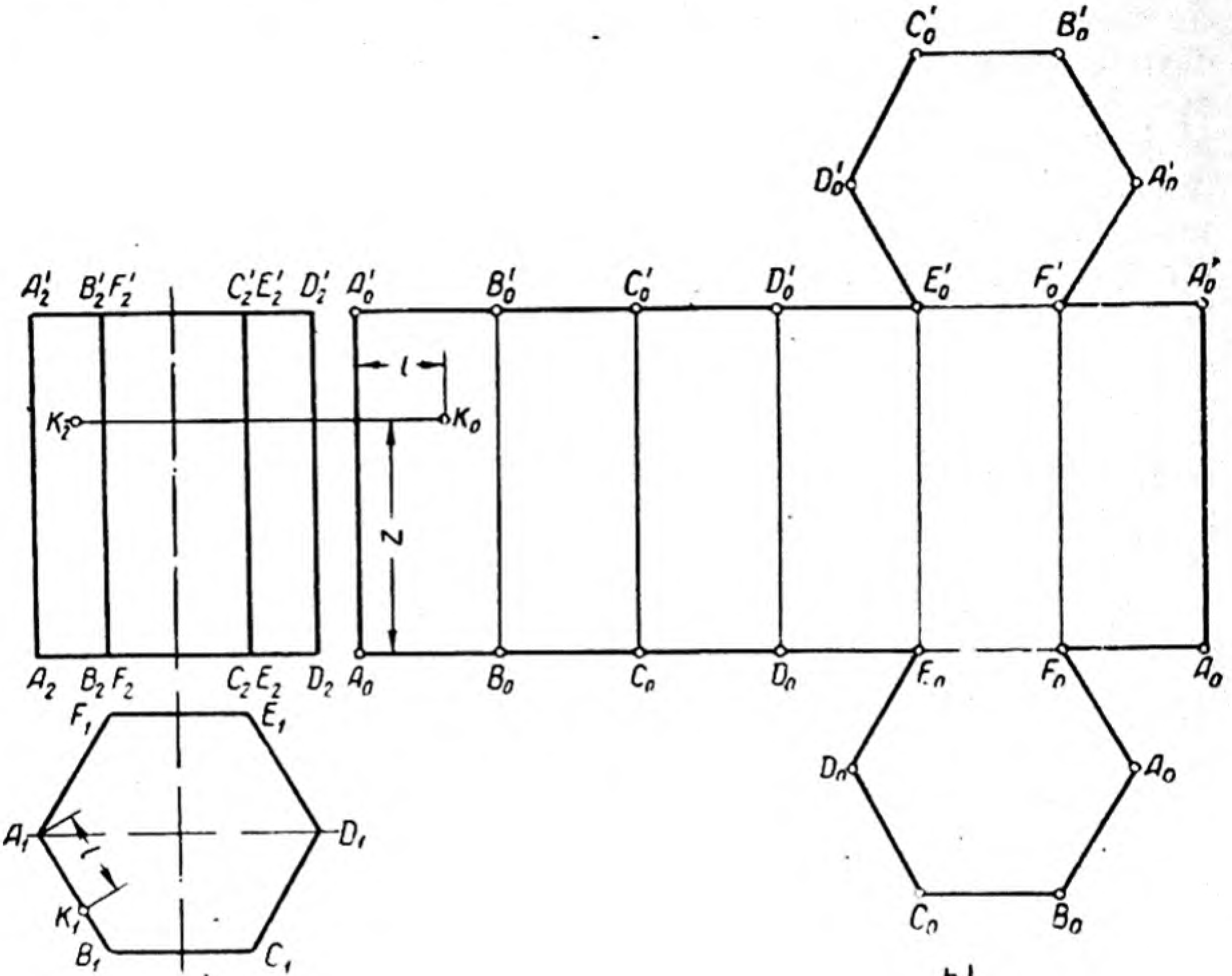


Fig. 290

equal to the side of the prism's base and then construct the rectangle $A_0A_0A'_0A'_0$. The height A'_0A_0 of this rectangle is equal to the altitude of the prism.

In this developed surface the lateral edges of the prism A'_0A_0, B'_0B_0, . . . are situated perpendicular to the sides A_0A_0 and $A'_0A'_0$ of the rectangle, since all the angles of each face of the prism are right angles.

The complete development of the prism will be obtained if, adjoining the top and bottom sides of one developed face, we draw the top and bottom surfaces of the prism as projected onto plane Π_1 in their true size in the form of regular hexagons.

The point K_0 marked on the lateral face $AA'B'B$ can easily be found. On the development of this face point K_0 lies at a distance from the line A_0B_0 equal to the value of z for point K and to the right of line $A_0'A_0$ at a distance $A_1K_1 = l$.

2. The development of an oblique triangular prism (Fig. 291) becomes complicated because the true distance between the edges of each lateral face cannot be read directly from the drawing. This distance may be determined by the transformation of projections. For example, by successive revolutions of the prism about two axes respectively perpendicular to the planes of projection Π_1 and Π_2, the prism can be made to lie perpendicular to plane Π_1. Then on plane Π_1 the prism will be projected as a triangle whose sides are equal in length to the true distances between the lateral edges.

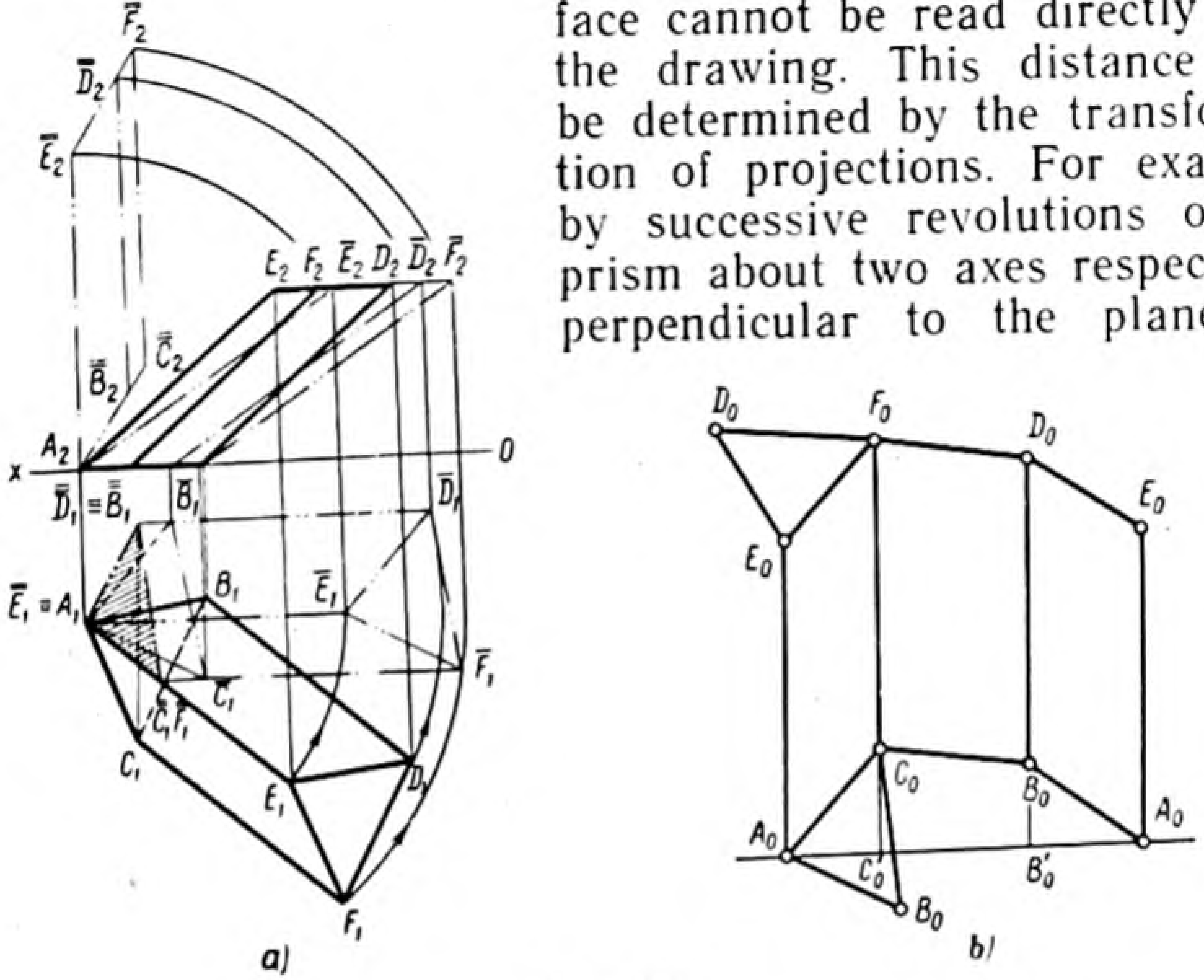

$\bar{\bar{F}}_2$
$\bar{\bar{D}}_2$
$\bar{\bar{E}}_2$
E_2 F_2 $\bar{E}_2$ D_2 $\bar{D}_2$ $\bar{F}_2$
$\bar{\bar{B}}_2$
$\bar{\bar{C}}_2$
A_2
x
0
$\bar{D}_1 \equiv \bar{\bar{B}}_1$
$\bar{B}_1$
$\bar{D}_1$
$\bar{\bar{E}}_1 \equiv A_1$
B_1
$\bar{E}_1$
$\bar{F}_1$
$\bar{C}_1$
$\bar{\bar{C}}_1\bar{\bar{F}}_1$
C_1
E_1
D_1
F_1
a)
D_0
F_0
D_0
E_0
E_0
C_0
B_0
A_0
A_0
C_0'
B_0'
B_0
b)

Fig. 291

The first revolution is about the axis perpendicular to Π_1 and passing through the vertex A. The axis is not shown in the drawing. This brings the prism into the position $A\bar{B}\bar{C}\bar{D}\bar{E}\bar{F}$. The projection in this position is drawn by single-dot-and-space lines. The lateral edges of a prism are now parallel to plane Π_2 and so are projected onto it as true lengths.

By second revolution about the axis passing through the same vertex A but perpendicular to Π_2, we bring the prism into the vertical position $A\bar{\bar{B}}\bar{\bar{C}}\bar{\bar{D}}\bar{\bar{E}}\bar{\bar{F}}$. The projection of this position is represented by double-dot-and-space lines. The lateral edges of the prism will then be projected onto plane Π_2 as segments $A_2\bar{\bar{E}}_2$, $\bar{\bar{B}}_2\bar{\bar{D}}_2$ and $\bar{\bar{C}}_2\bar{\bar{F}}_2$ perpendicular to ground line Ox. On plane Π_1 they will be projected as points $A_1 \equiv \bar{\bar{E}}_1$, $\bar{\bar{B}}_1 \equiv \bar{\bar{D}}_1$ and $\bar{\bar{C}}_1 \equiv \bar{\bar{F}}_1$. Joining these points by straight lines we obtain the triangle which after the second revolution of the prism is projected onto Π_1. This triangle is indicated by hatching.

We now have all the dimensions required to develop the surface of the prism.

Let us draw a horizontal line (Fig. 291, *b*) and take on it a point A_0 at random. From this point lay off the lengths of the sides of the hatched triangle in succession. We then obtain points C_0, B_0 and A_0 through which we erect perpendiculars to the horizontal line. Then we lay off on these perpendiculars the true lengths of the lateral edges of the prism, taking into account the distance of the ground line Ox from points $\overline{\overline{B}}_2$ and $\overline{\overline{C}}_2$.

Points A_0, C_0, B_0, A_0 and E_0, F_0, D_0, E_0 found in this way are joined by straight lines in the same order as on the prism. The figure $A_0C_0B_0A_0E_0D_0F_0E_0$ is then the development of the sides of the prism. To obtain the complete development it is necessary to adjoin the bottom and top of the prism to the corresponding edges of one of the developed faces. The dimensions of the sides of these ends are taken directly from the drawing of the prism since they lie parallel to plane Π_1 and are projected onto it in their true size. Thus, $\triangle A_0B_0C_0 = \triangle E_0F_0D_0 = \triangle A_1B_1C_1$.

3. Another method of developing the lateral surface of the same triangular prism is shown in Fig. 292. This method is known as the method of unwrapping. In this case the surface may be said to be "unwrapped". After converting the principal system of planes of projection Π_1-Π_2 by a new system Π_1-Π_4, where plane Π_4, perpendicular to Π_1, is parallel to the lateral edges of the prism, we obtain the projection of a prism on plane Π_4 in which these edges are seen in their true size as lines A_4E_4, B_4D_4 and C_4F_4. Then, in succession we revolve the lateral prism faces about the edges *AE*, *BD* and *CF* until they are parallel to the plane of the drawing. That gives us the development of the lateral surfaces of the prism.

The vertices *A*, *B*, *C*, etc., during such rotations in space, will move in planes perpendicular to the lateral edges of the prism. In Fig. 292 the movement of points A_4, B_4 and C_4 is shown by straight lines A_4A_0, B_4B_0 and C_4C_0 drawn perpendicular to the lines A_4E_4, B_4D_4 and C_4F_4. To facilitate the construction on these perpendiculars of points A_0, B_0 and C_0, i. e., the vertices of the lower base of the prism rotated into the plane of the drawing, from A_0 we describe an arc with radius $A_0C_0 = A_1C_1$ to intersect line C_4C_0 in point C_0. From C_0 we describe an arc with radius $C_0B_0 = C_1B_1$ to intersect line B_4B_0 in point B_0. Finally, from point B_0 an arc with radius $B_0A_0 = B_1A_1$ is drawn to intersect A_4A_0 in point A_0. Lines A_1B_1, B_1C_1 and C_1A_1 give the true length of the sides of the prism's base.

Then, through points A_0, B_0 and C_0, we draw lines parallel to the projections of the lateral edges of the prisms on plane Π_4. Next, we lay off on these lines, from points A_0, B_0 and C_0, the true lengths of the edges. Joining up the points A_0, B_0, C_0, E_0, D_0 and F_0 in the corresponding order we obtain the development of the lateral surfaces of the prism.

4. The development of the lateral surface of a pyramid can be reduced to the construction of a series of triangles by finding

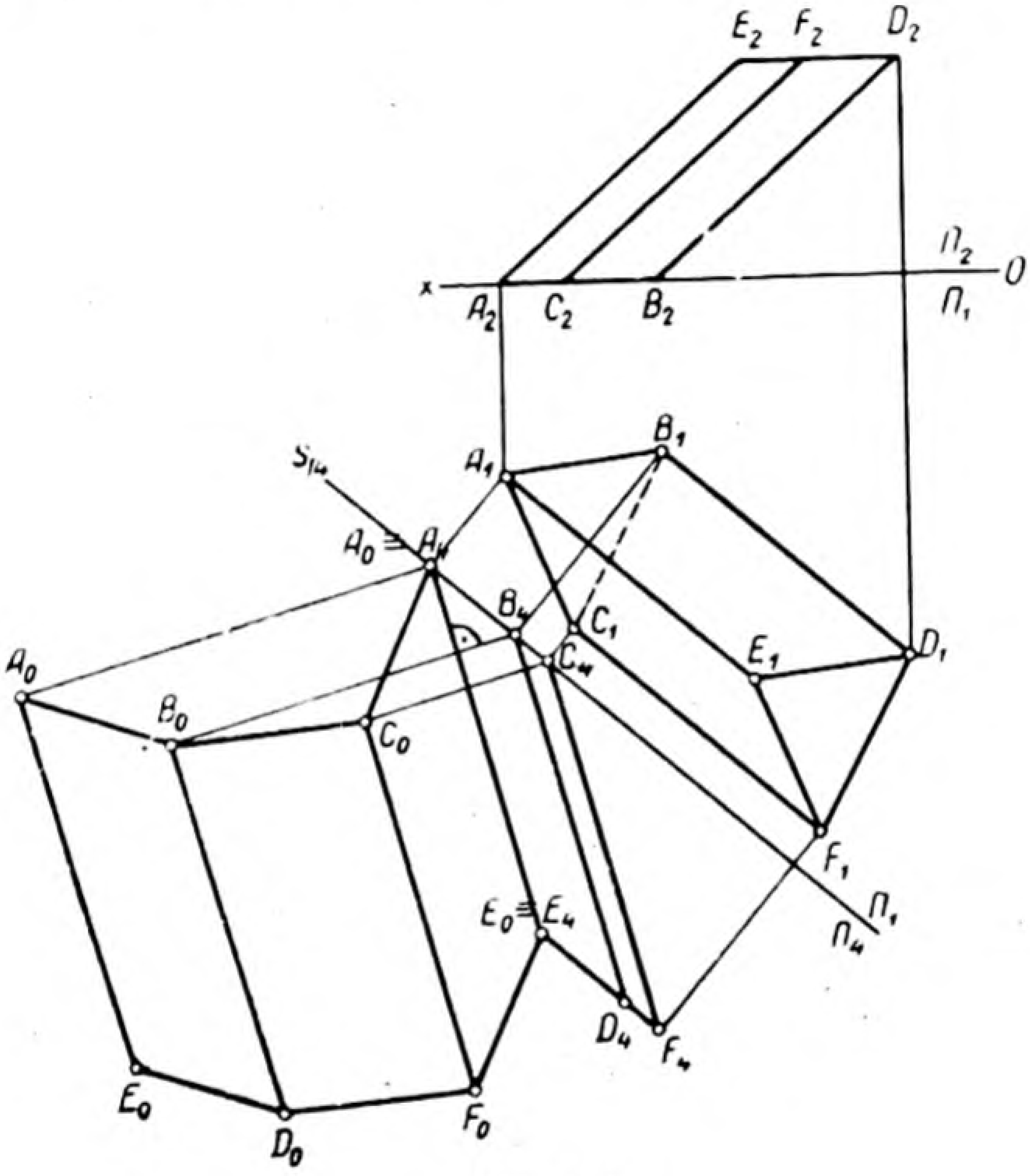
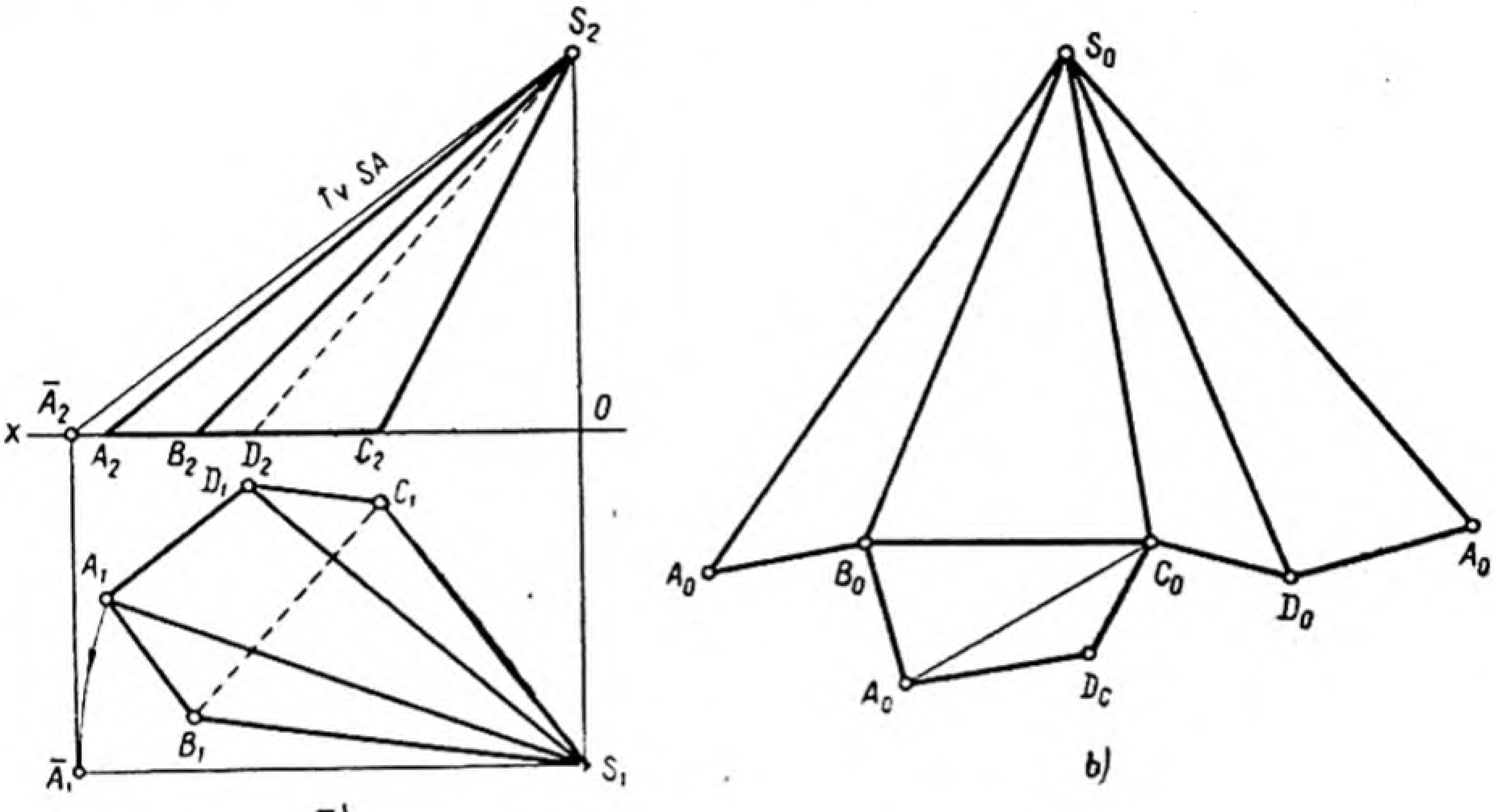


Fig. 292

their three sides. Each of these triangles will be a true view of the corresponding face of the pyramid.

The development of the surface of the oblique tetragonal pyramid *SABCD* (Fig. 293) is begun by determining the true lengths of the lateral edges. These edges are rotated about an axis passed through the vertex *S* of the pyramid and perpendicular to plane Π_1, to a position parallel to plane Π_2. All the lateral edges then are projected as true lengths. In Fig. 293 the construction is shown for the edge *SA*. Any convenient point is marked on the drawing. The triangle $S_0A_0B_0$ is constructed by marking off distances equal to the true length of the sides of face *SAB*. Then the triangles

$S_0B_0C_0$ and so on are built up. After this we add to the development the sides of the base of the pyramid $ABCD$. These are taken from plane Π_1 on which the base is projected as a true view. The construction of this base in the development is carried out with the help of two triangles, ABC and ADC, into which the base is first divided.



a) b)

Fig. 293

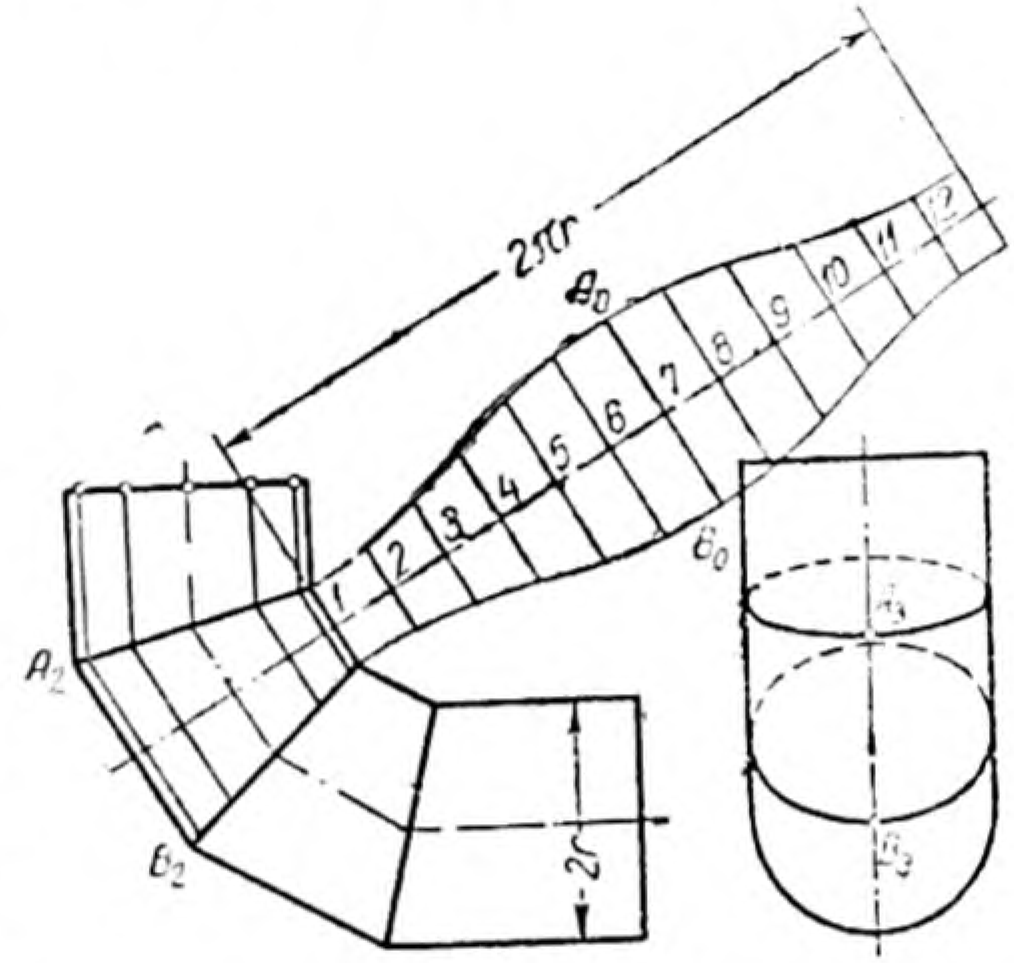
5. *The development of the lateral surface of a right circular cylinder* is a rectangle the base of which is equal to the circumference $2\pi r$ and the height of which is equal to the altitude of the cylinder.

The development of a cylindrical elbow pipe is shown in Fig. 294. The length of the development is equal to the length of the circumference of a normal cross-section, since the diameter of all portions of the pipe is the same. In order to determine the top and bottom outlines of the development the circumference of the upper portion is divided into 12 equal parts and through each dividing point we draw a generatrix on the cylindrical surface of the pipe. With the help of this on the surface of that part of the elbow to be developed, we draw the same number of generatrices. Then the central line of the development is also divided into 12 equal parts and through the points of division we draw perpendiculars on which we lay off upwards and downwards from the line distances equal to one half of the length of the corresponding generatrix of the pipe section. The curved outlines of the development or pattern are drawn through the points obtained.

6. The development of the lateral surface of an elliptical cylinder (Fig. 295) is carried out by "unwrapping" the surface.

First, the cylinder was projected onto plane Π_4 which is perpendicular to plane Π_1 and parallel to a generatrix. In this case all the generatrices of the cylinder will be projected onto plane Π_4 in their true length. Then through the projections 1_4, $2_4 \equiv 8_4$, $3_4 \equiv 7_4$, $4_4 \equiv 6_4$ and 5_4 of the eight points which divide the circumference into equal parts, perpendiculars are drawn to the projections of the generatrices on plane Π_4. It should be noted that the circumference is the projection of the lower end of the cylinder on plane Π_1.

The projections of the eight selected points, when the surface of the cylinder is being unwrapped, move onto the perpendiculars. The arcs are struck off successively on these perpendiculars beginning from point 1_4. The arcs are equal in length to one division of the projection of the base on Π_1 and determine the position of eight points on the development of the lower line of cylinder. Having drawn a continuous curve through these points and having constructed a second curve symmetrical to the first at a distance equal to the length of the generatrix of the cylinder, the development of the lateral surface of the given cylinder is completed.



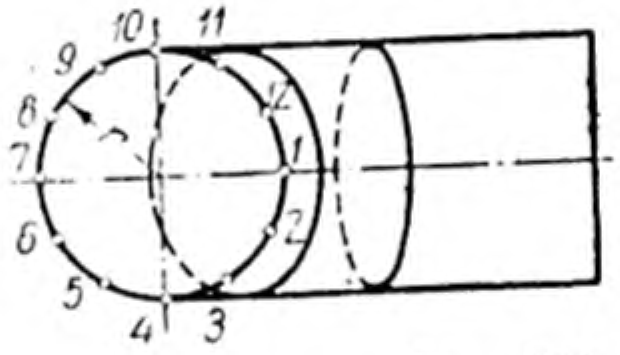


Fig. 294

7. The development of the surface of a right circular cone is a circular sector, the radius of which is equal to the length of the generatrix of the cone, and the central angle of which is determined by the formula:

$$\alpha^\circ = \frac{r \cdot 360^\circ}{l},$$

where r is the radius of the cone's base and l — the length of the generatrix.

A right circular cone truncated by the frontal-projecting plane

α and the development of its surface is shown in Fig. 296. After construction of sector $S_0O_012_0$ whose arc is equal to the length of the circumference of the circular base of the cone and whose radius is equal to the length of the generatrix of the cone, the curve $A_0B_0C_0D_0E_0F_0K_0L_0M_0N_0R_0V_0A_0$, which is the development of the elliptical section is plotted on the developed surface.

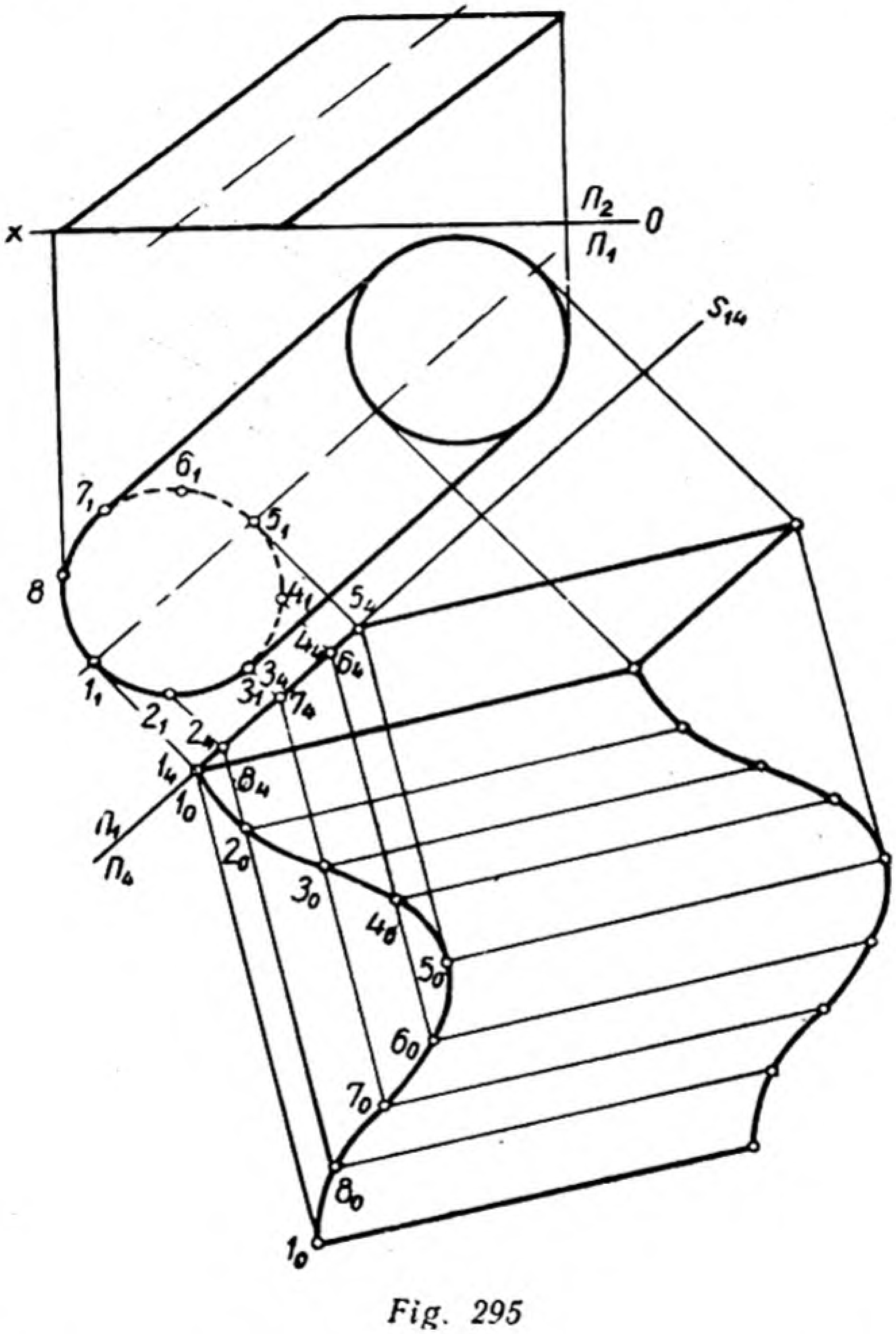


Fig. 295

The circle of the base of the cone is divided into 12 equal parts. Through these divisions we draw the 12 generatrices of the cone. The points A and K are contained in the limiting generatrices, or elements, of the cone. Both these elements are projected onto plane Π_2 in their true length, so we shall find these points on the developed surface if we lay off from point S_0, on straight lines S_0O_0 and S_012_0, a distance equal to S_2A_2 and on line S_06_0 a distance equal to S_2K_2. To determine the distances of the other points of the line of the ellipse to the vertex S, through their frontal projections we draw straight lines parallel to ground line Ox until they intersect one of the limiting generatrices of the cone, which corresponds to revolving them to a position parallel to plane Π_2. In Fig. 296 this is shown for point C_2 only. Laying off these distances from S_0 of the development, on the corresponding lines $S_0 1_0$, $S_0 2_0$, $S_0 3_0$, etc., we obtain on each of the generatrices a point.

By drawing a curve through these points on the development we obtain the development of the cone cut by plane α.

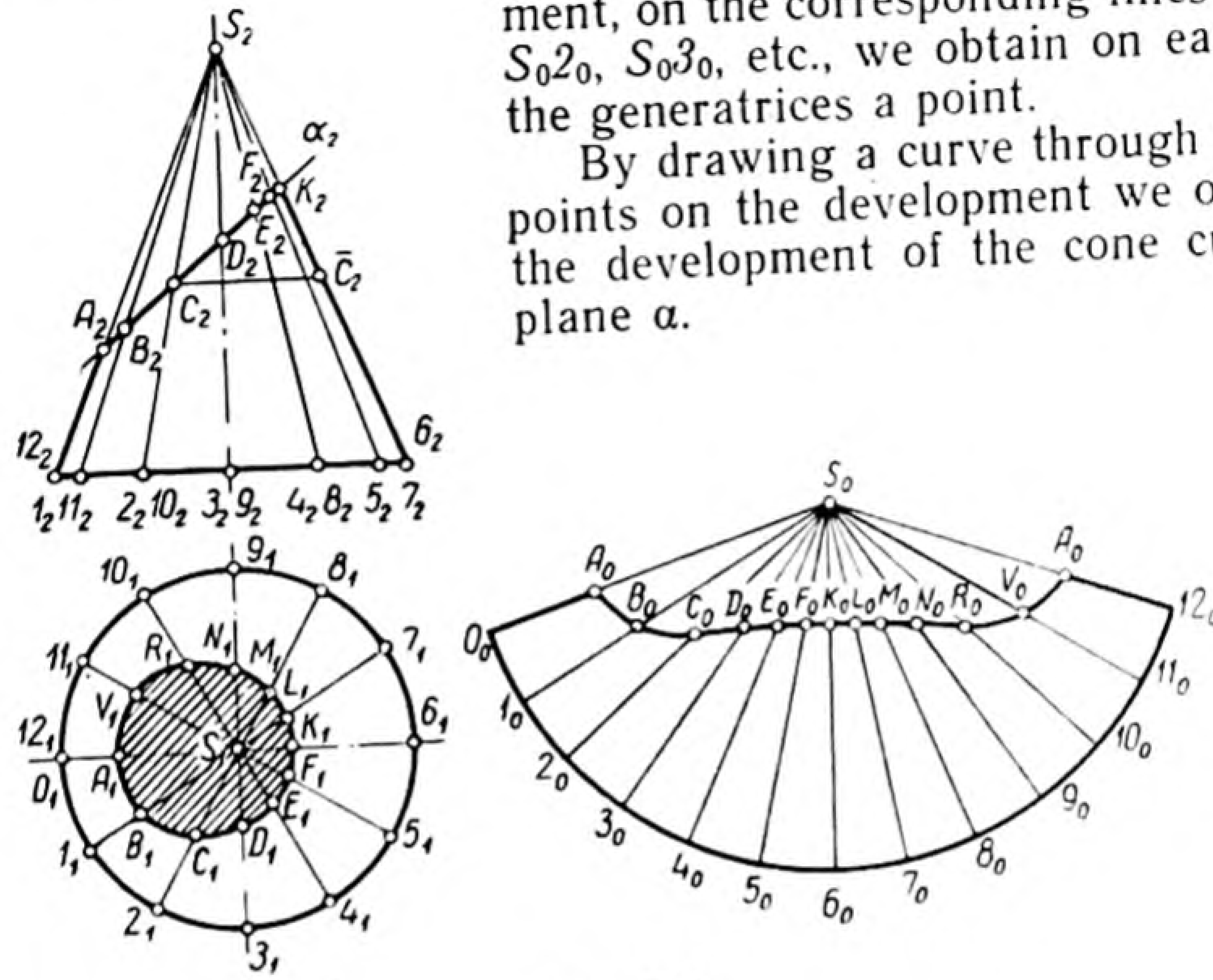

S2
α2
F2
K2
E2
D2
C̄2
A2
C2
B2
12_2
6_2
1_2 11_2
2_2 10_2
3_2 9_2
4_2 8_2
5_2 7_2
9_1
10_1
8_1
11_1
7_1
R1
N1
M1
L1
V1
K1
6_1
12_1
S1
F1
0_1
A1
E1
5_1
1_1
B1
C1
D1
2_1
4_1
3_1
S0
A0
A0
B0
C0
D0
E0
F0
K0
L0
M0
N0
R0
V0
0_0
12_0
1_0
11_0
2_0
10_0
3_0
9_0
4_0
8_0
5_0
6_0
7_0

Fig. 296

8. When the generatrices of a frustum of a cone make angles close to 90° with its base and the apex lies beyond the limits of the drawing, an approximation of the development of the cone may be obtained, as shown in Fig. 297.

We construct an equilateral trapezoid $KLMN$ whose sides KN and LM are equal to l, the length of the generatrix of the truncated cone. The top and bottom are respectively equal to the length of the circumferences of its base $2\pi R$ and its top $2\pi r$ (r is the radius of the top, R — the radius of the base). Divide sides KL and NM into 8 equal parts. Join the points obtained by straight lines *1-2*, *3-4*, etc.

Taking the sides KN and LM of the trapezoid as parts of the radius of a sector of a circle on the development, draw the arc of this sector passing through points M and N. The position of point A on the extension of line *7-8* is determined as follows: 1) from point *8* draw a perpendicular *8B* to side KN of the trapezoid; 2) construct the bisector *8D* of angle *N8B*; 3) a distance equal to ND is laid off on line *7-8* downwards from point *8*. The point A

thus obtained is one of the points of the lower arc of the development of the cone.

The construction of any other point of this arc, for example, point *D*, lying on straight line *5-6*, is carried out in the following sequence: 1) draw a perpendicular *NR* to line *5-6* from point *N*; 2) from point *R* we draw a perpendicular *RE* to the side *KN*;

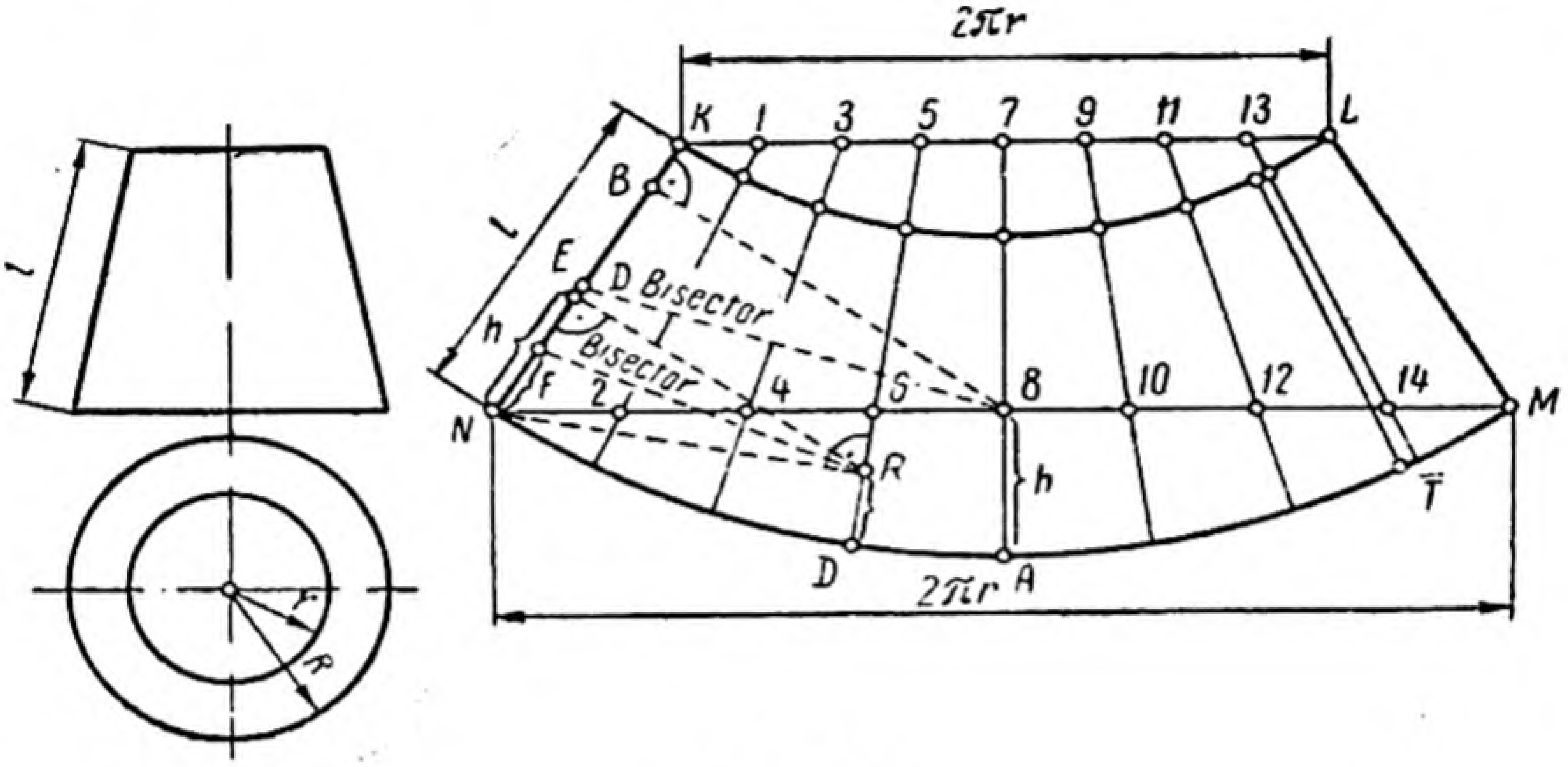


Fig. 297

3) draw the bisector *RF* of angle *NRE*. We find point *D* by stepping off a distance equal to *NF* on line *5-6* downwards from point *R*.

The points such as *C*, contained in arc *KL*, which is the development of the upper line of the cone, are found by laying off on the lines *7-8*, *5-6*, etc. from the points of the lower arc distances equal to the length *l* of the generatrix.

The figure *KNDAMLCK* constructed in this way is somewhat bigger than the true development. In order to obtain a more exact development, the divisions of the length of the circumference of the base and top of the cone should be laid off not on the chords but on the arcs *KL* and *NM*, each corresponding pair of points found should be joined by a straight line. In the drawing points *A* and *T* are shown. The figure *KNDATSCK* will then be a more exact development of the cone.

9. *The construction of the development of an oblique cone* is similar to that of an oblique pyramid. A cone is shown in Fig. 298, *a*, and in Fig. 298, *b*, the development of its surface. This development is begun by dividing the base of the cone into any suitable number of parts, sufficiently small in length to ensure accuracy of the construction. Through the points chosen draw the generatrices *SA*, *SB*, *SC*, etc. The rest of the construction is that of the pyramid, shown in Fig. 293. Through the points

obtained a continuous curve $F_0E_0D_0C_0B_0A_0N_0M_0L_0K_0F_0$ is drawn. This is the development of base line of the oblique cone.

So as to plot a point on this approximate development, say, point R lying on the surface of the cone, through point R draw a generatrix, then find this generatrix on the development and on it, from point S_0 lay off the true distance from point R to the vertex S

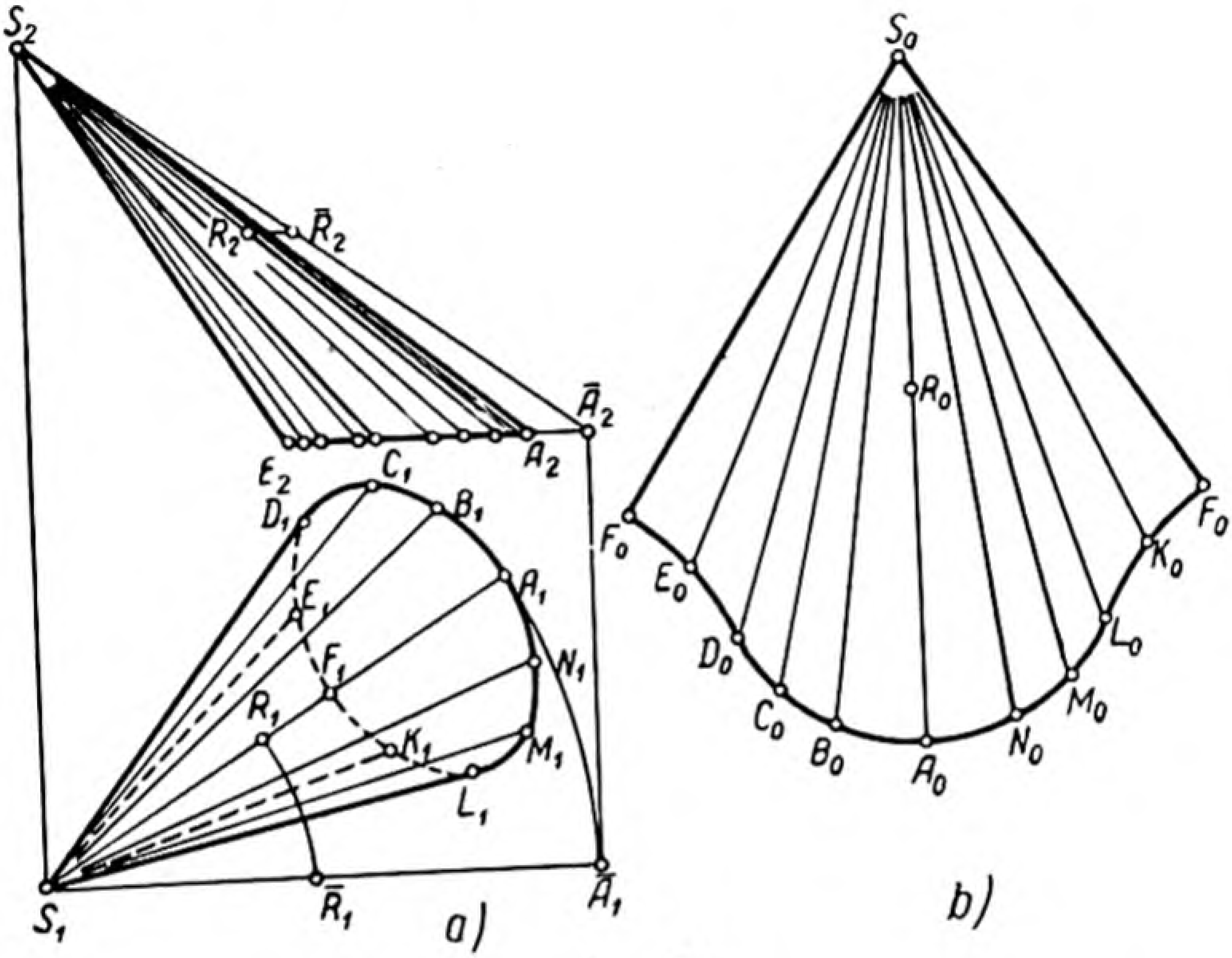


Fig. 298

of the cone. In Fig. 298 this distance is determined by the method of revolution.

10. The development of a sphere can only be approximate. Such a development is made by using either meridians or lines of latitude. Sometimes it is advantageous to develop one part of the sphere using the meridians and the other part by using lines of latitude.

If the sphere is divided up by meridians into a number of sections, or gores, then each section of the surface is "peeled" off the sphere as it were (Fig. 299, *a*). When the sphere is divided by lines of latitude parallel lines are drawn to divide the surface into a series of belts, or zones (Fig. 299, *b*). The middle equatorial belt is assumed to be a cylinder. The top and bottom belts are assumed to be cones and the medium belts, as frustums of cones. The development of the belts is then carried out according to the rules for developing a cylinder, a cone and frustums of cones.

The development of the sphere, shown in Fig. 300, is carried out with the use of meridians. The horizontal projection of the sphere is divided into 12 equal parts. This gives the horizontal projections of the 12 portions ("gores") cut by meridional planes. To develop the first gore we draw a horizontal line, and from point A_0 lay off on it A_0B_0 equal to the chord of arc A_1B_1. Then through the middle A_0B_0 we draw a perpendicular to it and step off on it points 0_0, 1_0, 2_0, 3_0, 4_0 and 5_0. The distance between these points is equal

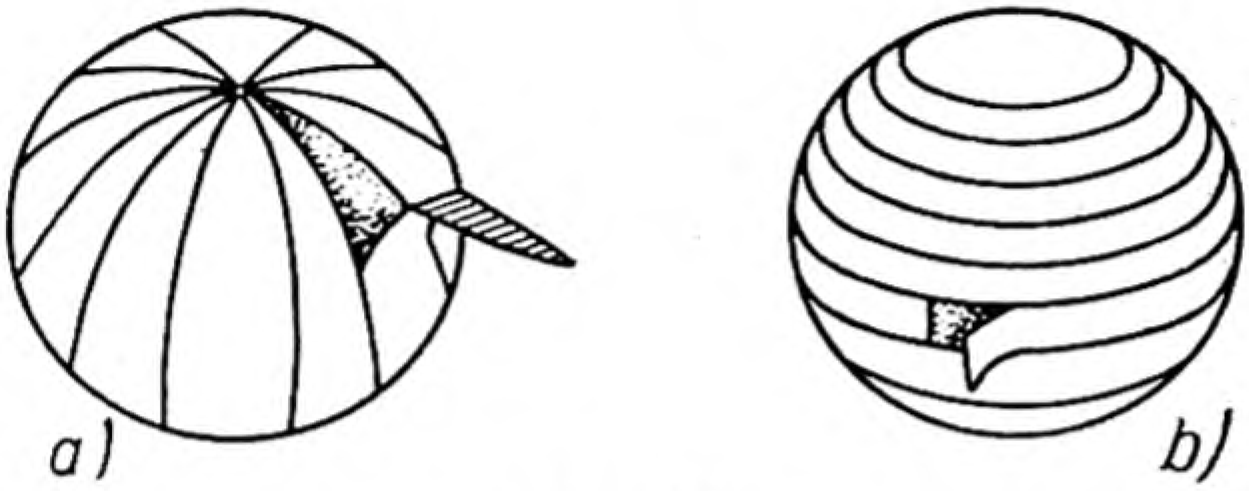
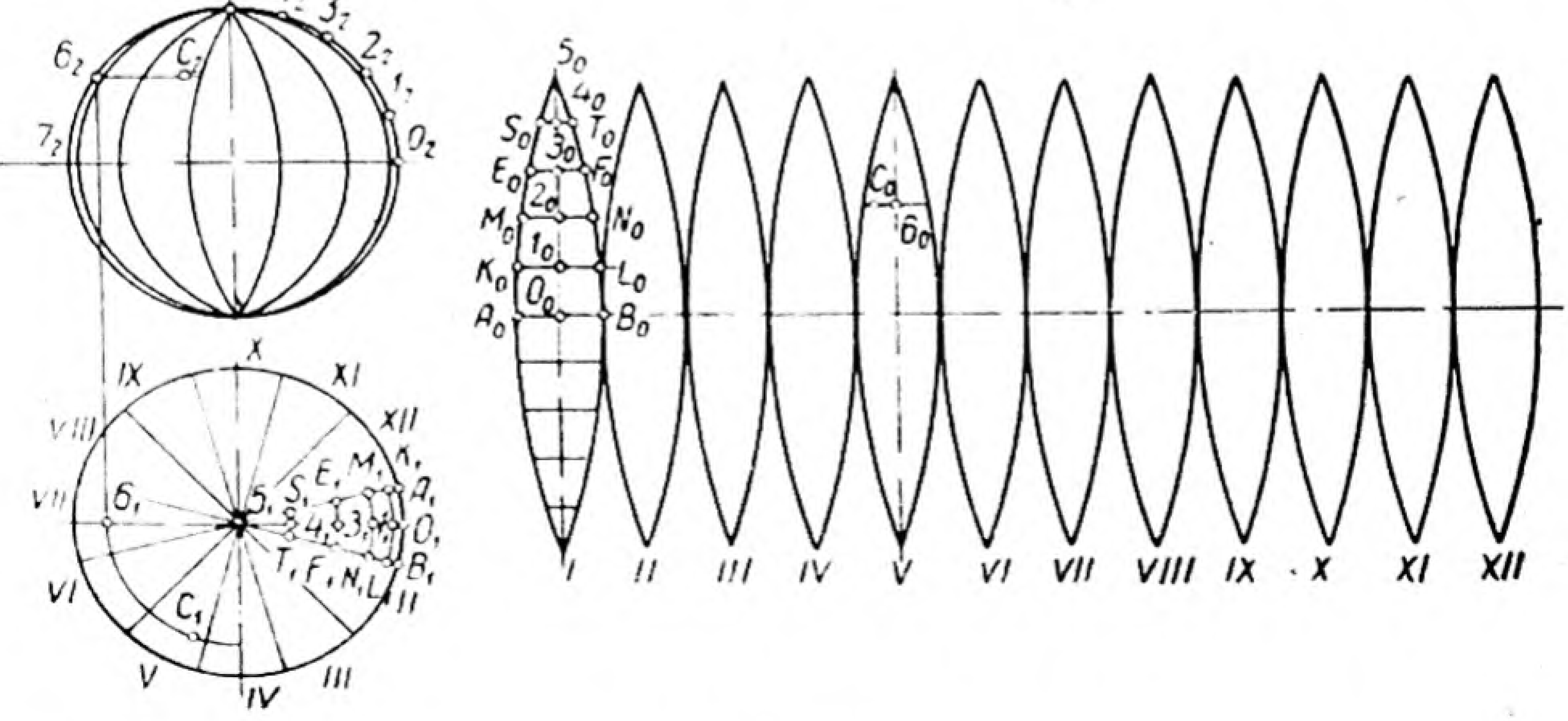


Fig. 299

respectively to the lengths of chords 0_21_2, 1_22_2, 2_23_2, 3_24_2 and 4_25_2, into which a quarter of the circumference of the sphere was divided. These divisions are to be seen on the projection of the sphere surface on plane Π_2. Through points 1_0, 2_0, 3_0 and 4_0 we draw straight lines perpendicular to axis 0_05_0 and lay off on them to either side of the axis, at equal distances from it, points K_0 and L_0, M_0 and N_0, E_0 and F_0, S_0 and T_0. K_0L_0 is equal to the chord of the arc K_1L_1 drawn in plane Π_1 through point 1_1 belonging to the first gore. M_0N_0 is equal to the chord of the arc M_1N_1. E_0F_0 is equal to the chord E_1F_1 and, finally, S_0T_0 is equal to the chord of arc S_1T_1.



5₂
4₂
3₂
2₂
1₂
0₂
6₂
7₂
C₂
IX
X
XI
XII
VIII
VII
VI
V
IV
III
II
6₁
5₁
S₁
E₁
M₁
K₁
A₁
0₁
B₁
4₁
3₁
T₁
F₁
N₁
L₁
C₁
5₀
4₀
S₀
3₀
T₀
E₀
2₀
F₀
M₀
1₀
N₀
K₀
0₀
L₀
A₀
B₀
C₀
6₀
I
II
III
IV
V
VI
VII
VIII
IX
X
XI
XII

Fig. 300

The points A_0, K_0, M_0, E_0, S_0 and 5_0 obtained in this way, and also points B_0, L_0, N_0, F_0, T_0 and 5_0 are then joined by curves. The figure received is then an approximate development of the top half of the first gore.

The bottom half is symmetrical to the top half with respect to line A_0B_0. The remaining eleven parts of the sphere surface development are identical with the first gore.

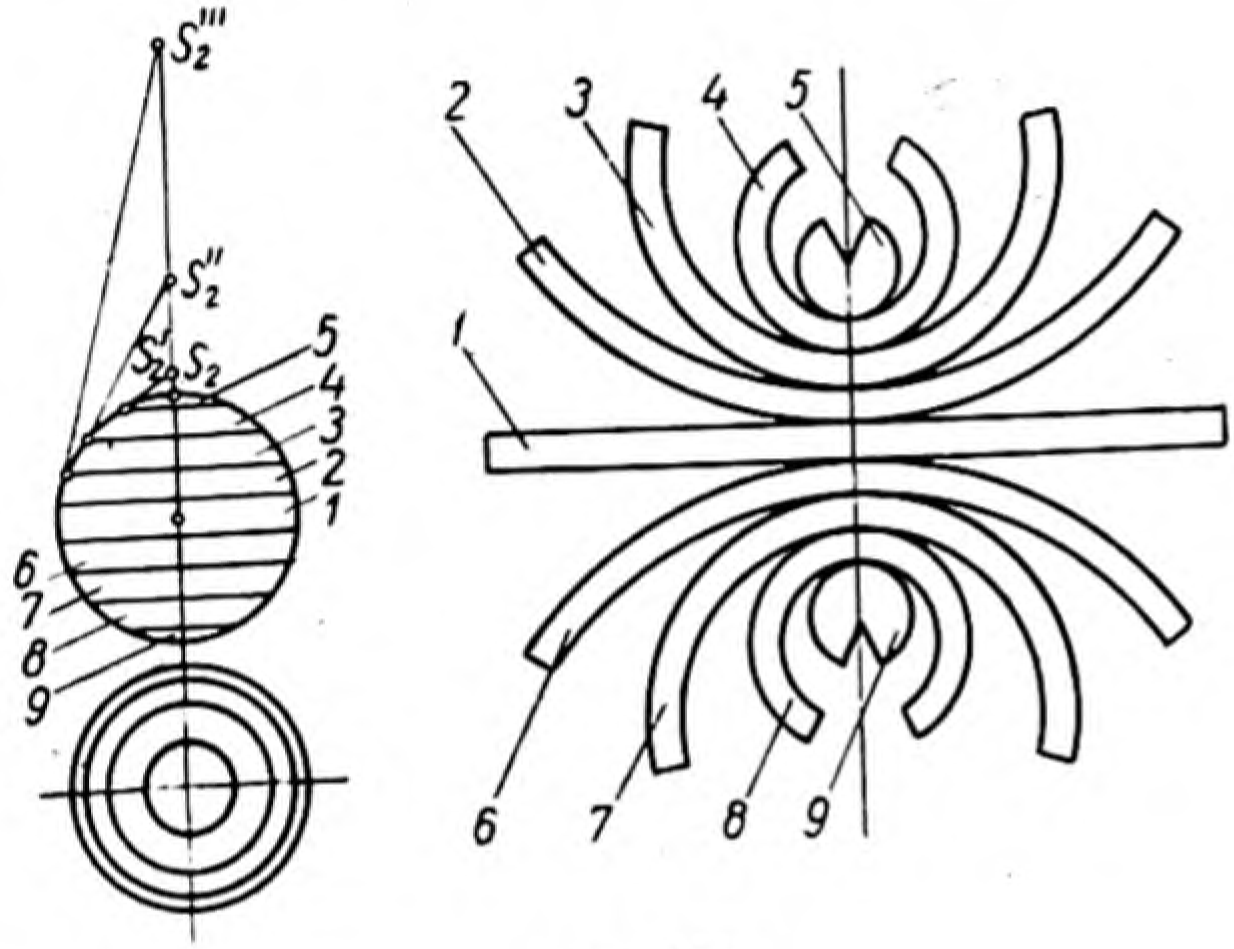


Fig. 301

If it is required to find the position of the development of the sphere of point C situated in the top part of the fifth gore, it is first necessary to find the altitude of this point C in relation to the horizontal axis of development. For this purpose through point C a parallel is drawn on the surface of the sphere. Then, having measured the length of the chord of arc $6_2 7_2$ we lay off this distance from line A_0B_0 on the axis of development of the fifth gore. Then we draw a line parallel to the horizontal axis of development through point 6_0 and lay off on it, to the left of the vertical axis of the fifth gore, the distance from point C_1 to the nearest boundary line of the fifth gore in the horizontal projection. The point C_0 is then the required point.

The development of the sphere with the use of latitudes is shown in Fig. 301. The sphere is divided by a series of parallel planes into seven belts, or zones, and two spherical segments (*5* and *9*). The middle zone (*1*) is developed as a right circular cylinder. The development of zones *2, 3, 4, 6, 7* and *8* is carried out according to the method for developing right circular frustums of cones. The development of segments *5* and *9* is carried out in conformity with method of developing cones.

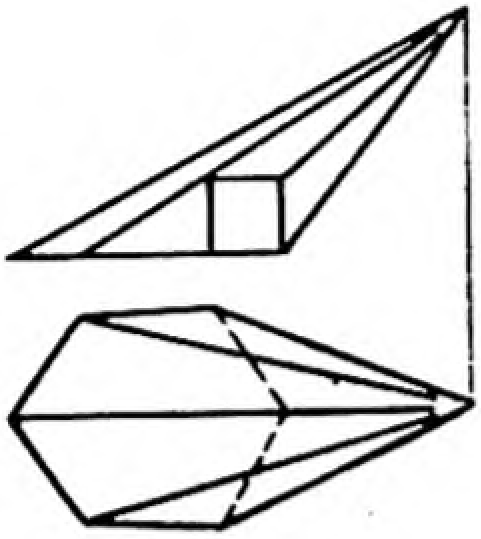


Fig. 302

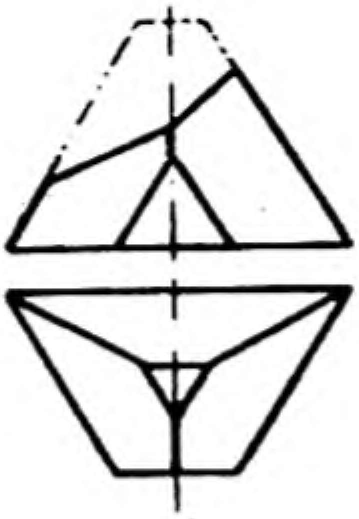


Fig. 303

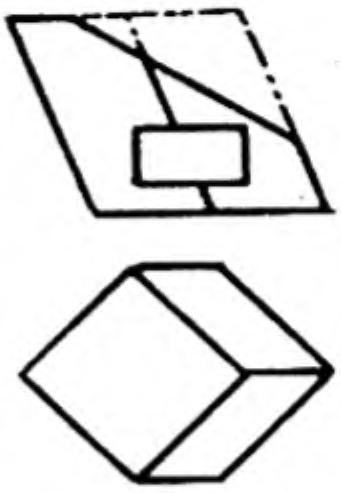


Fig. 304

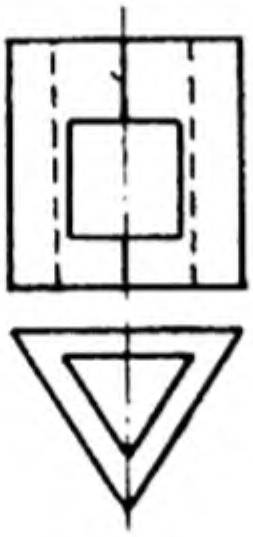


Fig. 305

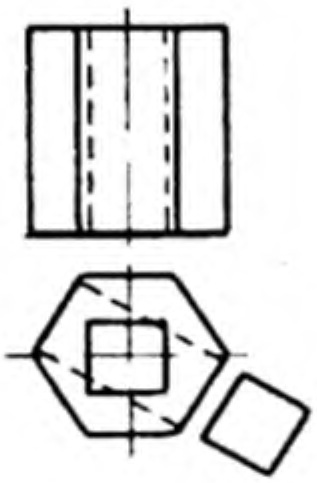


Fig. 306

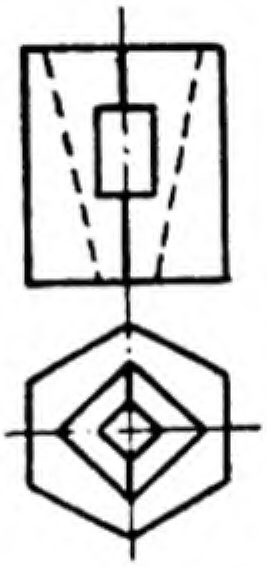


Fig. 307

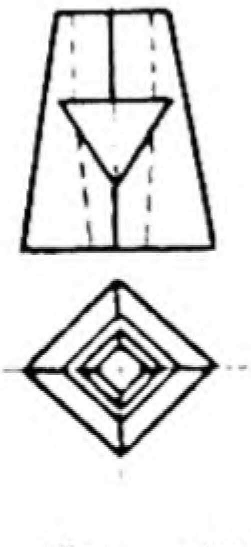


Fig. 308

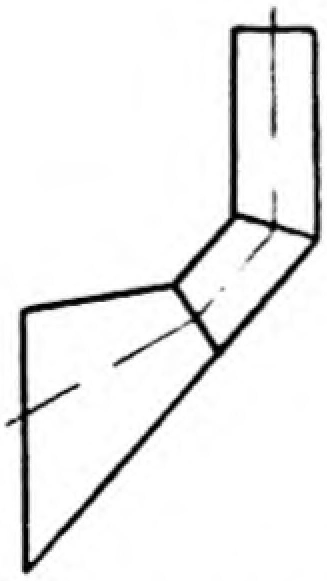


Fig. 309

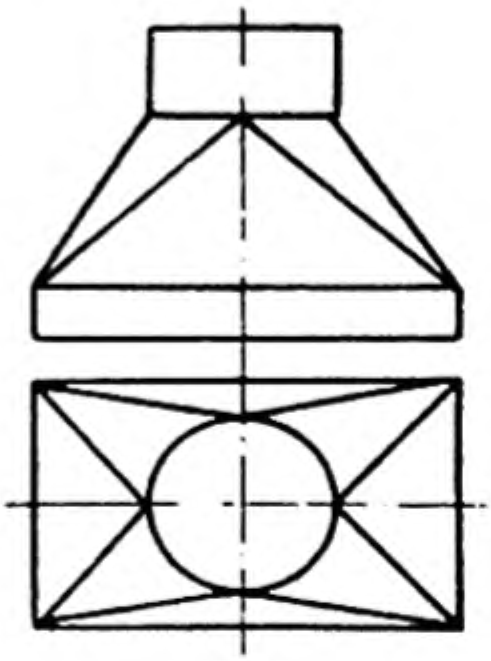


Fig. 310

Problems and Exercises

1. What is a development of the surface of a body?

2. What form is assumed by the development of the lateral surface a) of a right prism, b) of a right circular cylinder, c) of a right circular cone?

3. Construct the missing projections of the cut-outs and through holes of the bodies, depicted in Figs. 302-308. The construction should be carried out in three projections and on an augmented scale.

Hint. The procedure for constructing individual points is given in § 32.

4. Develop the surface of the pipe elbow (Fig. 309).

5. Develop the surface of the transition piece, shown in Fig. 310.

Hint. To carry out the development of the middle section, this surface should be divided into parts of triangular form.

Chapter VIII

INTERSECTION OF PLANES AND SURFACES

§ 34. Plane Sections of Polyhedrons

1. The construction of plane sections of polyhedrons may be divided into two groups. The first group covers those cases when the edges of the polyhedron and the cutting or section planes are oblique. The second includes cases when the edges of the polyhedron or the section plane itself, or both, are perpendicular to planes of projection.

In any case, as the result of the intersection of the edges of the polyhedron by the cutting plane, a plane polygon is formed and the number of its vertices is equal to the number of the edges of the polyhedron.

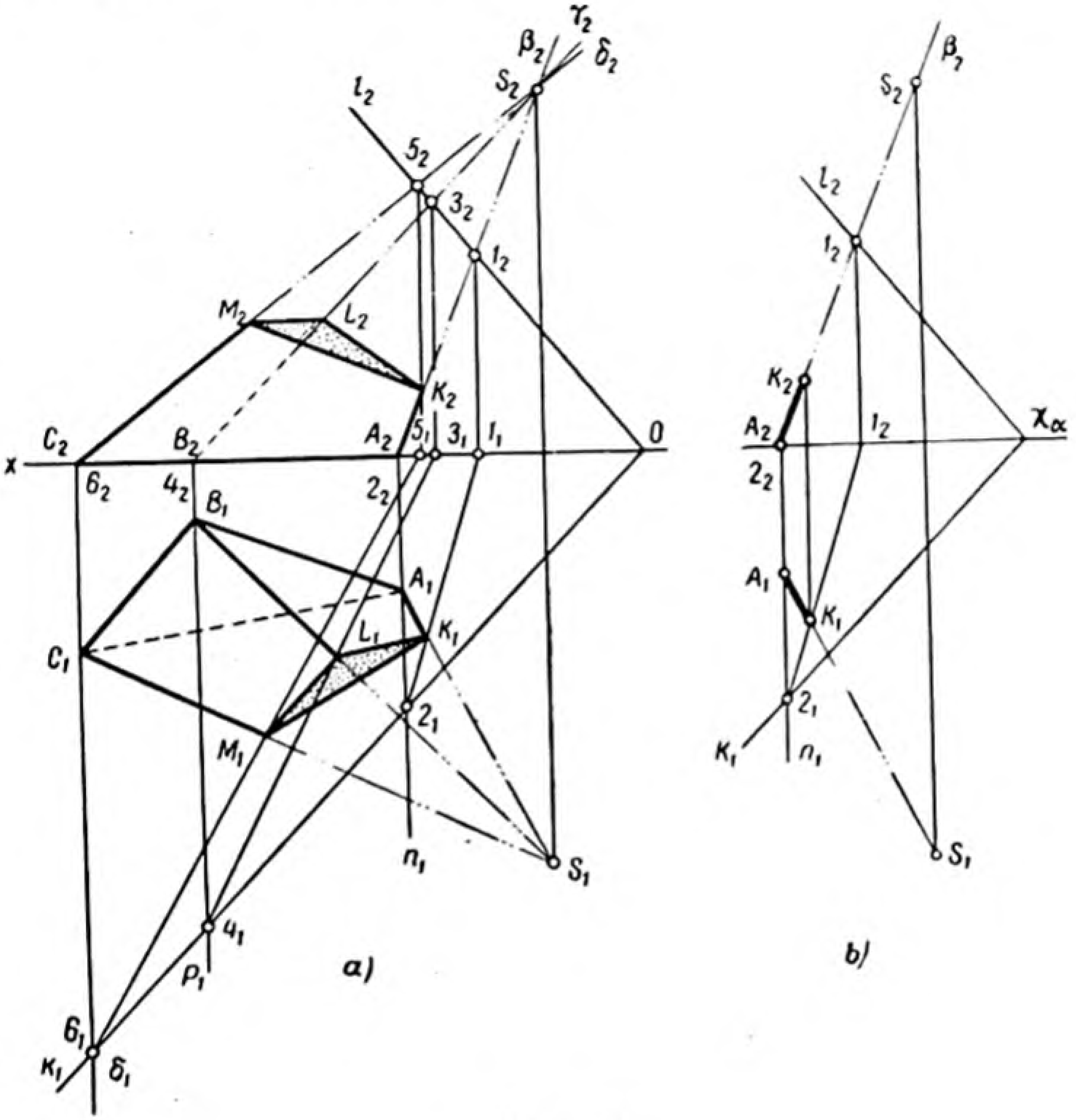
When determining plane sections of the first group, the vertices of the polygonal section are found by using auxiliary planes, as was done when determinating piercing points of a line in a plane and when finding the line of intersection of two planes.

The vertices of the polygons, received when solving problems of the second group, are constructed with the help of auxiliary lines drawn on the edges of the solids or simply by means of projectors.

2. The procedure for intersecting a triangular oblique pyramid by an oblique plane α. This case belongs to the first group. The vertices of the triangle *KLM* which represents a section figure have been found as meeting points *K*, *L* and *M* of the edges *SA*, *SB* and *SC* with plane α. For this purpose through each edge an auxiliary frontal-projecting plane is drawn. The line of intersection of each of these planes with the given plane α is constructed. The intersection of each edge with the corresponding line gives the points in which the edges pierce, as is shown in Fig. 311.

For example, point *K* is found at the intersection of edge *SA* with line *1-2*, in which plane α intersects the auxiliary frontal-projecting plane β passing through edge *SA*. The construction of point *K* is shown in Fig. 311, *b*.

The projection of the part *SKLM* of the pyramid which is cut away is shown by dashes and dots. The projections of triangle *KLM* of the section are shown by dots.



γ_2
β_2
δ_2
l_2
S_2
5_2
3_2
1_2
M_2
L_2
K_2
C_2
B_2
A_2
5_1
3_1
1_1
0
x
6_2
4_2
B_1
2_2
A_1
K_1
L_1
C_1
2_1
M_1
n_1
S_1
4_1
P_1
6_1
K_1
δ_1
β_2
S_2
l_2
1_2
K_2
A_2
1_2
x_α
2_2
A_1
K_1
2_1
K_1
n_1
S_1

a) *b)*

Fig. 311

3. The construction of plane figures, or sections, received when cutting polyhedra, in cases falling within the second group, is shown in Figs. 312 and 313.

In the first case (Fig. 312) the construction of the section *KLMN* of the right tetragonal prism by an oblique plane α is reduced to the construction of the frontal projection $K_2L_2M_2N_2$ of this figure using its horizontal projection $K_1L_1M_1N_1$, which coincides with the horizontal projection of the prism itself. This construction is done with the help of frontals of plane α drawn through points *K, L, M, N*. At the intersection of the frontal projections of the frontals with the frontal projections of the respective edges of the prism points K_2, L_2, M_2 and N_2 are found. These points are then joined by straight lines. The tetragon $K_2L_2M_2N_2$ is the frontal

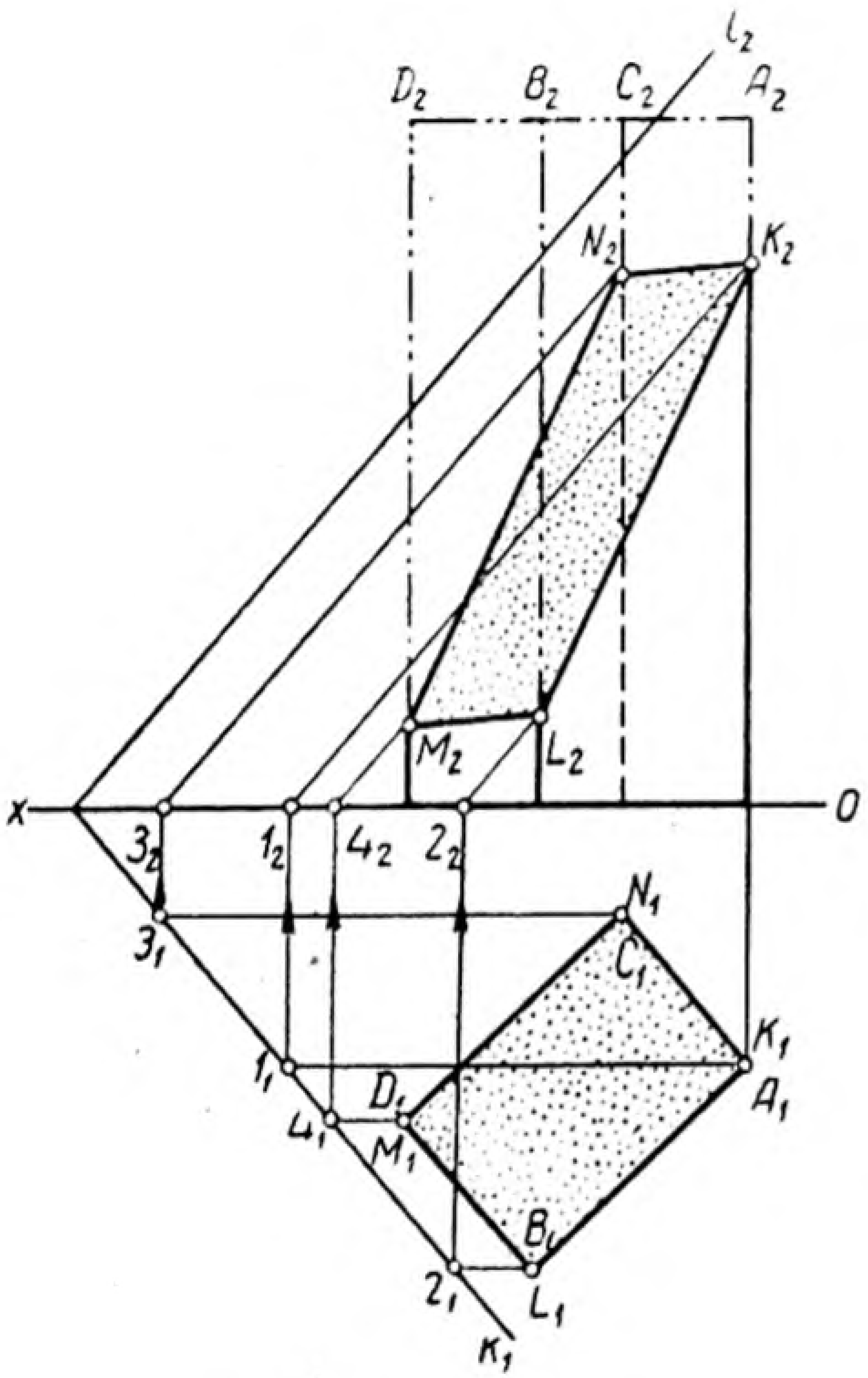

l2
D2
B2
C2
A2
N2
K2
M2
L2
x
0
32
12
42
22
N1
C1
31
K1
11
D1
A1
41
M1
B1
21
L1
K1

Fig. 312

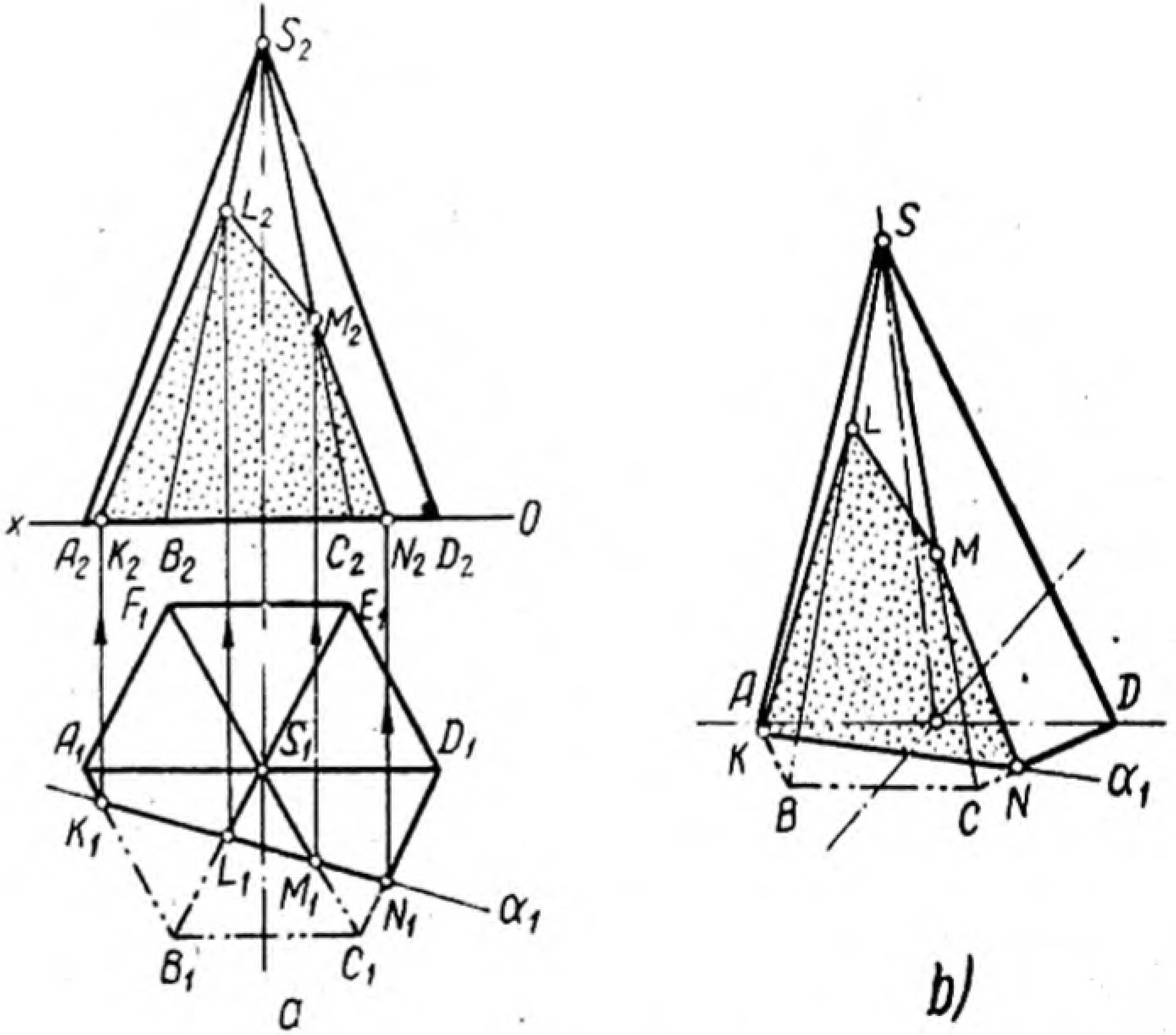

S2
L2
M2
x
0
A2
K2
B2
C2
N2
D2
F1
E1
A1
S1
D1
K1
L1
M1
N1
α1
B1
C1
a
S
L
M
A
D
K
B
C
N
α1
b)

Fig. 313

projection of the section of the prism by plane α. The portion of the prism which has been cut off is shown by dashes and dots and the projections of the tetragon section is shaded in by dots.

If the section plane is a projecting plane, then one projection of the section of the polyhedron coincides with the trace-projection of this plane which possesses collecting properties. The second projection is constructed with the help of the first using projectors. A right regular hexagonal pyramid cut by a horizontal-projecting plane α is shown in Fig. 313, *a*. The points of intersection of the trace-projection α_1 of this plane with the projections of the lateral edges S_1B_1 and S_1C_1 and the edges of the base A_1B_1 and C_1D_1 determine the horizontal projections of the four vertices K_1, L_1, M_1 and N_1 of the section. By drawing projectors through these points to the corresponding projections S_2B_2, S_2C_2, A_2B_2 and C_2D_2, points L_2, M_2, K_2 and N_2 are found. These are the frontal projections of the vertices of the section $KLMN$.

The way the pyramid is intersected by plane α is illustrated by the axonometric representation, Fig. 313, *b*, in which the tetragon of the section is indicated by dots.

§ 35. Plane Sections of Curved Surfaces

1. The construction of plane sections of curved surfaces is similar to that of plane sections of polyhedra. In some cases the points contained in the intersection of a curved surface by a plane are constructed by drawing auxiliary lines, in others by using auxiliary planes.

Nevertheless, there are differences between the construction of plane sections of curved surfaces and the constructions of plane sections of polyhedra.

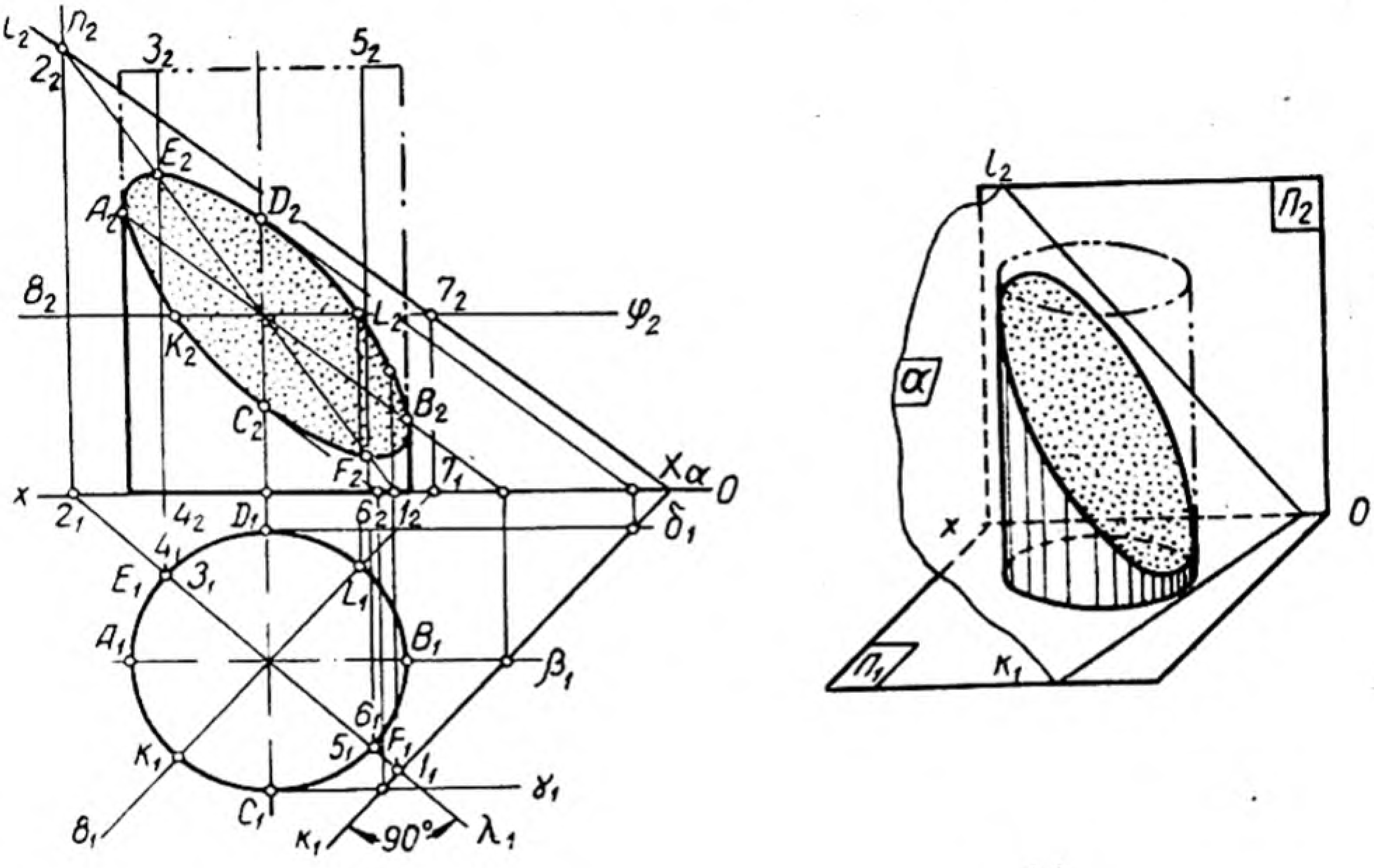
The difference is that the construction of points in a plane section of a curved surface is usually begun by finding on the figure so called *basic* or *special points* whose specific characteristics differ from those of other points. Special points include the highest and lowest points of a figure, points where the limiting elements are tangential to the curve of the section, the extreme points of the section, i. e., those nearest to and those farthest away from the observer, and certain other points.

2. Let us consider the intersection of a right circular cylinder by an oblique plane α (Fig. 314, *a*).

Since plane α is neither parallel to the axis of the cylinder nor perpendicular to it, the section will be an ellipse.

Let us pass a frontal plane β through the axis of the cylinder. This plane will cut the surface of the cylinder along its limiting generatrices or elements, and the plane α along a frontal. The intersection of the frontal with the limiting elements will determine two special points A and B contained in the ellipse of intersection.

Other special points are C, the point nearest to the observer, and D, the point farthest away. They are found with the help of the two frontal planes, γ and δ, passing respectively through the front and rear generatrices of the cylinder. These planes also intersect plane α in frontals. The intersection of the frontal projections of these frontals with the frontal projection of the axis of the



a) b)

Fig. 314

cylinder determines the frontal projections C_2 and D_2 of these special points. The axis of the cylinder coincides with the projection on plane Π_2 of the front and rear generatrices.

The highest point E and the lowest point F of the ellipse given by the intersection are found with the help of the horizontal-projecting plane λ passed through the axis of the cylinder perpendicular to trace k_1. The planes λ and α intersect in line *1-2*. This is a line of maximum inclination of plane α with respect to plane Π_1. Plane λ will intersect the cylinder along the generatrices *3-4* and *5-6*. The intersection of line *1-2* with lines *3-4* and *5-6* gives the special points E and F, respectively.

Besides the special points A, B, C, D, E and F, the intermediate points K and L are also found. These points are constructed with the help of the horizontal plane φ. The section of this plane with the cylinder is a circle and it intersects plane α in the horizontal

7-8. Points K_1 and L_1 lie at the intersection of the horizontal projection of the horizontal 7_1-8_1, and the circle which coincides with the projection of the cylinder on Π_1. Having determined points K_1 and L_1, with the help of projectors, we find points K_2 and L_2 on the trace φ_2 of plane φ.

An axonometric representation of the cylinder cut by plane α is shown in Fig. 314, *b*.

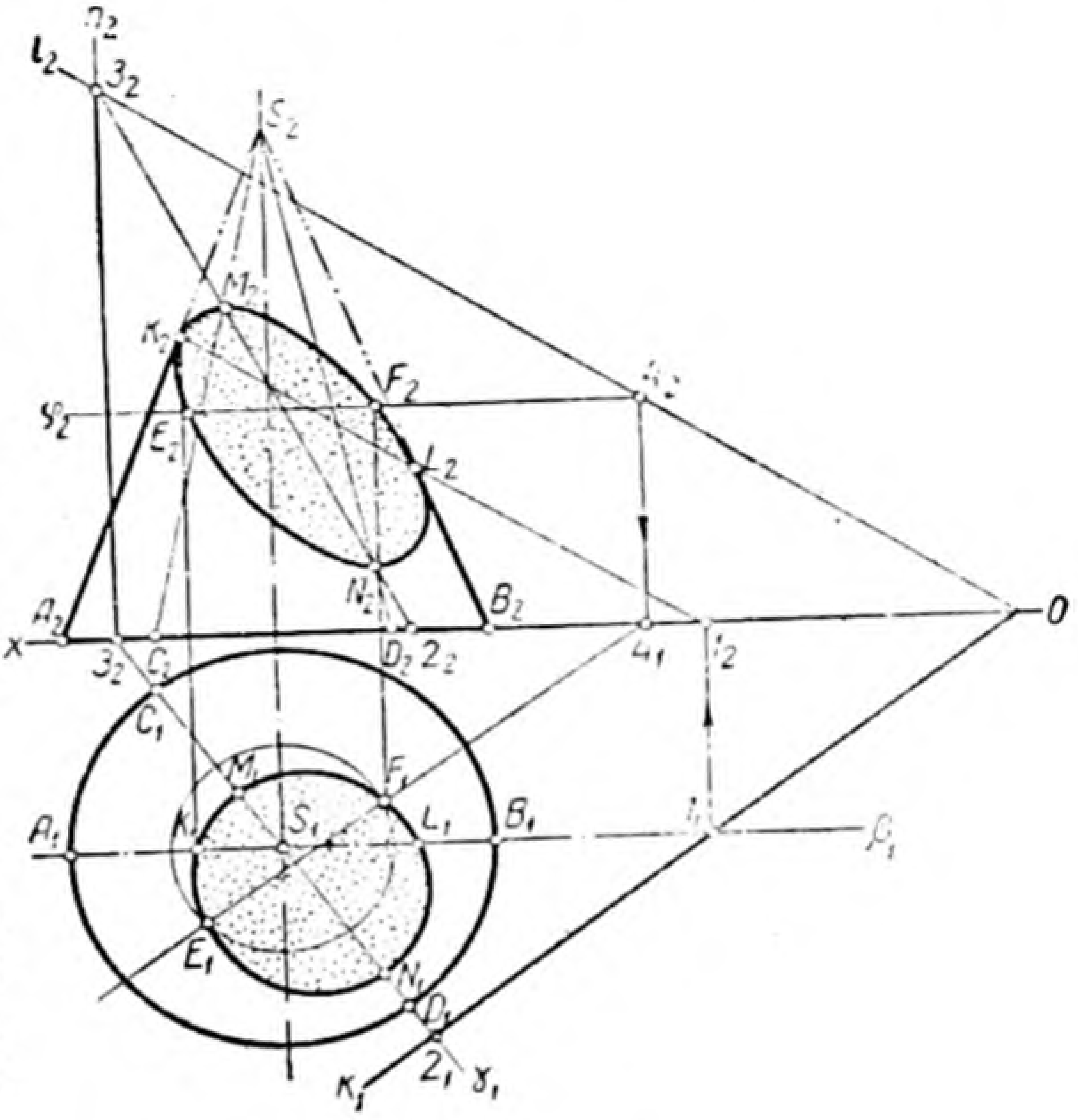


Fig. 315

3. The intersection of a plane with a straight circular cone may give the following lines:

1) Two straight generatrices, when the cutting plane passes through the vertex of the cone.

2) A circle, when the cutting plane is perpendicular to the axis of cone.

3) An ellipse, when the plane intersects all the generatrices of the cone and yet is not perpendicular to the axis of the cone (Fig. 315).

4) A parabola, when the cutting plane is parallel to one of the generatrices (Fig. 316). In this case the angle α between the cutting plane and the axis of the cone is the same as the angle β between the generatrix and the axis of the cone.

5) A hyperbola, when the cutting plane is parallel to any two generatrices of the cone or, as a special case, when this plane is parallel to the axis of the cone (Fig. 317). In the first case the angle α is smaller than angle β. In the second, the angle α is zero.

4. The construction of the section of a cone given by an oblique plane α (Fig. 315). We begin by determining special points. The special points K and L are found with the help of the frontal plane β passed through the axis of the cone. This plane cuts the cone along the limiting elements SA and SB, and the plane α in one of its frontals. At the intersection of the frontal with the limiting elements points K and L, in which the ellipse touches these elements, are found.

We then pass a plane γ perpendicular to the cutting plane α through the axis of the cone. The trace γ_1 is perpendicular to k_1. The plane γ intersects the cone along the generatrices SC and SD. It intersects plane α along the line *2-3*, which is a line of maximum

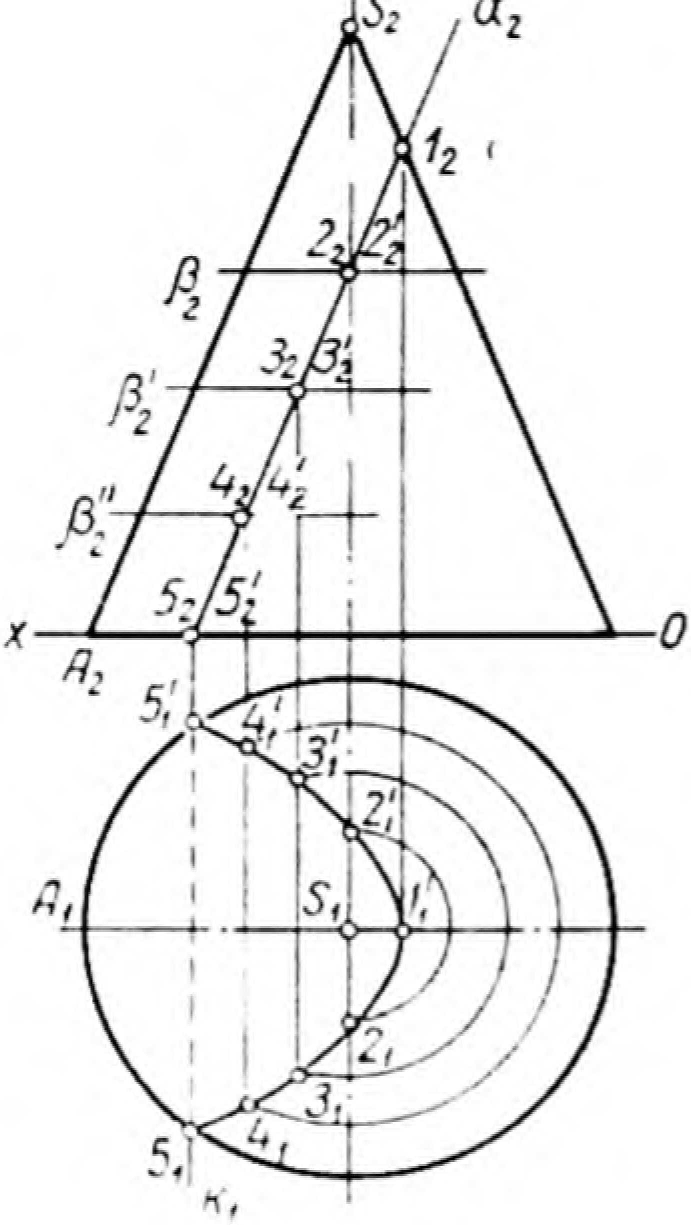


Fig. 316

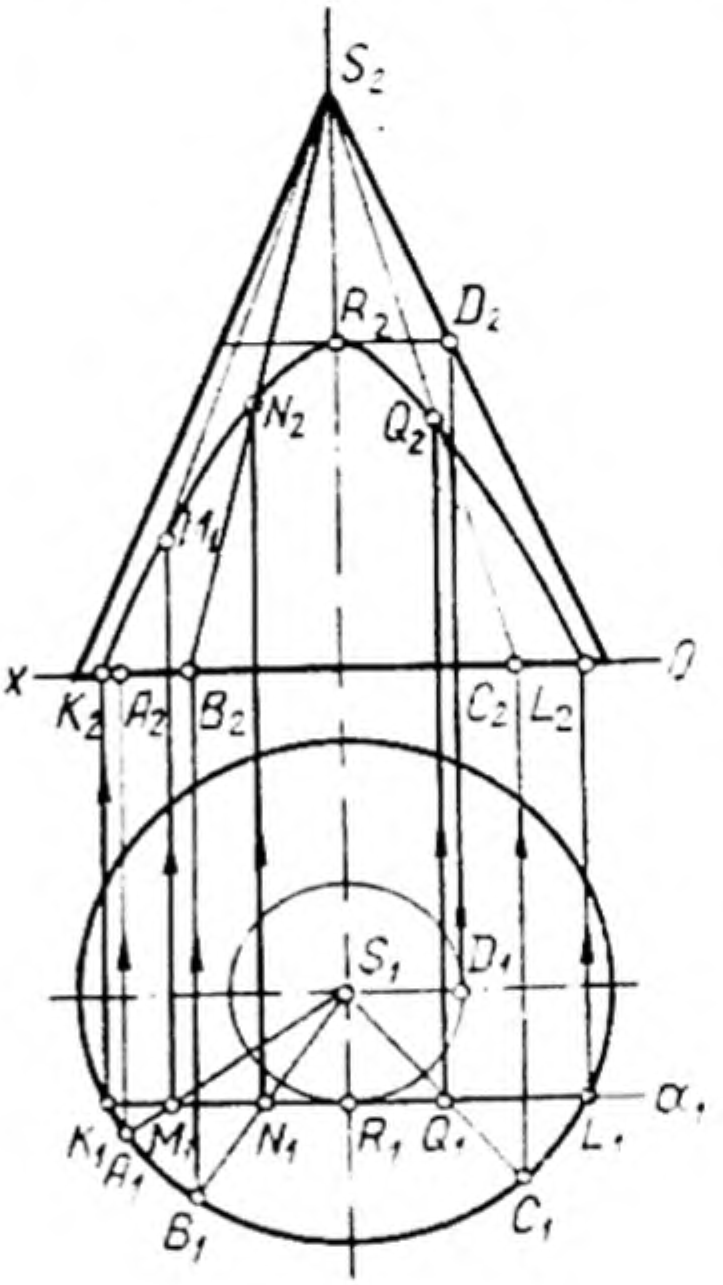


Fig. 317

inclination of plane α to plane Π_1. This line contains points M and N in which the generatrices SC and SD of the cone will respectively intersect plane α. In this case point M is the highest point and N the lowest point, a point of the ellipse given by the intersection of the cone by plane α.

A series of other points of the ellipse may be constructed either with the help of the horizontal-projecting planes passed through the vertex S, or by means of horizontal planes. The points E and F of the ellipse are obtained by drawing the auxiliary horizontal plane φ. Plane φ intersects plane α in a horizontal. The corresponding section of the cone is a circle, the diameter of which is equal to the length of the part of trace φ_2 contained between the frontal projections S_2A_2 and S_2B_2 of the limiting elements of the cone. By describing this circle, with S_1 as the centre, we obtain points E_1 and F_1 at its intersection with the horizontal projection of the horizontal of plane α. Points E_2 and F_2 on trace φ_2 of plane φ are determined with the help of projectors.

Joining points K_1, M_1, F_1, L_1, N_1, E_1 and K_1 by a curve we obtain the ellipse which is the horizontal projection of the intersection of the cone by plane α. The frontal projection of this intersection is also an ellipse, passing through points K_2, M_2, F_2, L_2, N_2, E_2 and K_2.

5. A cone intersected by a frontal-projecting plane α, parallel to the limiting element SA of the cone, is shown in Fig. 316. In this case the plane section of the cone is a parabola. The frontal projection of the parabola coincides with trace-projection α_2 of the cutting plane.

To construct the horizontal projection of the parabola we draw a series of auxiliary horizontal planes (β, β', β''), each of which intersects the cone in a circle and the plane α in a horizontal perpendicular to plane Π_2. At the intersection of the horizontal projections of the horizontals with the horizontal projections of the respective circles, we obtain the points 2_1, $2'_1$, 3_1, $3'_1$, 4_1, $4'_1$. Points $5'_1$ and 5_1 are the horizontal projections of points 5 and $5'$ of the parabola and are contained in the circle of the base of the cone. The horizontal projection 1_1 of the highest point of the parabola, as well as points $5'_1$ and 5_1, are obtained by drawing projectors from points 1_2 and 5_2.

By drawing a curve through points 5_1, 4_1, 3_1, 2_1, 1_1, $2'_1$, $3'_1$, $4'_1$ and $5'_1$ we obtain the horizontal projection of the parabola. The line $5_15'_1$, shown by dashes, is the horizontal projection of the line along which plane α cuts the plane of the cone's base.

6. The intersection of a cone by a frontal plane α which does not pass through the cone vertex S is shown in Fig. 317. In this case the section of the cone will be a hyperbola which is projected onto plane Π_1 as a straight line coinciding with the trace α_1 and onto plane Π_2 in its true size.

The points K and L of the hyperbola, in which it intersects plane Π_1, are determined by intersection of the base circle with trace α_1 of the cutting plane α. The frontal projections K_2 and L_2 of these points are contained in ground line Ox.

To construct the frontal projection R_2 of the special point R, the vertex of the hyperbola, with point S as the centre, we draw a circle whose radius is equal to the distance from point S_1 to trace α_1. This circle is the horizontal projection of the section of the cone given by a horizontal plane passing through point R. To find the frontal projection of this circle through point D_1 we draw a projector to intersect the frontal projection of the right-hand cone generatrix in point D_2. A line is then drawn through point D_2 parallel to ground line Ox and that part of it, which lies between the frontal projections of the limiting elements, is the projection on plane Π_2 of the auxiliary circle of radius S_1D_1. Point R_2 is the midpoint of this line.

The projections of the other points, M, N and Q, on plane Π_2 may be found by drawing a series of generatrices of the cone, or horizontal auxiliary planes. In Fig. 317 points M, N and Q contained in the hyperbola have been found by drawing the generatrices SA, SB and SC.

7. The construction of the horizontal projection of a sphere cut by a horizontal-projecting plane α (Fig. 318) is begun by finding the axis of the ellipse, which is the circle of the section projected onto plane Π_1. The minor axis A_1B_1 of the ellipse coincides with the horizontal projection of the major meridian of the sphere, the extremes A_1 and B_1 of this axis being determined by points A_2 and B_2 in which the frontal trace α_2 of the cutting plane intersects the frontal projection of the major meridian of the sphere. The projection E_2D_2 of the major axis of the ellipse of the section on plane Π_2 is the midpoint of line A_2B_2. Let us draw the auxiliary horizontal plane β so that its frontal trace β_2 passes through point $E_2 \equiv D_2$. Having determined radius r of the circle given by the intersection of plane β and the sphere, with point C_1 as the centre we describe a circle with this radius to intersect the projector drawn through point $E_2 \equiv D_2$ in points E_1 and D_1. Line E_1D_1 is the major axis of the ellipse.

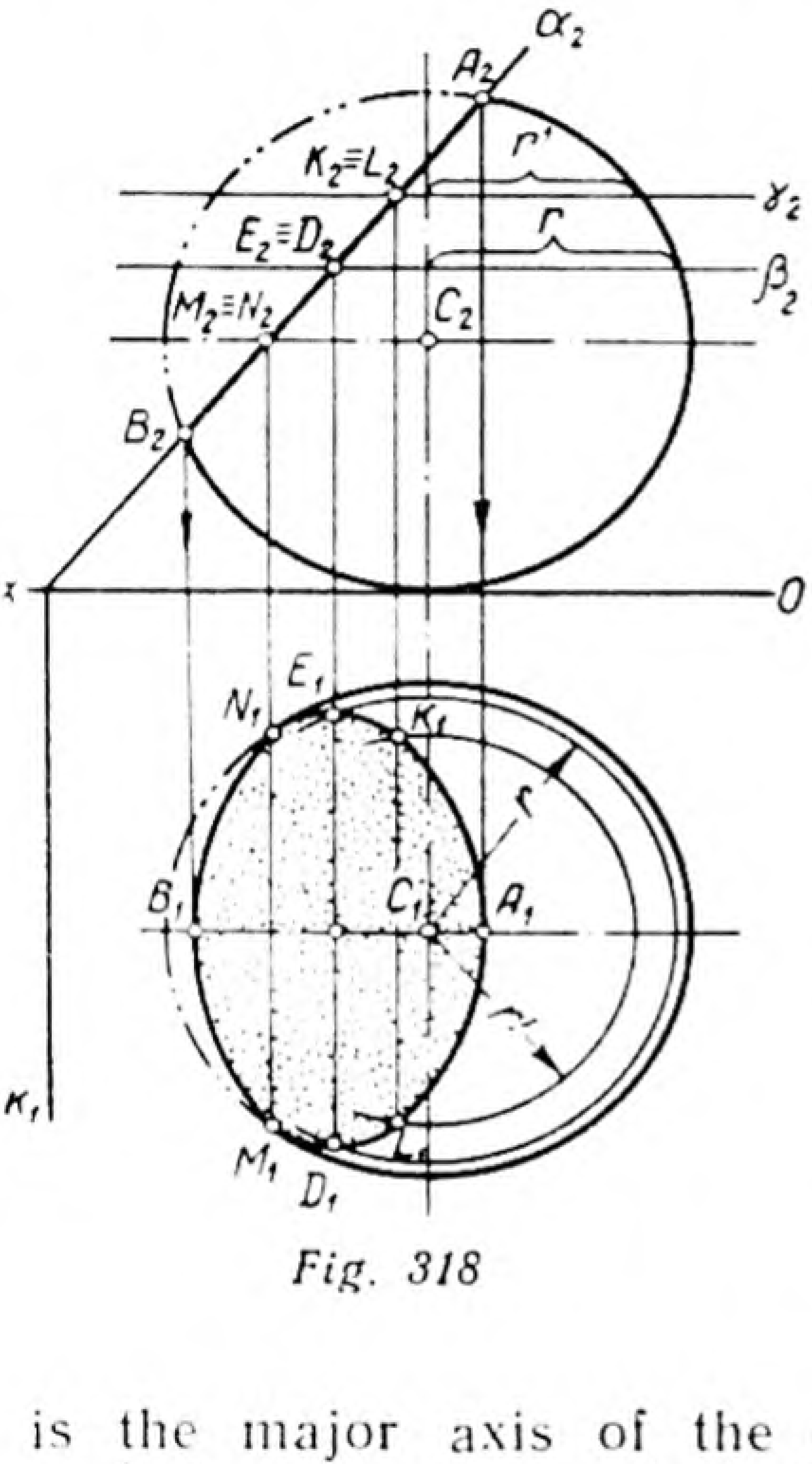


Fig. 318

Let us draw a projector through point $M_2 \equiv N_2$, i. e., through the intersection of trace α_2 of the cutting plane with the frontal projection of the equator of the sphere. This line will cut the circle, which is the horizontal projection of the equator of the sphere, in points M_1 and N_1. It follows that, in points M and N, the circle given by the intersection of the sphere by plane α touches the equator.

A series of other points of the section may be constructed with the help of auxiliary horizontal planes.

8. It should be noted that when constructing projections of plane sections, also in other cases, as few auxiliary lines as possible should be used. The drawing is simplified by using as

auxiliary lines any of the projections of the elements which happen to lie in a convenient position for this purpose.

An example is given in Fig. 319. This is the construction of the intersection of an elliptic cylinder by an oblique plane α. The drawing in this case is simplified because all the auxiliary frontal-projecting planes, which are passed through the generatrices of the cylinder to find the points of their intersection with plane α, are parallel.

So first the line of intersection of one such auxiliary plane, for example β, with plane α is found. This is done by finding points *1* and *2* of the intersection of the corresponding traces of planes α and β. The lines of intersection of other auxiliary frontal-projecting planes, γ, δ and φ, with plane α will then be parallel to line *1-2* and, when constructing these lines, it is necessary to determine only one of their points in each case.

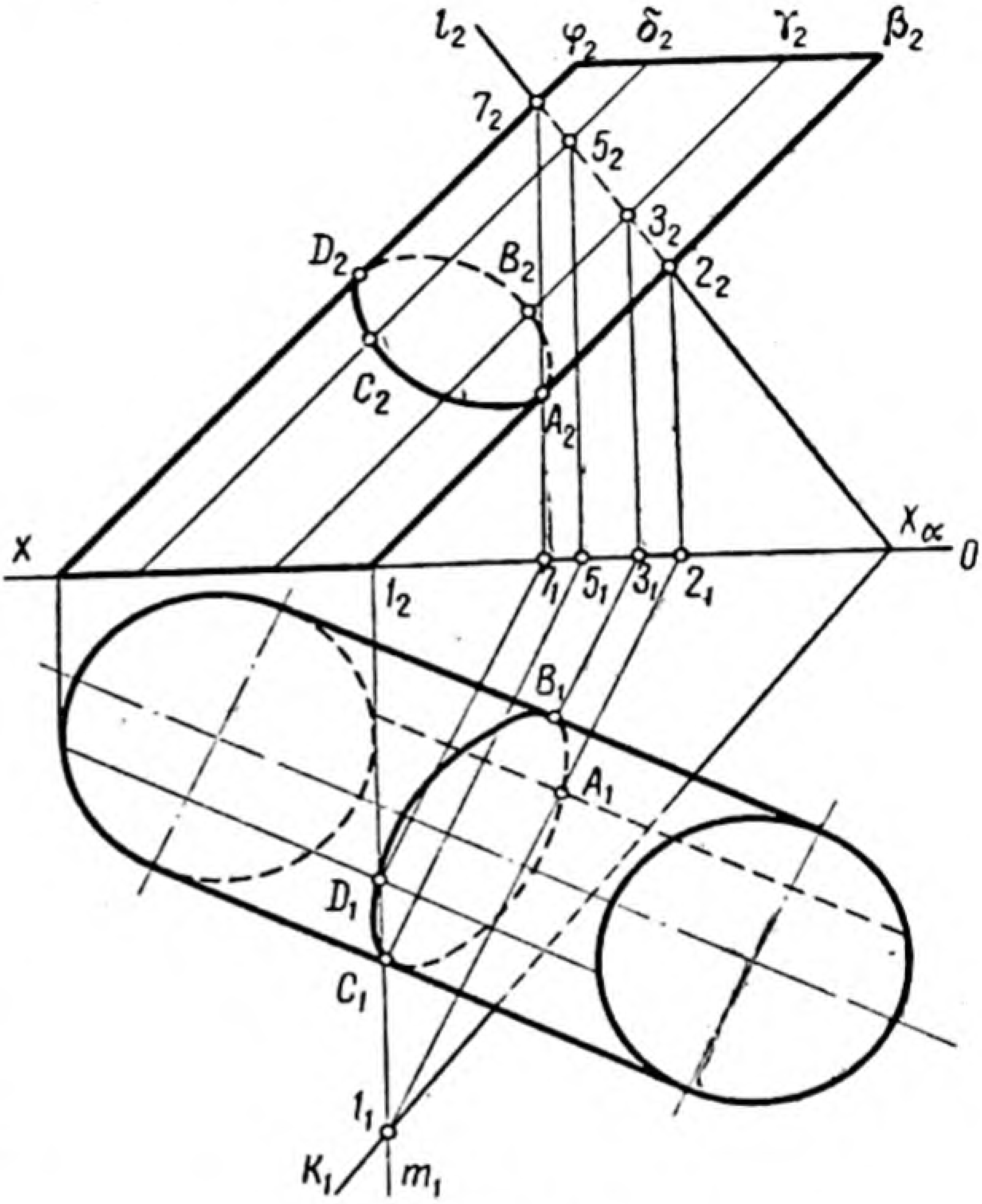


Fig. 319

9. In engineering practice constructions are widely used in which planes are made to cut curved surfaces. For example, in Fig. 320 the drawing of a connecting rod head is shown, and the way in which the elements of its surface flare into each other is shown by the lines of intersection of a complex body of revolution by two frontal planes, α and β.

Let us subdivide the given surface into components of simple geometrical form, i. e., a sphere, a torus and a cylinder. The spherical surface of radius r_1 is intersected by planes α and β in arcs of circles projected onto plane Π_2 without distortion. These planes cut the intermediate cylinder along generatrices.

The horizontal trace of the frontal plane α intersects the horizontal projection of the equator of the sphere in point A_1. The radius r_2 of the section of the spherical surface is determined by the distance at which point A_1 lies from the horizontal projection C_1 of the axis passing through the centre C of the sphere, perpendicular to plane Π_2. In a similar manner the radius r_5 of the semi-circle of intersection of the spherical surface, seen in the frontal projection as lying on the right, is determined.

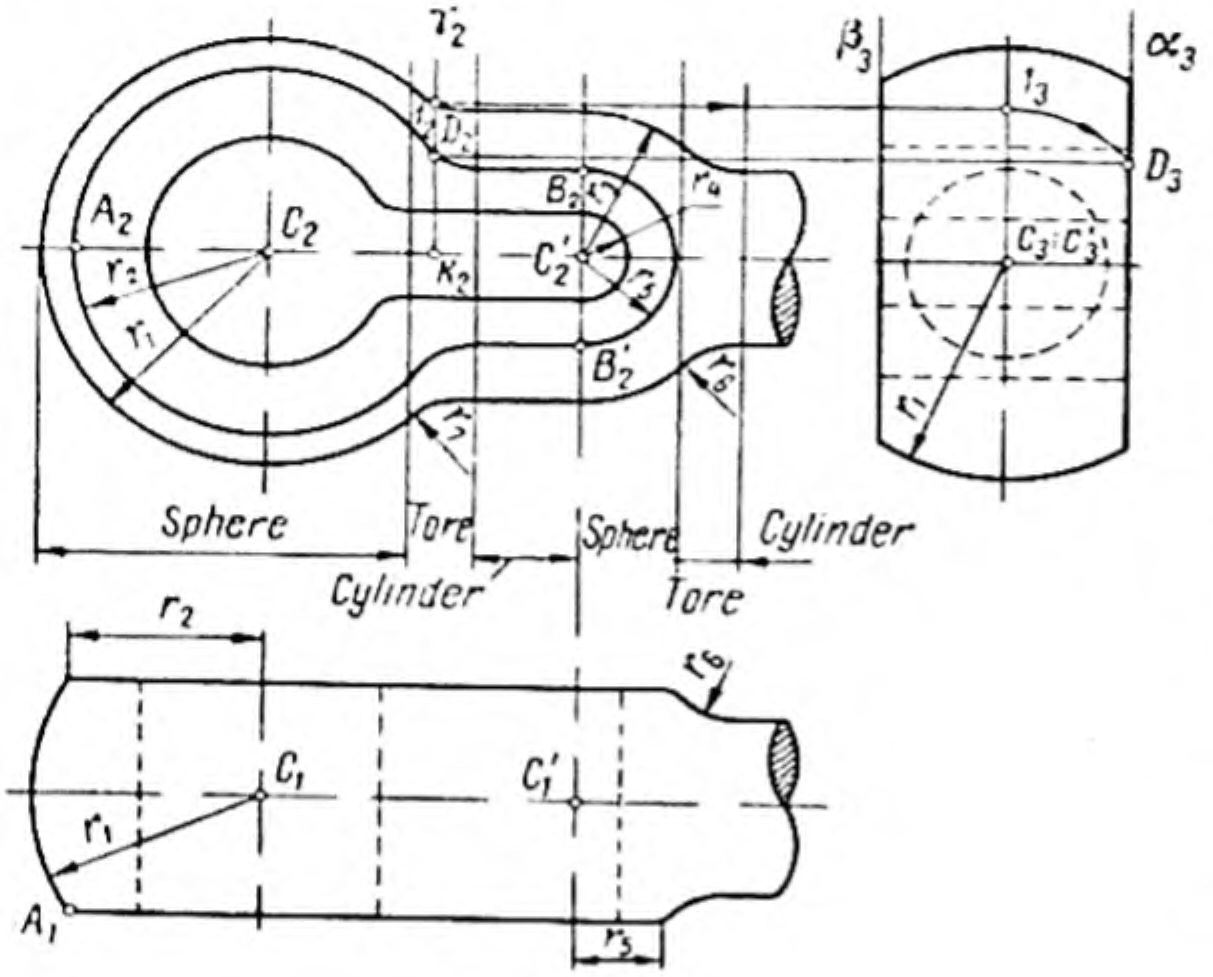


Fig. 320

The generatrices along which the cylindrical surface is intersected by planes α and β may easily be constructed. The projections of these generatrices on plane Π_2 are determined by points B_2 and B_2' in which the semi-circle of radius r_5 intersects the vertical line dividing the frontal projections of the surfaces of the cylinder itself and sphere of radius r_3.

The position of the intermediate point D_2 on the frontal projection of the torus is determined by drawing an auxiliary profile plane γ. This plane cuts the torus in a circle, of radius $K_2 1_2$, which projects in its true size on plane Π_3.

By describing an arc of this circle from point C_3 ($C_3 1_3 = K_2 1_2$), we find point D_3 in which this circle cuts the trace α_3 of the cutting plane. Through point D_3 we draw a horizontal projector to intersect trace γ_2 of the auxiliary plane γ in point D_2. In Fig. 320 these constructions are indicated by arrows.

§ 36. The Transformation of Projections by Oblique Projection and Relative Conformity When Constructing Plane Sections of Surfaces

1. We shall construct the section of an elliptical cone cut by an oblique plane α by transformation of its projections (Fig. 321).

First, plane Π_2 should be substituted by plane Π_4 perpendicular to planes Π_1 and α. The ground line s_{14} of the new system of planes Π_1-Π_4 should be drawn at right angles to trace k_1 of plane α. In the system of planes Π_1-Π_4 the plane α becomes a projecting plane, therefore its trace α_4 will then possess collecting properties. The trace α_4 is constructed with the help of point 1, taken on the frontal trace l of plane α.

Having built the projection of the cone on plane Π_4, we may find the part $M_4K_4N_4$ of trace α_4. This is the projection of the required section on plane Π_4. By drawing a series of generatrices of the cone we find first the horizontal, and then the frontal projections of the points of the section, including points M, K and N in the Π_1-Π_2 system of planes of projection. After this, through the frontal projections of the points obtained an ellipse is drawn. This is the projection on plane Π_2 of the section of the cone cut by plane α. Through the horizontal projections of these points another ellipse, the projection of the section on plane Π_1, is drawn.

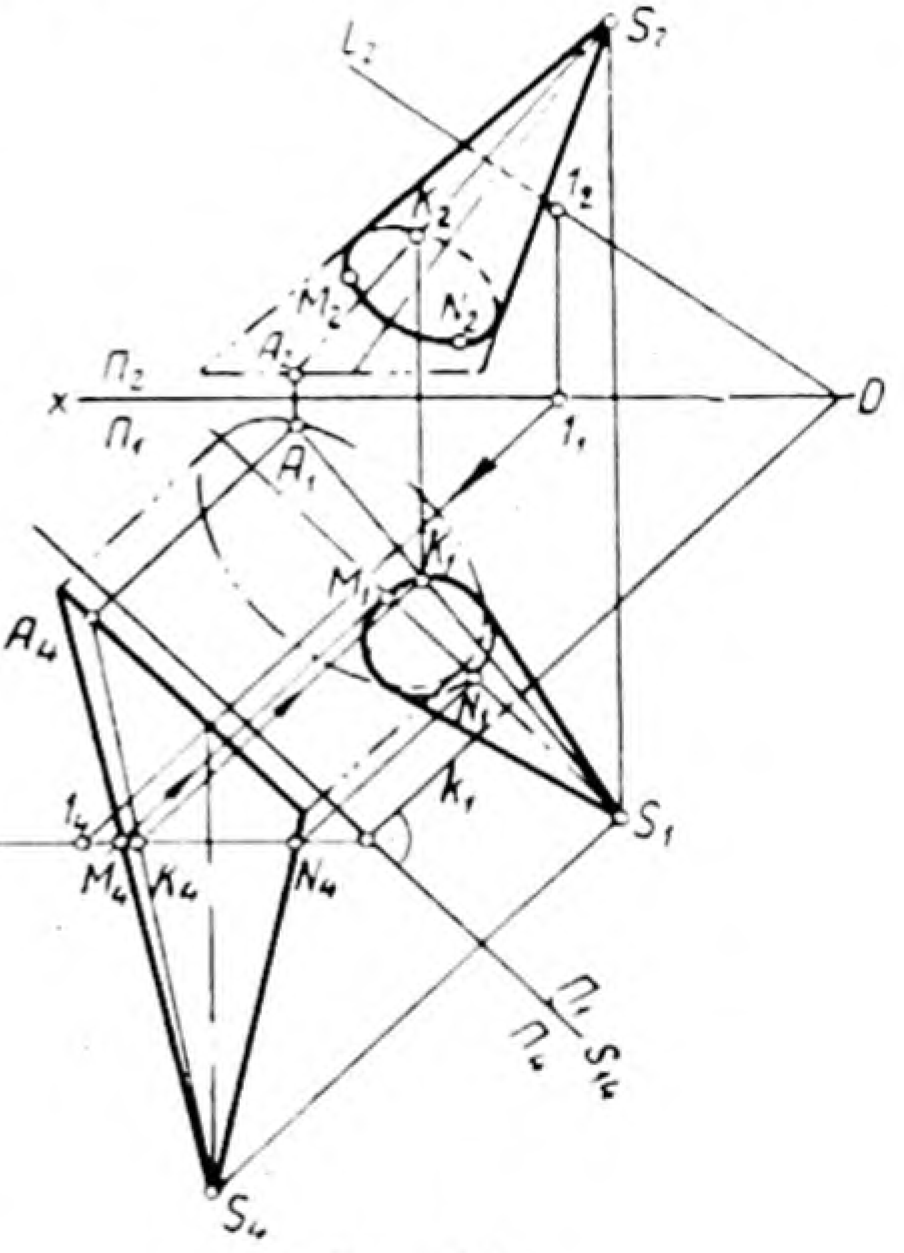


Fig. 321

2. The multiview drawing (Fig. 322) shows the construction of the section of a sphere given by an oblique plane α. The method of transformation of projections is also used in this case.

We project the sphere and plane α onto a new plane of projections Π_4 perpendicular both to plane Π_1 and plane α. The ground line s_{14} of the system of planes of projection Π_1-Π_4 is drawn perpendicularly to trace k_1 of plane α. The sphere is projected onto plane Π_4 as a circle with its centre at point C_4 which lies at the same distance from ground line s_{14} as the point C_2 from ground line Ox.

The plane α is projected onto plane Π_4 as the line α_4, and the section of the sphere as the line A_4B_4. This line coincides with line α_4, the trace-projection of the plane on plane Π_4.

Line A_4B_4 determines the length of the major axes of the ellipses in which the section of the sphere, a circle, is projected onto planes Π_1 and Π_2. The projection of the centre C' of the circle of the section on plane Π_4 is point C'_4, the midpoint of line A_4B_4. We find the projection of this point C' on planes Π_1 and Π_2. The projection C'_1 of this point on plane Π_1 is given by the intersection of the projector drawn through C'_4 perpendicular to s_{14} and the line passing through point C_1 parallel to ground line s_{14}. The projection C'_2 of point C' lies on the projector from C'_1 perpendicular to ground line Ox, at a distance $z_{C'}$ from that axis. This distance $z_{C'}$ is obtained from the projection on plane Π_4.

Having found points C_1' and C_2' through them we draw lines respectively parallel to traces k_1 and l_2 and lay off on them to either side of points C_1' and C_2' distances equal to one half of the length of line A_4B_4. The lines D_1E_1 and S_2F_2 obtained are the major axes of the ellipses which are projections of the circular section on planes Π_1 and Π_2.

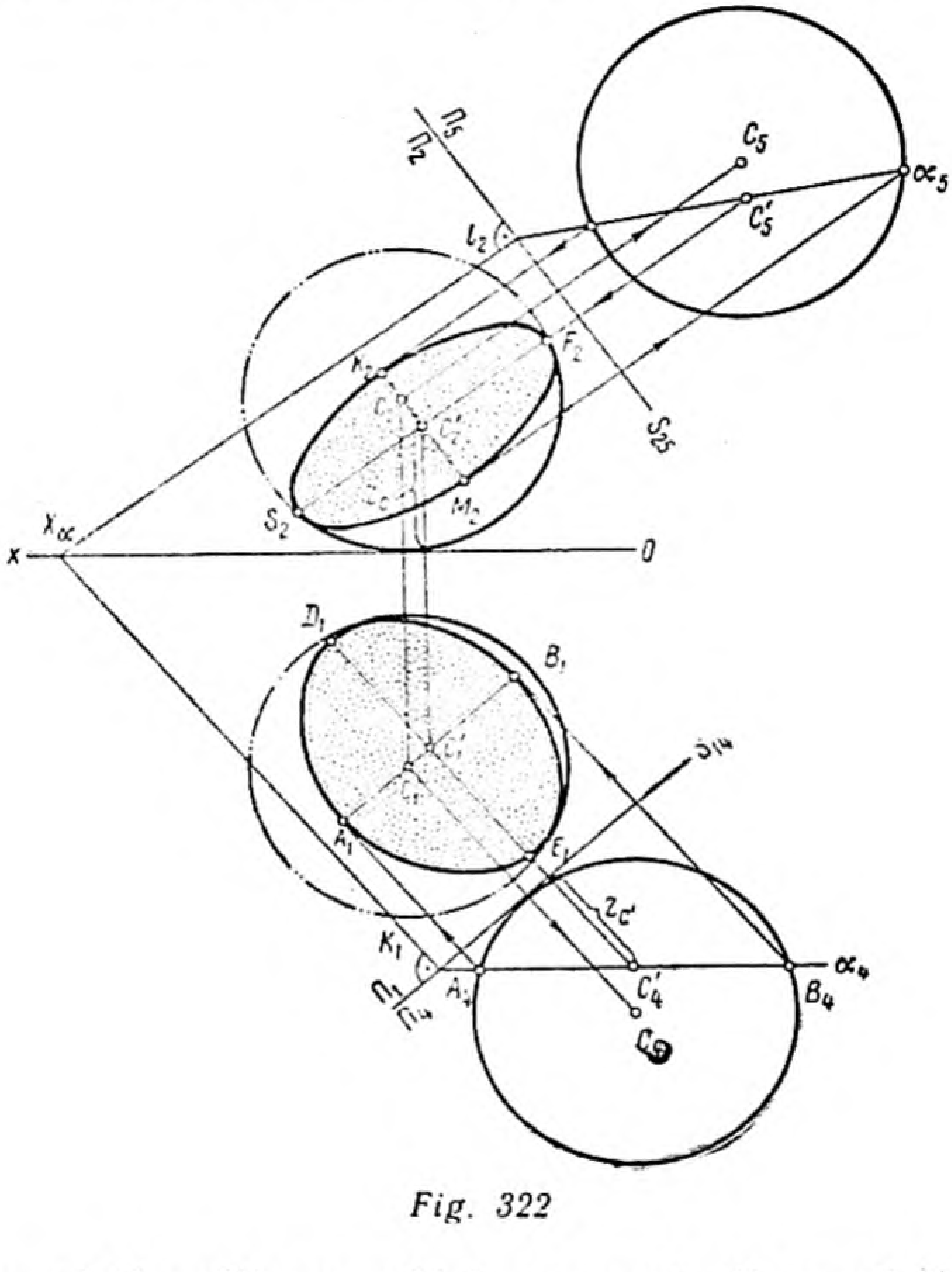


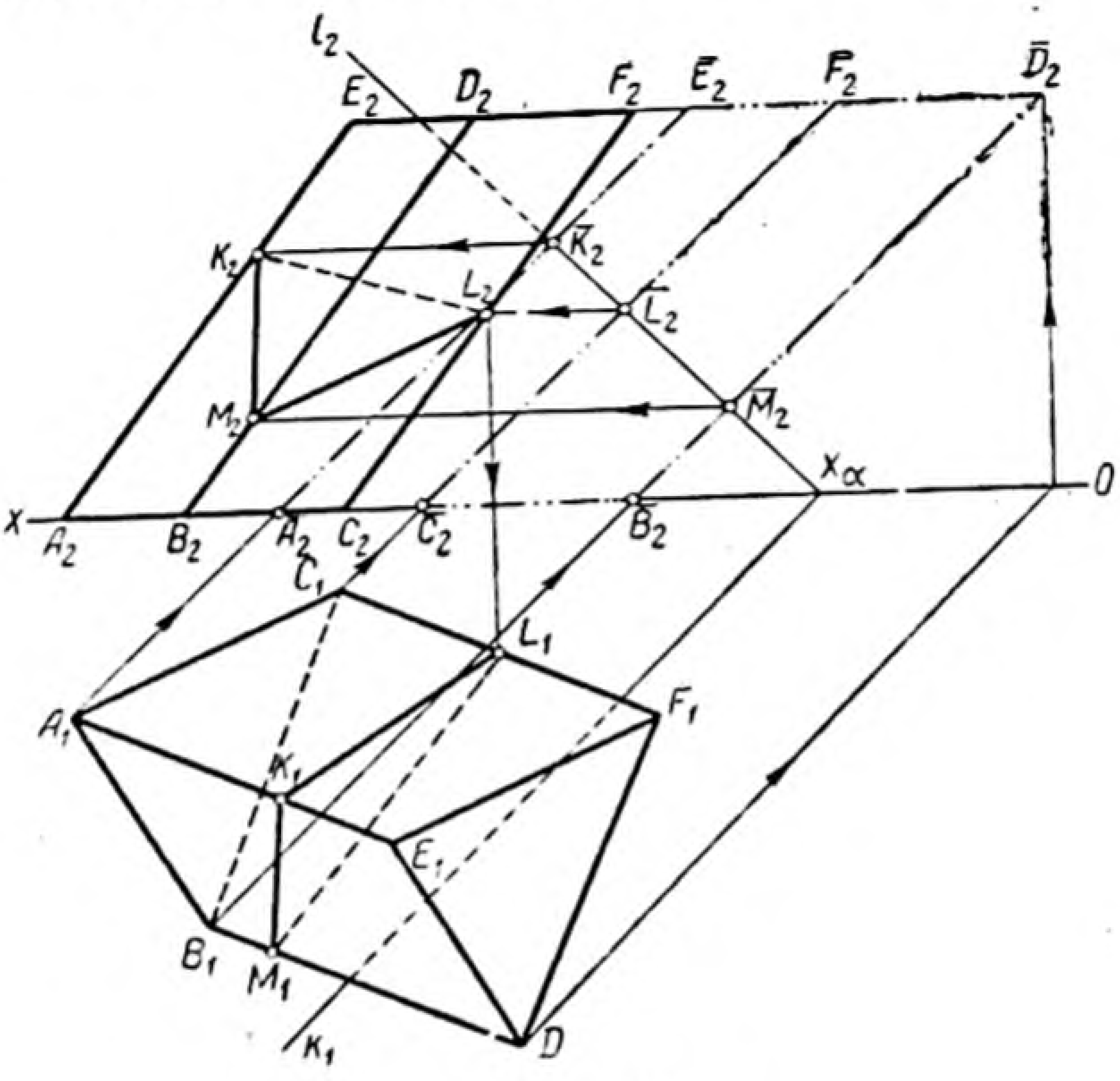
Fig. 322

The minor axis A_1B_1 of the ellipse which is the projection of the circular section on plane Π_1 is given by the projection of line AB on Π_1. To obtain the minor axis through points A_4 and B_4 draw projectors perpendicular to ground line s_{14} to intersect the line passing through point C_1' parallel to ground line s_{14}.

The length of the minor axis K_2M_2 of the ellipse on plane Π_2 is obtained by a similar transformation of projections, however in this case, a new plane Π_5, perpendicular both to plane Π_2 and the cutting plane α, is substituted for plane Π_1.

Having found the position and length of the axes of the ellipses which are projections of the circular section of the sphere on planes Π_1 and Π_2, we can proceed to draw in the curves themselves.

3. We shall now examine the method of constructing plane sections by means of oblique projection. This type of projection has



l_2
E_2
D_2
F_2
$\overline{E}_2$
$\overline{F}_2$
$\overline{D}_2$
K_2
$\overline{K}_2$
L_2
$\overline{L}_2$
M_2
$\overline{M}_2$
x_α
0
x
A_2
B_2
$\overline{A}_2$
C_2
$\overline{C}_2$
$\overline{B}_2$
C_1
L_1
A_1
F_1
K_1
E_1
B_1
M_1
k_1
D

Fig. 323

already been used to determine the piercing point of a line with a plane (see Fig. 170).

The construction of the section of an oblique triangular prism given by an oblique plane α is shown in Fig. 323.

The plane α and prism are projected onto plane Π_2 in a direction parallel to trace k_1. The projection of plane α on plane Π_2 is then line l_2, i. e., the frontal trace of plane α. The prism is projected onto plane Π_2 as $\overline{A}_2\overline{B}_2\overline{C}_2\overline{E}_2\overline{F}_2\overline{D}_2$. In Fig. 323 this projection is shown by dashes and two dots. The intersection of the oblique projections of the lateral edges of the prism with the trace l_2, points $\overline{K}_2$, $\overline{L}_2$ and $\overline{M}_2$ determine the vertices of the triangle KLM received when the plane α intersects the prism. Reverse projection gives the frontal projections K_2, L_2 and M_2 of the vertices of the intersection triangle KLM and the horizontal projections K_1, L_1 and M_1.

4. The construction shown in Fig. 324 is carried out by using the related conformity properties of triangle KLM in which the surface of the pyramid $SABC$ is intersected by plane α.

In § 31 it was already noted that there exists a relationship between the projection of a plane figure and the figure itself. This is called the related conformity and this conformity exists between any two projections of the figure.

Therefore, problems which are solved by conventional methods of descriptive geometry may also be solved by using the method

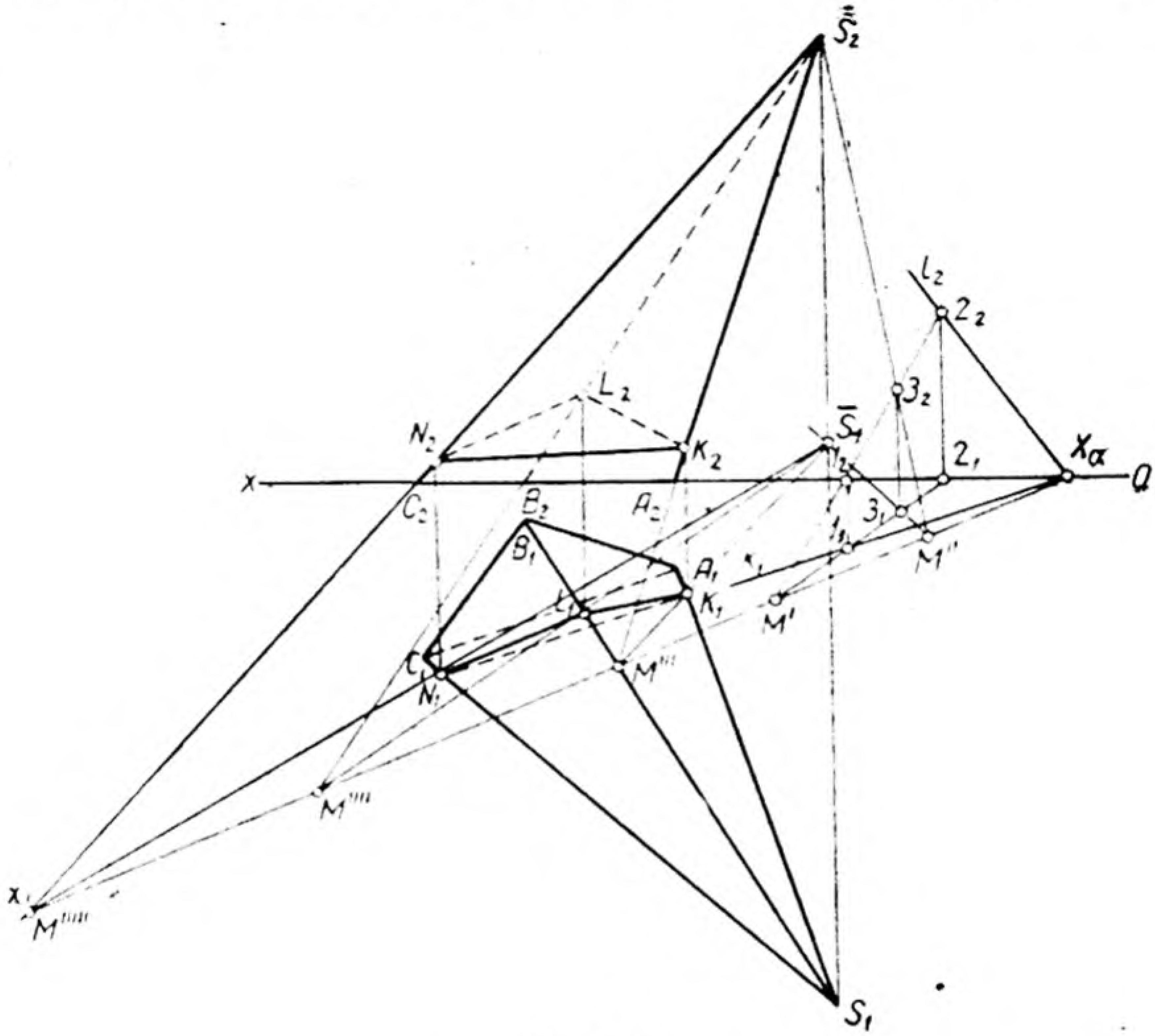


Fig. 324

of related conformity for any two projections of the projected objects.

Tackling this problem (Fig. 324), we must establish the related conformity between the section in plane α and the horizontal projection of that same section of the pyramid. We begin by determining the axis and direction of relativity. To determine the direction of the relativity it is necessary to find one pair of related points.

The first duplicate point through which the axis of relativity will pass X_α, is the point of convergence of traces of the plane. To determine the second duplicate point we draw in plane α any line *1-2*, the projections of which, $1_1 2_1$ and $1_2 2_2$, intersect in point M' which corresponds to itself. After this we pass the axis of relativity through points X_α and M'. The direction of relativity, in this particular case of related conformity is perpendicular to ground line Ox.

Point $\overline{S}_1$, related to point S_2, is now found by joining point S_2 by a straight line with projection 3_2 of a point 3 taken at random on line 1-2. Line $S_2 3_2$ is produced to intersect the axis of relativity in point M''. Then by drawing the straight line $M''3_1$, which is related to line $S_2 3_2$, to intersect with the line S_2S_1, which runs in the direction of relativity, we find point $\overline{S}_1$ corresponding to the related point S_2.

Then we extend the projections S_2A_2, S_2B_2 and S_2C_2 of the lateral edges of the pyramid to cut the axis of relativity in points M''', M'''' and M'''''. By joining up these points with point $\overline{S}_1$ we obtain lines $\overline{S}_1M'''$, $\overline{S}_1M''''$ and $\overline{S}_1M'''''$ which are related to lines S_2A_2, S_2B_2 and S_2C_2.

The intersections of lines $\overline{S}_1M'''$, $\overline{S}_1M''''$ and $\overline{S}_1M'''''$ with the horizontal projections S_1A_1, S_1B_1 and S_1C_1 of the lateral edges of the pyramid give the horizontal projections K_1, L_1 and N_1 of the vertices of triangle KLN which is the figure received when the pyramid $SABC$ is cut by plane α.

The frontal projection $K_2L_2N_2$ of this triangle is constructed in the usual way with the help of projectors.

Problems and Exercises

1. Which planes are generally used as auxiliary planes when constructing plane sections?
2. Describe the sequence of the constructions used to find the section of a polyhedron given by an oblique plane?

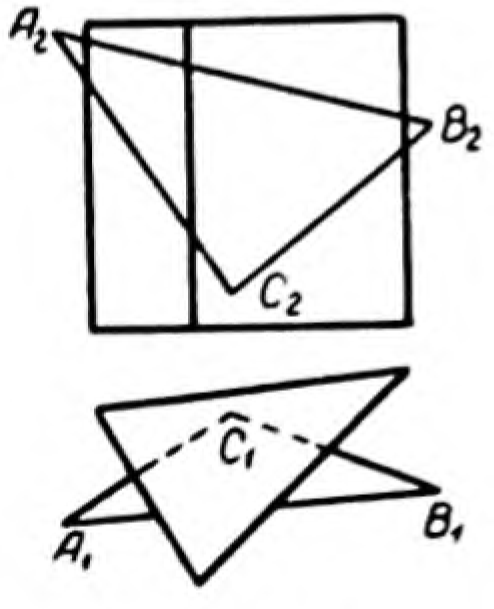


Fig. 325

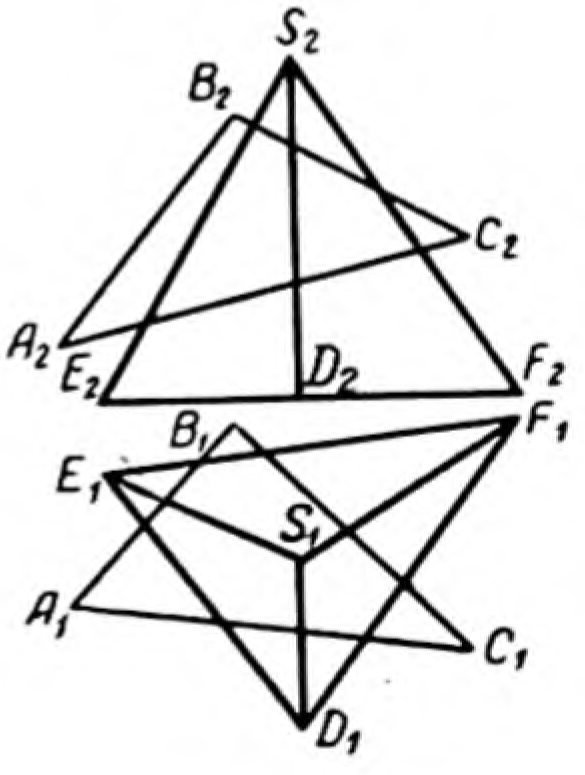


Fig. 326

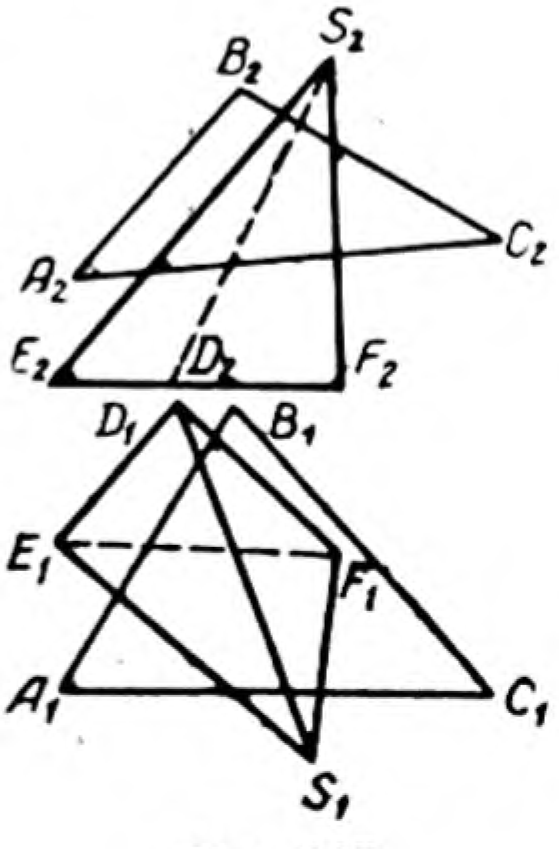


Fig. 327

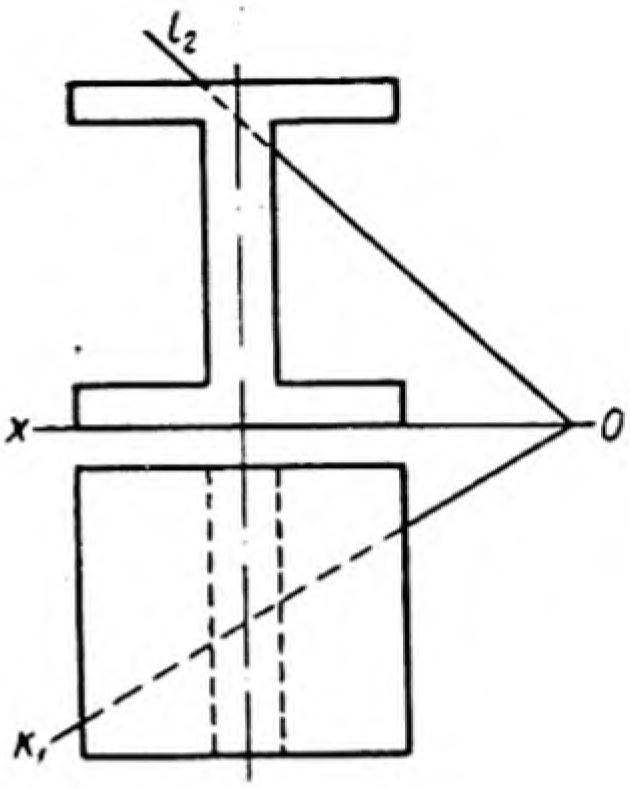


Fig. 328

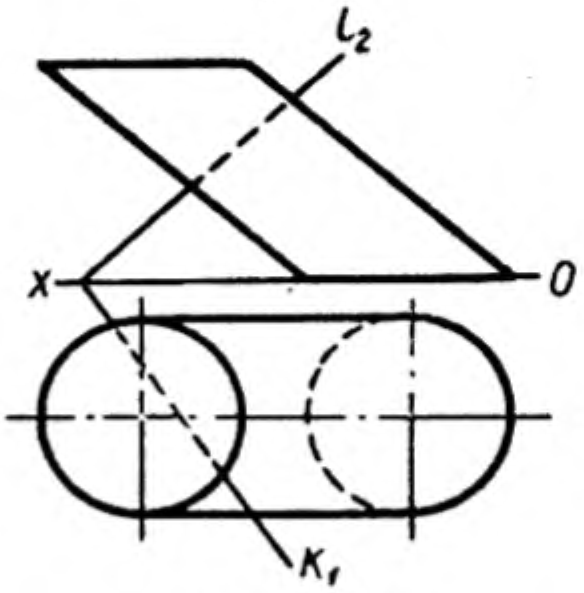


Fig. 329

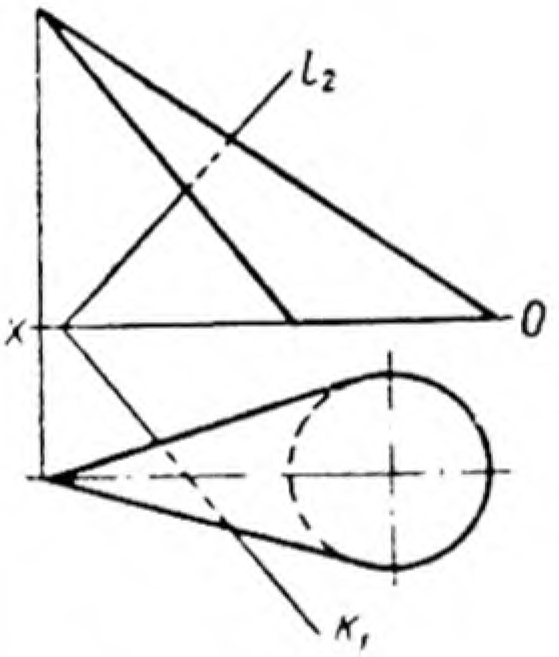


Fig. 330

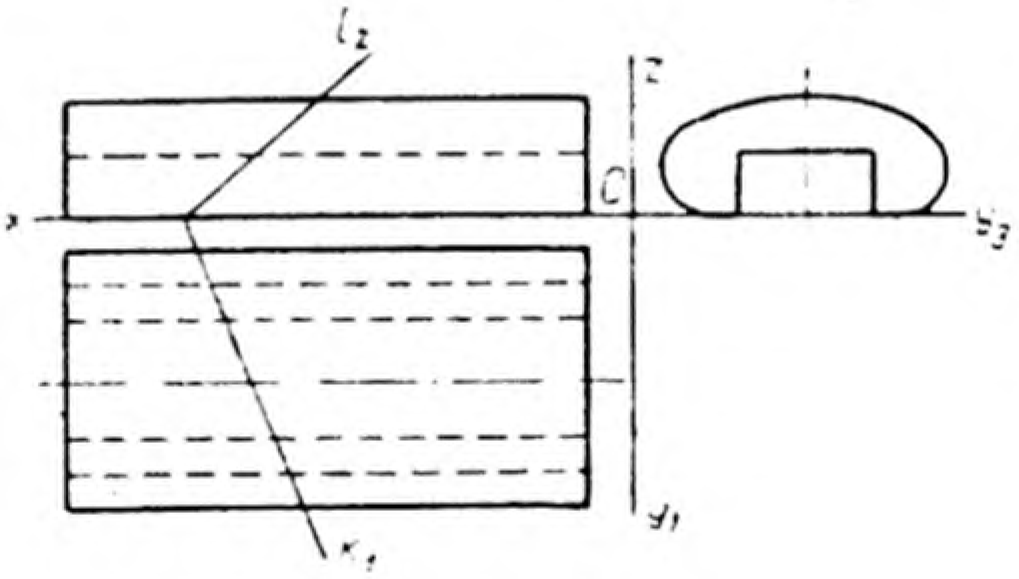

l_2
z
O
x
y_3
y_1
K_1

Fig. 331

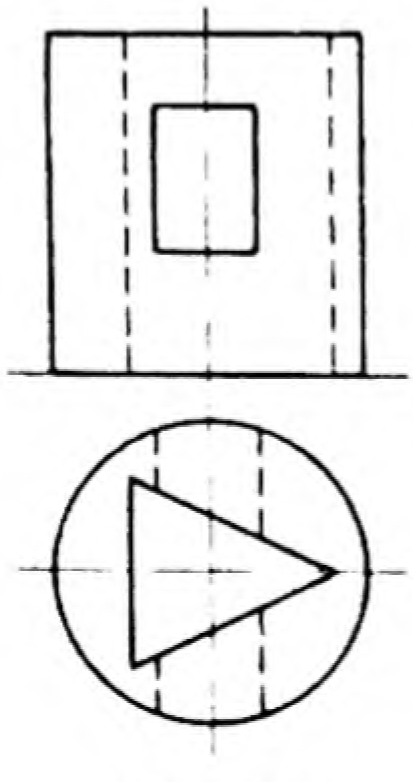


Fig. 332

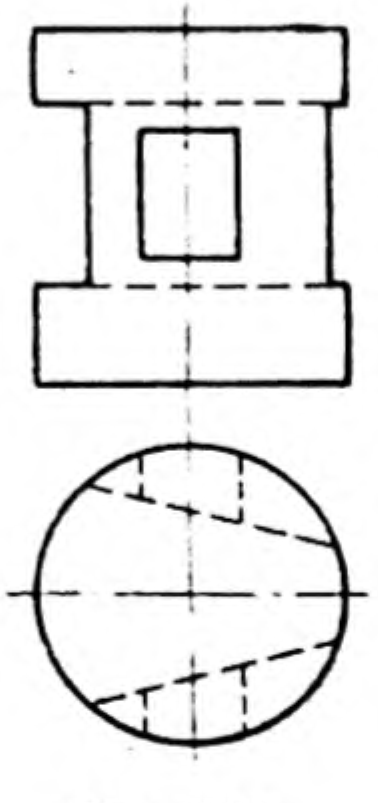


Fig. 333

3. Which points are called special points when constructing a plane section of a curved surface?

4. Construct the section given by an oblique plane when it cuts a triangular right prism. The traces of the plane make angles 45° and 30° with the ground line *Ox*.

5. Construct the intersection of a triangular plate *ABC* with a prism. Show the visible parts of the plate by hatching (Fig. 325).

6. Construct the intersection of a triangular plate *ABC* with a pyramid (Figs. 326 and 327).

Hint. This problem should be solved by two methods, first, by using an auxiliary plane, and then by substituting planes of projection (see Fig. 321).

7. Construct the intersection of a given space figure by plane α (Fig. 328).

8. Construct the intersection of a cylinder by a plane α (Fig. 329). First, solve the problem in the conventional auxiliary planes, and then, by using oblique projection.

9. Construct the intersection of a cone by a plane α (Fig. 330). First, solve the problem conventionally by passing auxiliary planes, then, employ the method of related conformity.

10. Construct the section of the handrail, shown in Fig. 331, when it is cut by plane α. Find the true view of the section.

11. Construct the third projection of the cylinders with through holes, shown in Figs. 332 and 333.

Chapter IX

INTERSECTION OF LINES AND SURFACES

§ 37. Construction of Points of Intersection of Lines and Surfaces with the Help of Projecting Planes

1. In general it may be said that the construction of the point of intersection of a straight line with a surface consists of three operations, these are as follows: 1) an auxiliary surface is passed through the line; 2) the section cut by the auxiliary plane is found; 3) the point, or points, of intersection of the given line with the section which has been constructed is determined. Such points are the points of *intersection* of the line with the surface, i. e., the *piercing points*. The number of piercing points of a line depends on the form of the surface. A line generally pierces the surfaces of simple geometrical bodies, such as prisms, cylinders, sphere, etc., in two points, one of which may be called the *point of entry*, and the other, the *point of exit*.

If the surface which the line pierces is perpendicular to one of the planes of projection, then the piercing points may be determined very easily. For example, points K and L, in which line DE pierces the right triangular prism ABC (Fig. 334), are projected onto plane Π_1 as points K_1 and L_1 in which the projection D_1E_1 of line DE intersects the triangular horizontal projection of the two ends of the prism. Once points K_1 and L_1 are known, projectors are used to find the frontal projections K_2 and L_2 of points K and L.

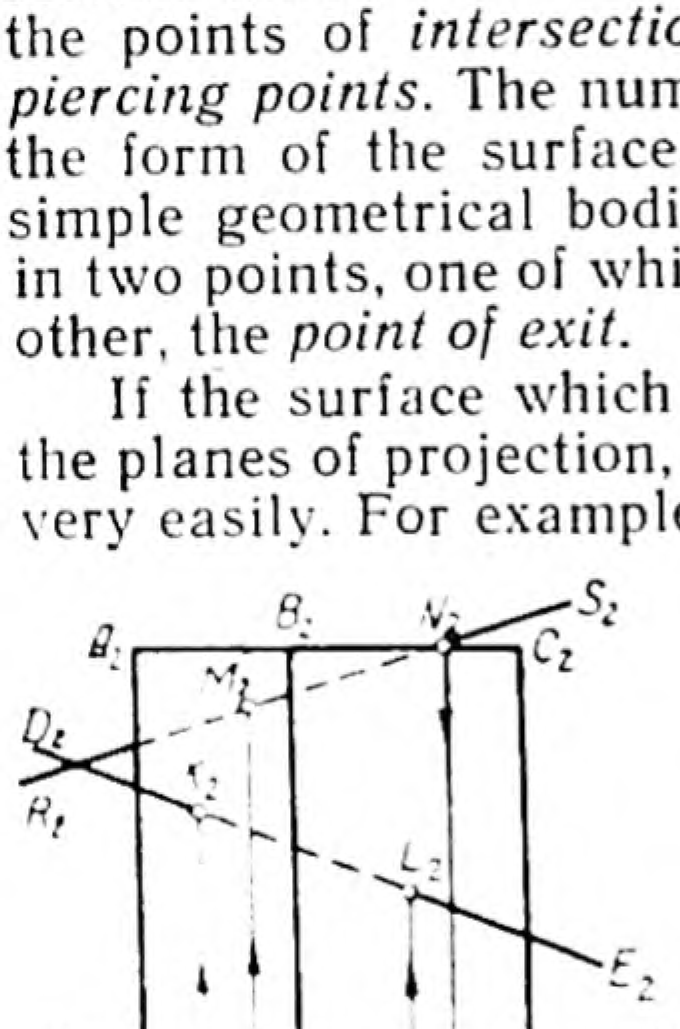


Fig. 334

The construction of points M and N, where another line, line RS, pierces the surface of the prism, is shown in Fig. 334. In this case point M lies in the hidden side of the prism, and point N lies in its upper face. The sequence of the construction is indicated by arrows.

The intersection points of a line and a right cylinder, the axis of which is perpendicular to plane of projection Π_1, may also be found without auxiliary constructions (Fig. 335).

It should be noted that when surfaces such as right prisms or right cylinders are intersected by curved lines, the piercing points are also found in this way. A space curve CD is shown as intersecting the surface of a cylinder in Fig. 335. The horizontal projections K_1' and L_1' of piercing points K' and L' are the points of intersection of curve C_1D_1 with the circle which is the horizontal projection of the cylinder. The frontal projections K_2' and L_2' are determined by projectors drawn to meet the frontal projection C_2D_2 of curve CD, as shown.

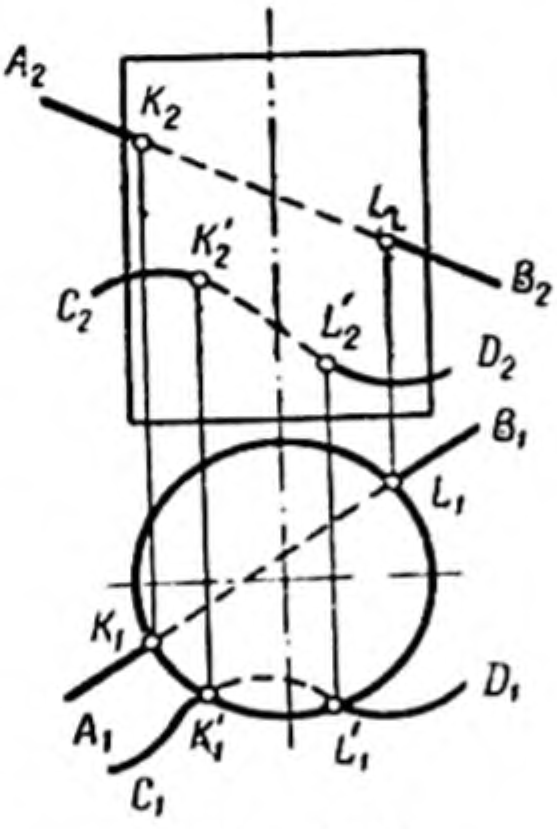


Fig. 335

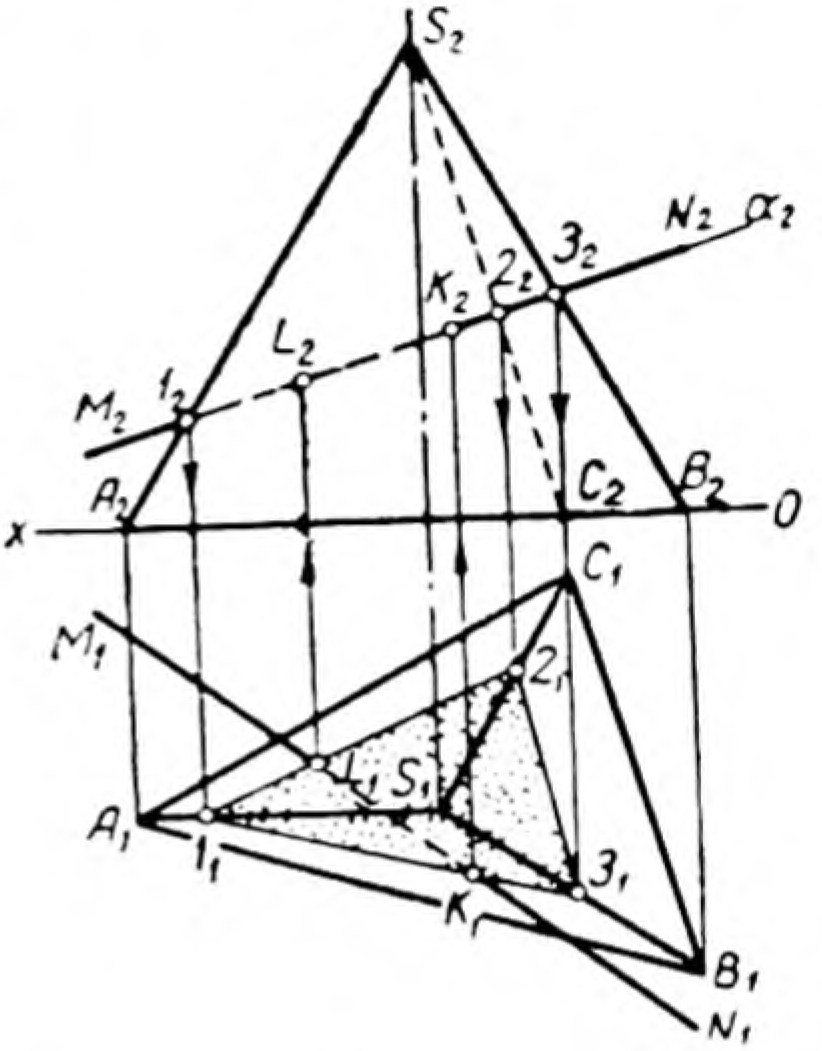


Fig. 336

2. The construction of the points of intersection of line MN and the triangular pyramid $SABC$ is shown in Fig. 336.

Draw an auxiliary frontal-projecting plane α through line MN. The section of the pyramid cut by this plane is triangle *1-2-3*. First, points 1_2, 2_2 and 3_2, where trace α_2 intersects frontal projections of the edges S_2A_2, S_2C_2 and S_2B_2, are found. Then, by means of these points and projectors, points 1_1, 2_1, 3_1 are determined. These points are joined by straight lines to obtain the triangle $1_12_13_1$, which is the horizontal projection of the section of the pyramid cut by the auxiliary plane α.

The straight line MN contained in plane α, together with triangle *1-2-3*, intersects the sides of this triangle in points K and L which are the points of intersection of this line with the surface of the pyramid. Points K_1 and L_1 are the points of intersections of projection M_1N_1 of line MN with the lines 1_12_1 and 1_13_1, and the latter are projections of the sides of triangle $1_12_13_1$. By using projectors we find on M_2N_2 the frontal projections K_2 and L_2 of the points K and L where line MN pierces the surface of pyramid.

When determining the visibility of separate parts of line MN, assuming this line is being projected onto planes Π_1 and Π_2, it is necessary to take into account the visibility of the faces of the pyramid on these planes of projection. When projecting onto plane Π_1 all three lateral faces are visible, so point K, contained in face SAB, and point L, contained in face SAC, are also visible. Therefore, the portions M_1L_1 and K_1N_1 of the horizontal projection of line MN are drawn as continuous lines, whereas the projection L_1K_1 of the part of the line MN within the surface is shown by dashes. When projecting onto plane Π_2 the face SAB is visible, but face SAC is not visible. Therefore the part K_2N_2 of the frontal projection of line MN is shown as a continuous line. The portions 1_2L_2 and L_2K_2 are shown by dashes, since they are the frontal projections of the part $1L$, which is hidden by the surface of the pyramid, and of the part LK, which is within the surface of the pyramid.

3. The points of intersection of a straight line a with a surface of revolution are also determined by passing a projecting plane through line a (Fig. 337).

The line in which the frontal-projecting plane α, passing through line a, cuts the surface of revolution is a curve. The frontal projection of this curve coincides with trace α_2.

To construct the horizontal projection of the curve of intersection a series of auxiliary planes are drawn to cut the surface of revolution and plane α. The sections received will be circles and horizontals perpendicular to plane Π_2. Having found the horizontal projections of these sections we determine points 2_1 and $2_1'$, 3_1 and $3_1'$, in which horizontal projections of the circles and horizontals intersect. These points and the projections 1_1, 4_1 and $4_1'$ are joined by a curve. The intersection of curve $4_13_12_11_12_1'3_1'4_1'$ with projection a_1 of line a gives points K_1 and L_1. These are the horizontal projections of the points of intersection of line a with the surface of revolution. Having obtained points K_1 and L_1 we find the points K_2 and L_2 on the frontal projection a_2 of line a. Finally, we determine the visibility of the separate parts of line a.

This solution demands the construction of an auxiliary curve. Although this involves extra work, it is unavoidable, since special constructions, not dealt with in this course, would have to be used to eliminate the construction of the curve.

4. The method of transforming projections to determine the points of intersection of a line AB and a sphere is shown in Fig. 338.

An auxiliary horizontal-projecting plane α is passed through line AB. The circular section of the sphere cut by this plane, together with line AB, is projected onto a new plane of projection Π_4. This plane Π_4 is perpendicular to plane Π_1 and parallel to plane α. The centre C' of the circular section lies at the same altitude as the centre C of the sphere itself, therefore the projection C_4' of point C' on plane Π_4 lies on the projector C_1C_4' at a distance from

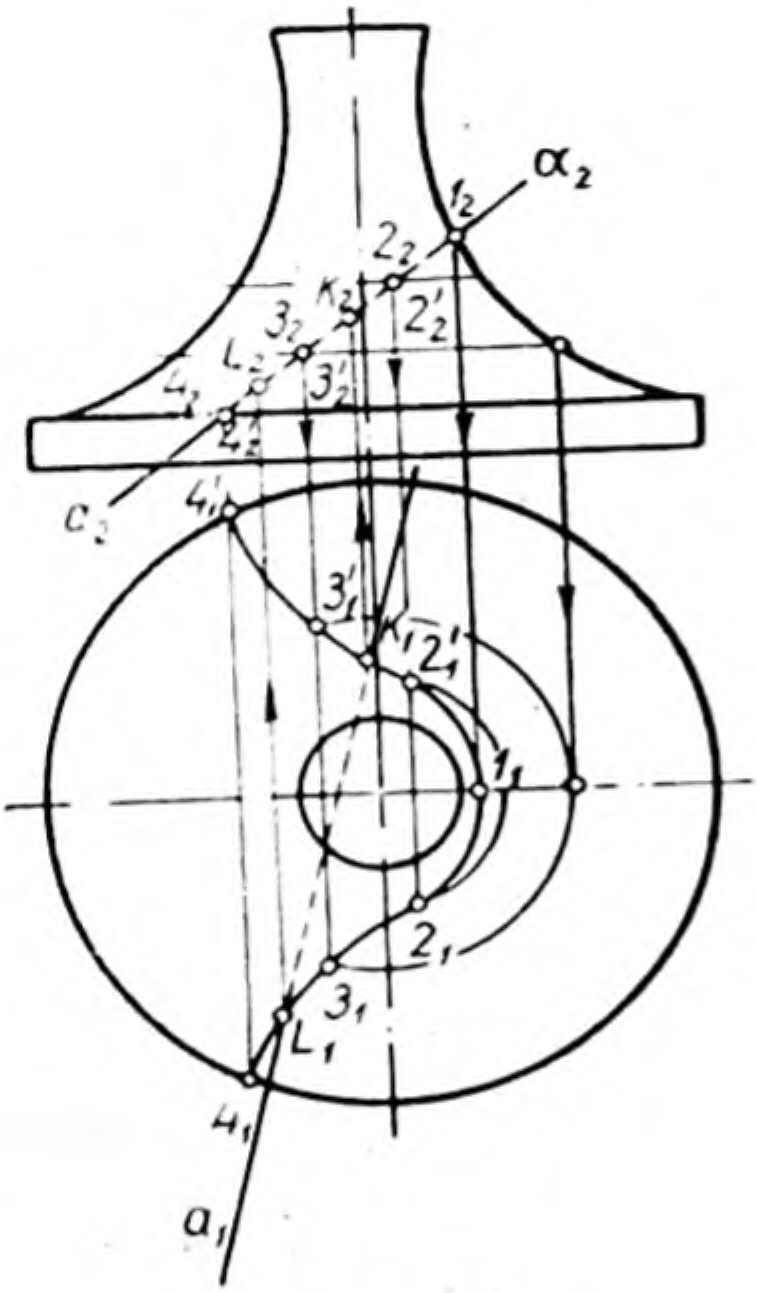


Fig. 337

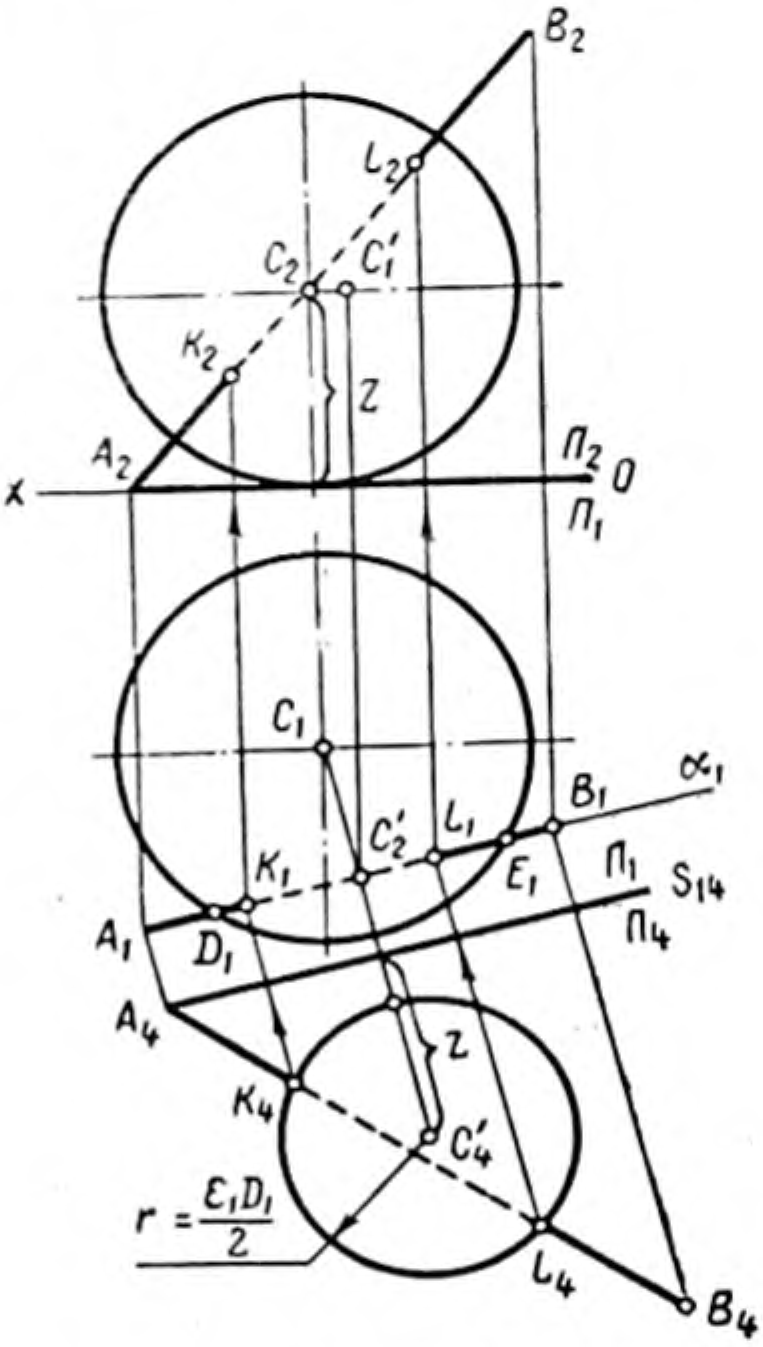


Fig. 338

the ground line s_{14} equal to the distance z for point C. A circle with a radius r, equal to one half of the length of line E_1D_1, is described with point C_4' as its centre. This circle intersects projection A_4B_4 of line AB on plane Π_4 in points K_4 and L_4. These points are the projections on plane Π_4 of points K and L in which line AB pierces the surface of the sphere.

Having obtained points K_4 and L_4 we first find the horizontal projections K_1 and L_1 and then the frontal projections K_2 and L_2 of points K and L. Point K is visible when projected onto plane Π_2, but it is hidden when projected onto plane Π_1. Point L is visible both when projected onto plane Π_1 and onto plane Π_2. This is taken into account when determining the visibility of the separate parts of the projections of line AB on the multiview drawing. It is quite evident that the part KL of line AB is hidden both when projected onto plane Π_1 and onto plane Π_2 since it penetrates the sphere.

§ 38. The Use of Oblique Planes to Determine Points of Intersection of Lines and Surfaces

1. In many cases the construction of the points of intersection of a straight line and a surface is simplified by passing an oblique plane through the line, instead of a projecting plane.

For example, if a plane perpendicular to one of the planes of projection were to be used for the construction of the points of intersection of line *AB* with the surface of an elliptical cylinder (Fig. 339), the section obtained would be an ellipse and one of its projections would have to be plotted by points. However, by passing an oblique plane parallel to the generatrices of the cylinder through line *AB* section of the cylinder will be given by two generatrices.

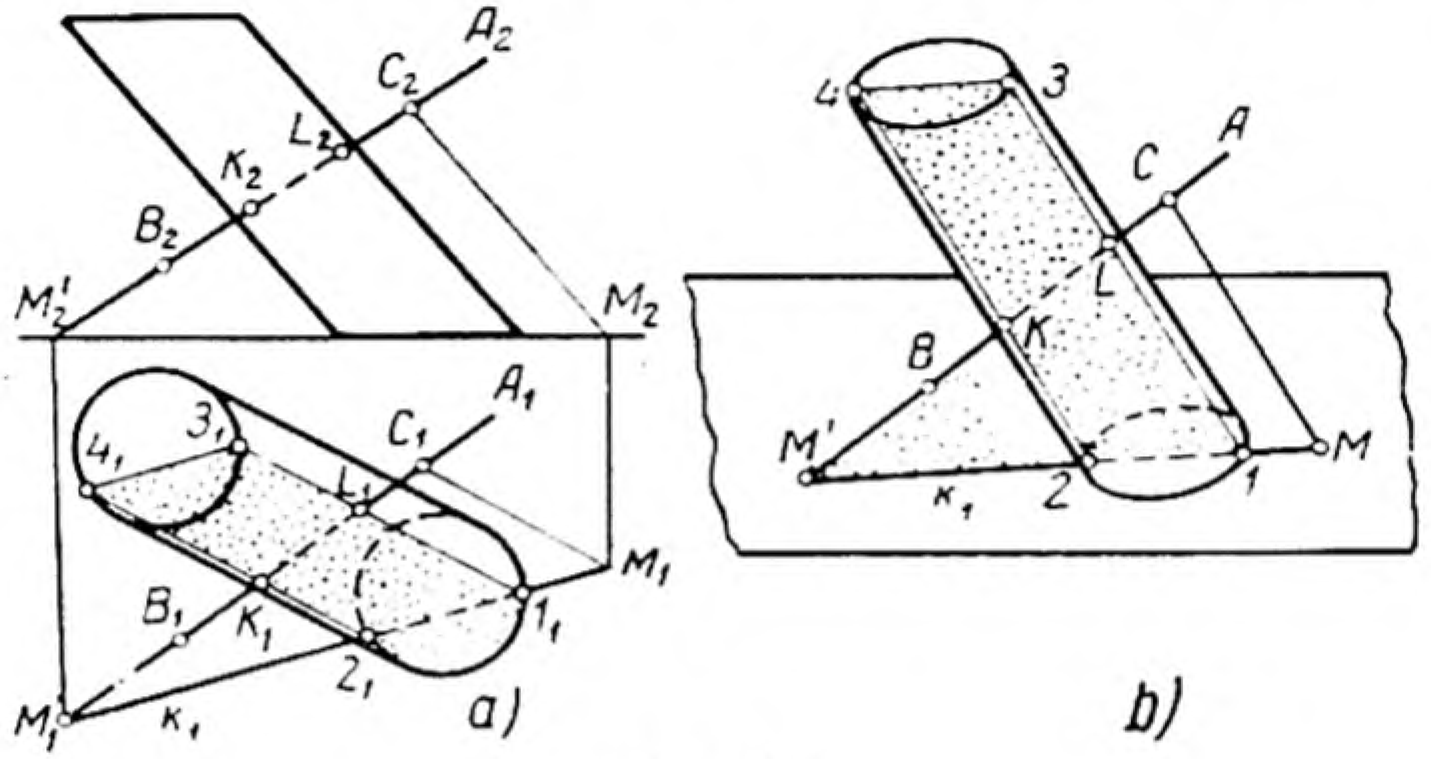


Fig. 339

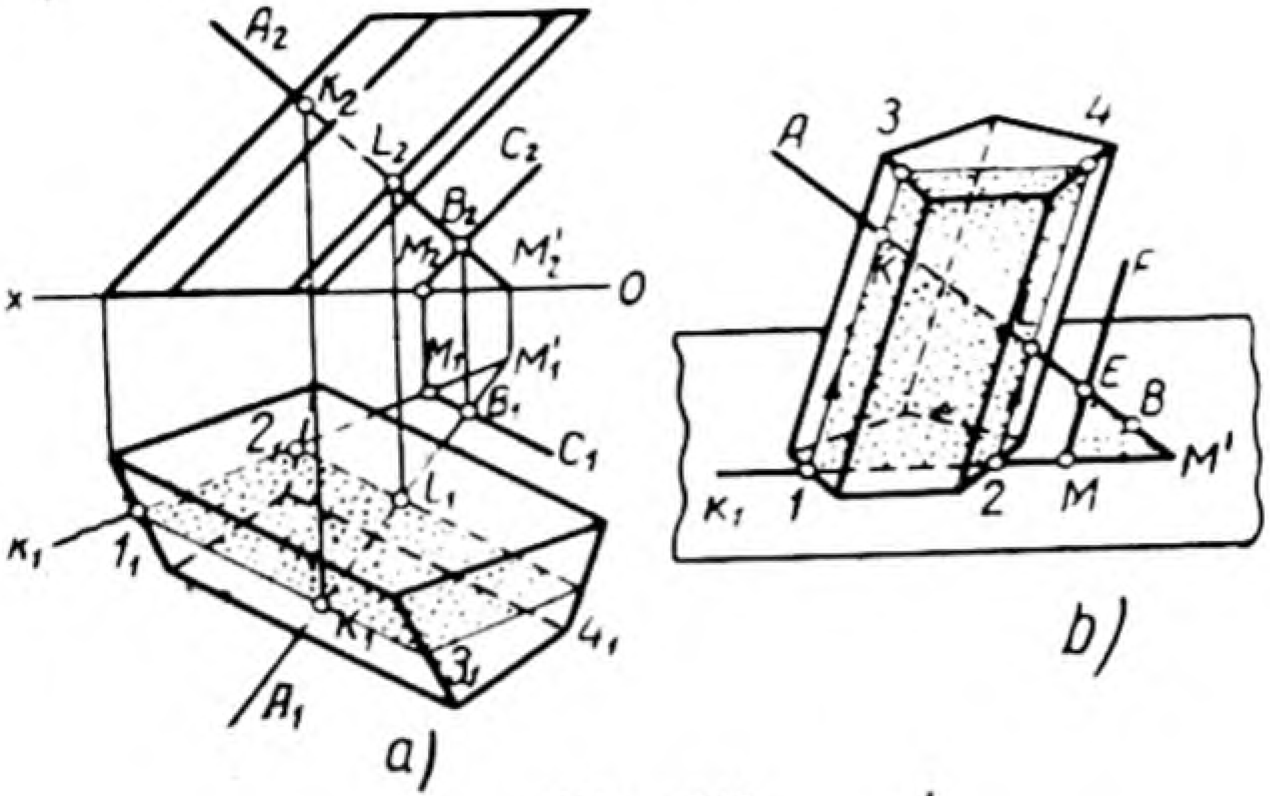
Such a plane may easily be constructed. Let us take any point *C* on line *AB* and through it draw line *CM* parallel to the generatrices of the cylinder. Then the plane determined by the intersecting lines *AB* and *CM* will be parallel to the generatrices of the cylinder.

The horizontal trace *M* of line *CM* and the horizontal trace *M'* of line *AB* are then found. The horizontal trace of the auxiliary plane determined by straight lines *AB* and *CM* passes through points *M* and *M'*. This trace k_1 intersects the base of the cylinder contained in plane Π_1 in points *1* and *2*. The horizontal projections of the generatrices *1-3* and *2-4*, along which the auxiliary plane cuts the lateral surface of the cylinder, are drawn through points 1_1 and 2_1.

The points K_1 and L_1, where lines $2_1 4_1$ and $1_1 3_1$ intersect line A_1B_1, are the horizontal projections of the required points *K* and *L* where line *AB* pierces the surface of the cylinder. The frontal projections K_2 and L_2 of points *K* and *L* are found by drawing projectors through K_1 and L_1 to intersect the frontal projection A_2B_2 of line *AB*.

Planes, such as those considered, which reduce the construction lines to a minimum are the most simple oblique auxiliary planes.

2. A simple oblique auxiliary plane is also used in Fig. 340, *a* for constructing the points of intersection of line *AB* with the faces of an oblique prism, although in this case it would also have been possible to pass a projecting plane through line *AB*.



a)

b)

Fig. 340

As in the preceding construction, a line *CM* is drawn through point *B* on line *AB* parallel to the lateral edges of the prism. The plane α, determined by the intersecting lines *AB* and *CM*, intersects the faces of the prism along lines *1-3* and *2-4* parallel to the lateral edges of the prism. The intersection of these lines with line *AB* gives the required points *K* and *L* where line *AB* pierces the prism.

This method of finding the points of intersection of a line and a pentagular prism can be seen in axonometric projection in Fig. 340, *b*.

3. When determining the points of intersection of line *AB* and a cone (Fig. 341) the most simple oblique auxiliary plane will pass through line *AB* and the vertex *S* of the cone. Such an auxiliary plane intersects the lateral surface of the cone along two generatrices. To construct this plane we take on line *AB* points *1* and *2* at random and join them by straight lines to the vertex *S* of the cone. Having found the horizontal traces *M* and *M′* of lines *S1* and *S2* we pass the horizontal trace k_1 of the plane through them and draw the generatrices *SE* and *SD* along which the plane intersects the surface of the cone. The points *K* and *L* in which generatrices *SE* and *SD* cut line *AB* are the points of intersection of this line and the cone.

4. When the base of the cone does not lie in the Π_1 plane of projection, but in some frontal-projecting plane, as shown in

Fig. 342, the construction of the points of intersection of line *AB* and the cone remains almost unchanged. By drawing auxiliary lines *S1* and *S2* through the vertex *S* of the cone and points *1* and *2* taken on line *AB* we find the traces *M* and *M′* of these lines and trace *MM′* of the auxiliary plane on the plane in which the base of the cone lies. We obtain the points *C* and *D* at the intersection of trace *MM′* with the base of the cone. The generatrices *SC* and *SD*, along which the auxiliary plane intersects the surface of the cone, pass through these points. Points *K* and *L*, in which the line *AB* pierces the cone, lie at the intersection of the generatrices *SC* and *SD* with line *AB*.

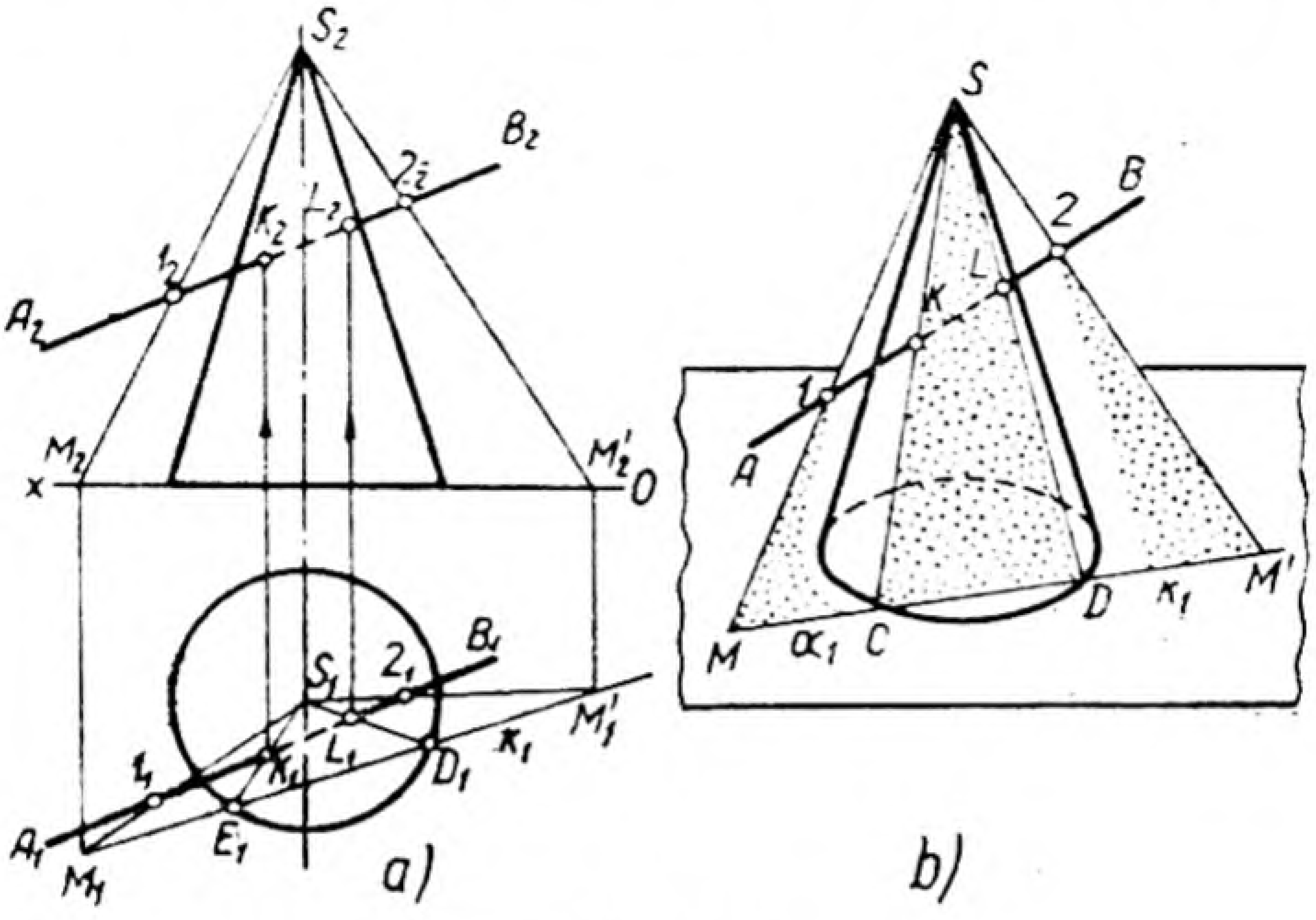

S_2
B_2
2_2
L_2
K_2
1_2
A_2
M_2
M'_2
x
0
S_1
2_1
B_1
1_1
K_1
L_1
D_1
α_1
M'_1
E_1
A_1
M_1
a)
S
B
2
L
K
1
A
M'
α_1
D
M
α_1
C
b)

Fig. 341

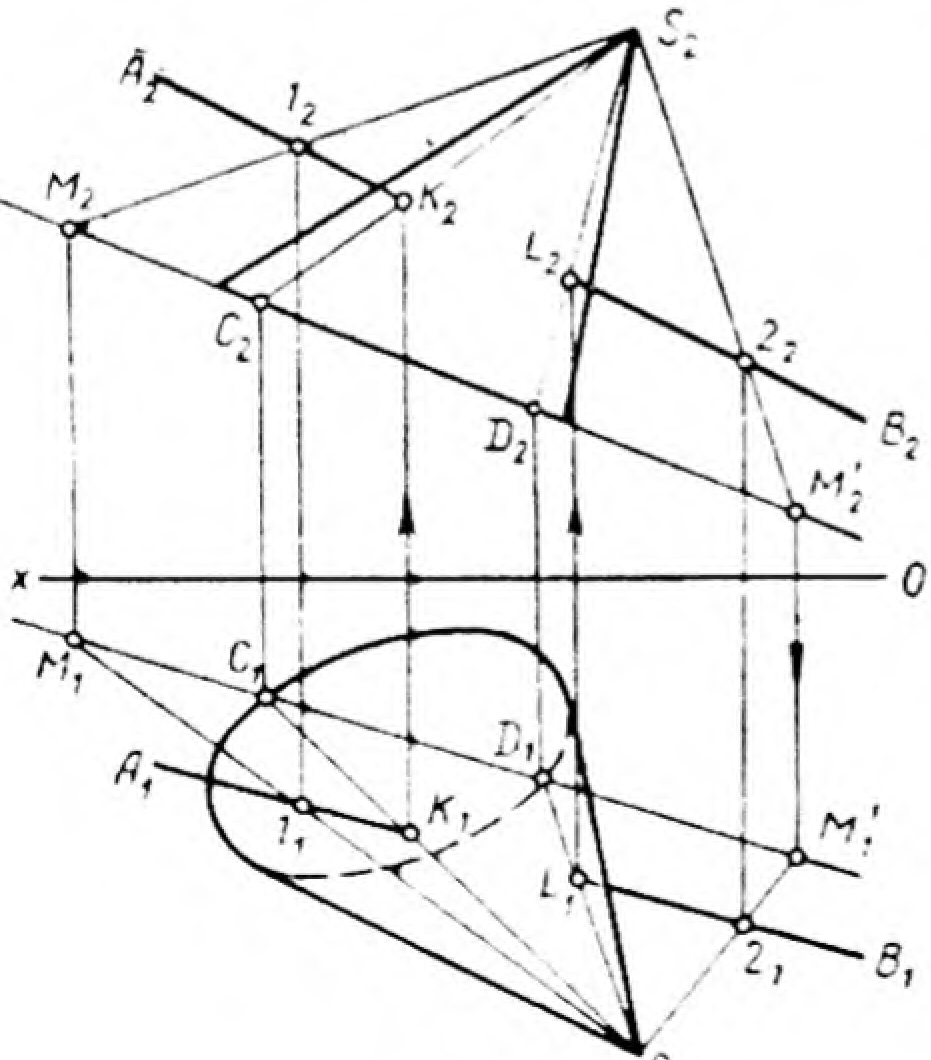

S_2
A_2
1_2
α_2
M_2
K_2
L_2
C_2
2_2
D_2
B_2
M'_2
x
0
M_1
C_1
D_1
A_1
K_1
1_1
M'_1
L_1
2_1
B_1
S_1

Fig. 342

5. Auxiliary planes which cut curved surfaces should be selected, whenever possible, so that the sections received are straight lines or circles and not curves in general, since the construction of the latter overloads the drawing with auxiliary lines and makes it less precise and legible.

Let us construct points K and L where line AB intersects a frustum of a cone by passing an auxiliary plane to intersect the cone along its generatrices (Fig. 343). Such a plane must pass through the vertex S of the cone. The projection of the vertex on plane Π_2 will lie at a point obtained by extending the frontal projections of the limiting elements to intersect. However, in this case the frontal projection S_2 of the vertex S lies outside the limits of the drawing.

In order to find the horizontal trace $k_1 \equiv M_1M_1'$ of the auxiliary plane α passing through line AB and the vertex S of the cone, it is necessary to obtain the horizontal trace M of line AB and the horizontal trace of some other line contained in the auxiliary plane α. For this purpose let us take line SF which passes through point F of line AB and through the vertex S of the cone. The frontal projection F_2 of point F is taken as the intersection of projection A_2B_2 and the frontal projection of the left-hand generatrix. Having found point F_2 we determine point F_1 on A_1B_1. Next the projection S_1F_1 of line SF is drawn. The frontal projection of this line coincides with the frontal projection of the left-hand generatrix of the cone because both these projections pass through point S_2.

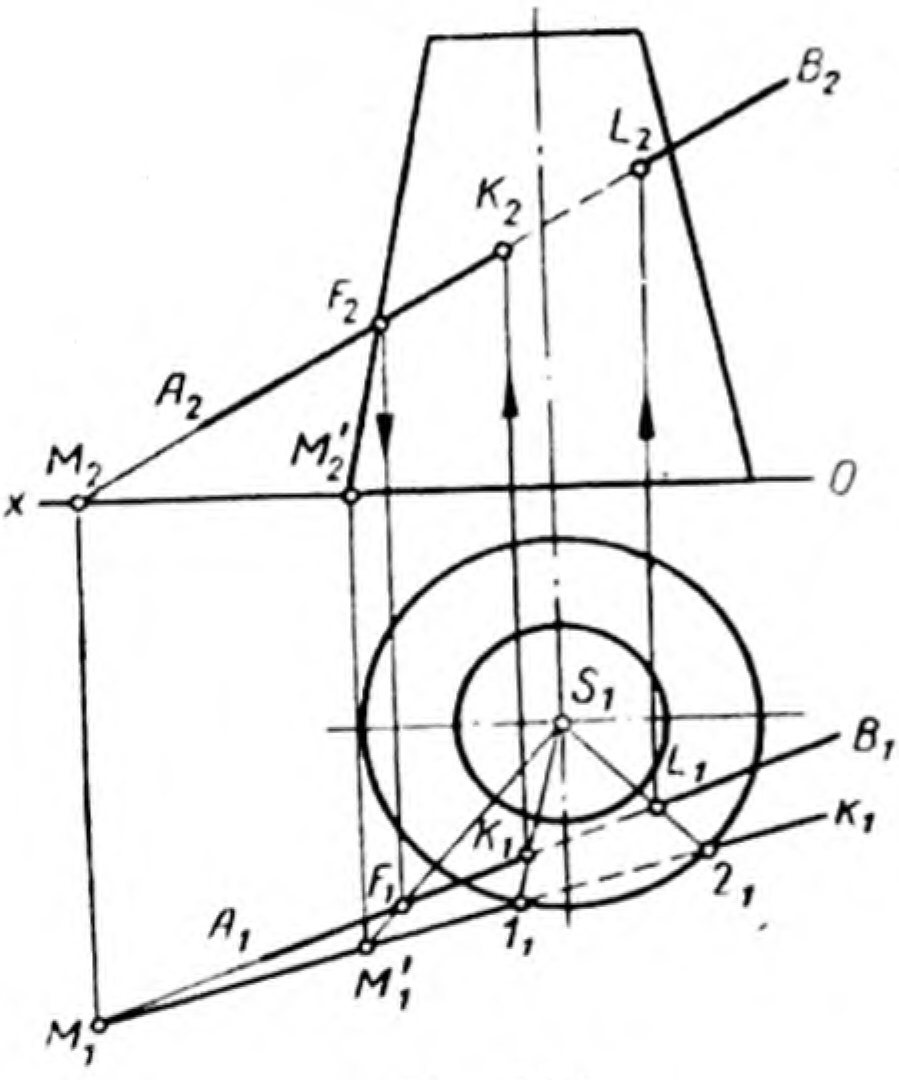


Fig. 343

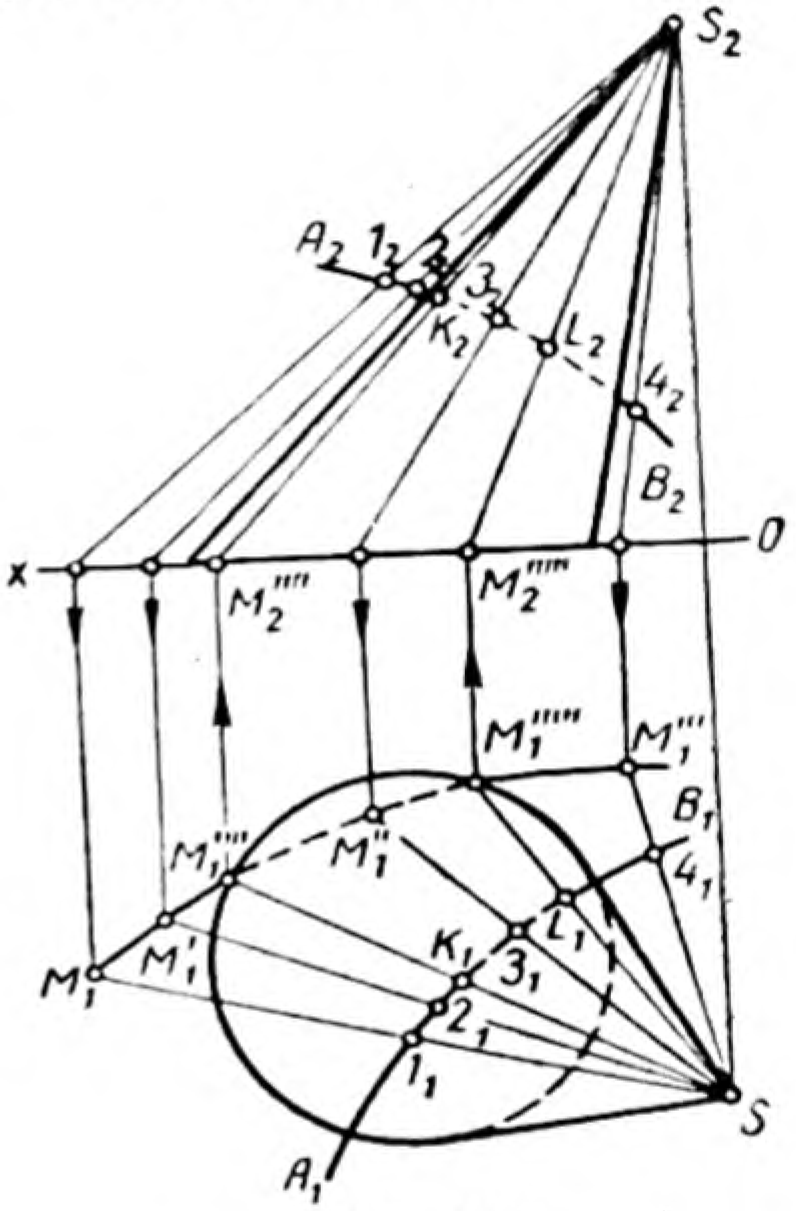
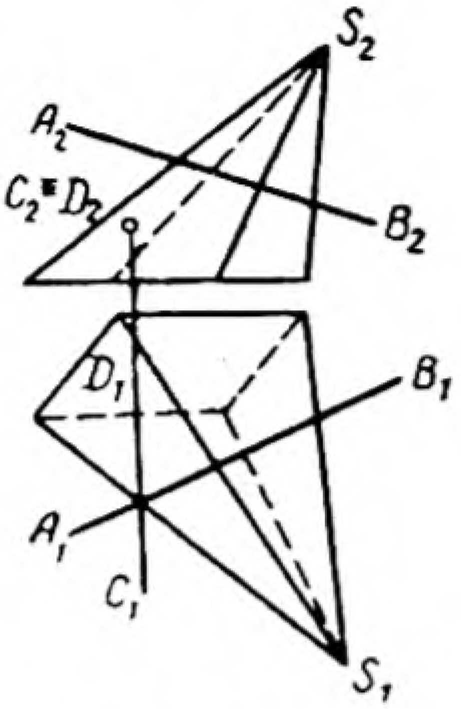


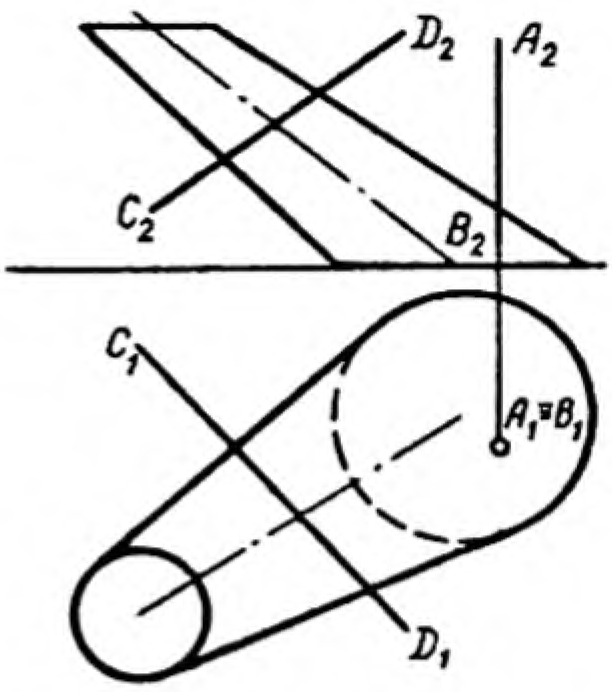
Fig. 344

Using projections S_2F_2 and S_1F_1 we construct the horizontal trace M' of line SF. The horizontal trace k_1 of the auxiliary plane α is drawn through points M_1 and M_1'. The horizontal projections $S_1 1_1$ and $S_1 2_1$ of the generatrices, along which the auxiliary plane cuts the surface of the cone, are then constructed. The points of



S_2
A_2
$C_2 \equiv D_2$
B_2
D_1
B_1
A_1
C_1
S_1

Fig. 345



D_2
A_2
C_2
B_2
C_1
$A_1 \equiv B_1$
D_1

Fig. 346

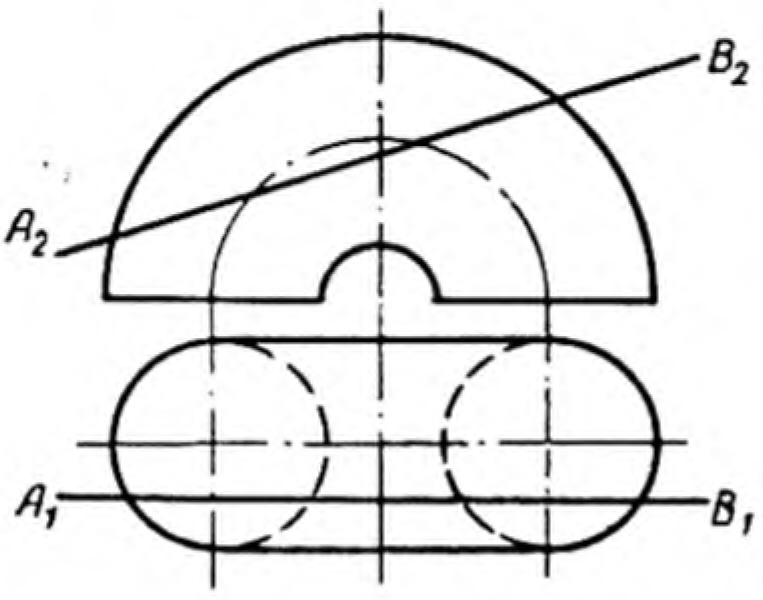


Fig. 347

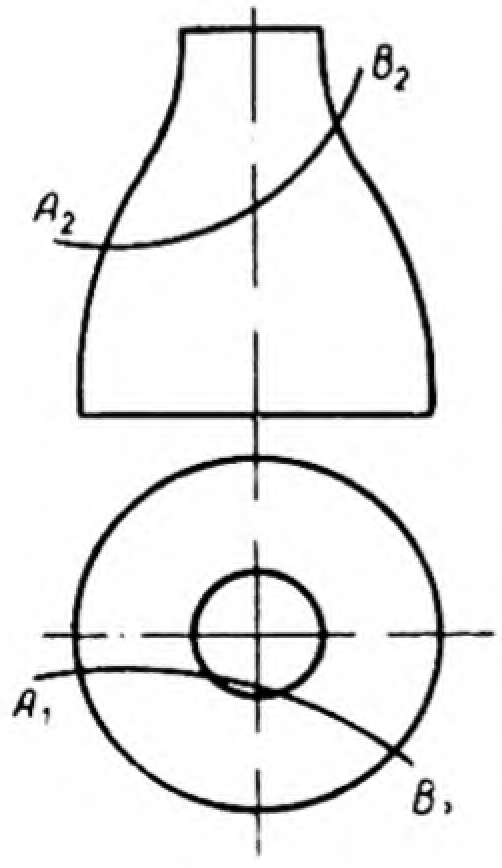


Fig. 348

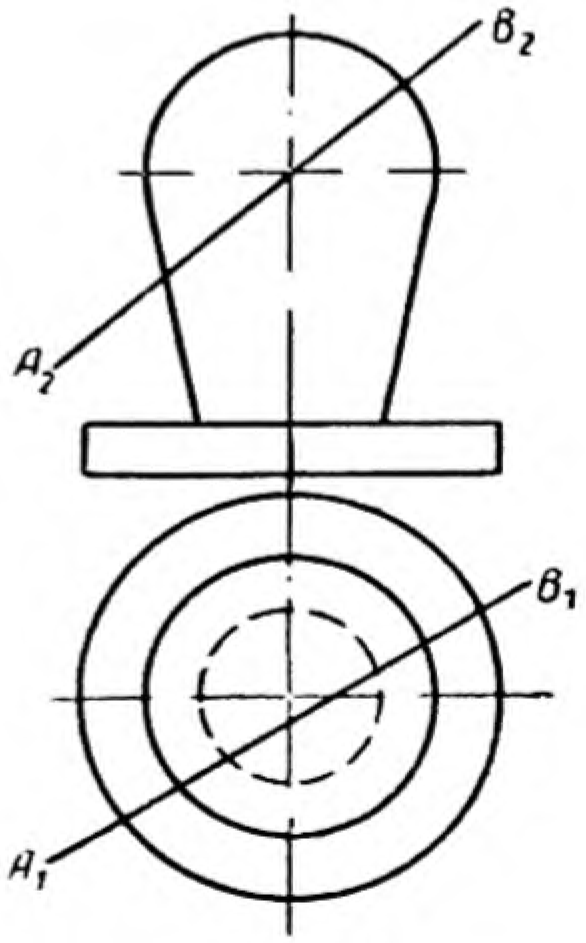


Fig. 349

intersection $S_1 1_1$ and $S_1 2_1$ with A_1B_1 determine the horizontal projections K_1 and L_1 of the required points K and L in which the line AB intersects the cone. The frontal projections K_2 and L_2 of points K and L are found on A_2B_2 with the help of projectors.

6. The use of an auxiliary conical surface for determining the intersection points of a space curve AB with a cone is shown in Fig. 344. The auxiliary conical surface passed through curve AB and vertex S of the given cone intersects the latter along its generatrices.

First the horizontal trace of the auxiliary conical surface should be found. For this purpose straight lines are drawn through vertex S of cone and points *1, 2, 3, 4* of curve AB. Through the traces, or piercing points M_1, M_1', M_1'' and M_1''' of these lines a smooth curve is drawn. This curve intersects the circle of the base of the given cone in points M'''' and M'''''.

The straight lines SM'''' and SM''''', drawn through the points obtained and the vertex of the cone, are the generatrices along which the two conical surfaces intersect. The points K and L, in which curve AB cuts these two generatrices, are the required points in which curve AB pierces the given cone.

Problems and Exercises

1. What types of auxiliary planes are used for determining the points of intersection of the surfaces of solid figures by straight lines?

2. What auxiliary surfaces are used for determining the points of intersection of surfaces of solid figures by curved lines?

3. What, in general, is the sequence in which the points of intersection of straight or curved lines with the surfaces of solid figures are constructed?

4. Find the points of intersection of lines AB and CD with the surface of the pyramid, shown in Fig. 345.

5. Find the points of intersection of lines AB and CD with the surface of the elliptical cone, shown in Fig. 346.

6. Find the points of intersection of line AB with the surface of the annular torus, shown in Fig. 347.

7. Find the points of intersection of curve AB with the surface of revolution, shown in Fig. 348.

Hint. Use an auxiliary cylindrical surface of intersection passing through curve AB.

8. Find the points of intersection of line AB with the surface of revolution, shown in Fig. 349.

Chapter X

INTERSECTION OF SURFACES

§ 39. Construction of the Intersection of Surfaces Using Projecting Planes

1. The construction of points contained in lines in which surfaces intersect each other, also called *transition lines,* is carried out by introducing various auxiliary elements such as projecting planes, oblique planes and spherical surfaces. Transformation of projections and oblique projection are also used for this purpose. The choice of the auxiliary method to be employed is determined by the form and position of the intersecting surfaces.

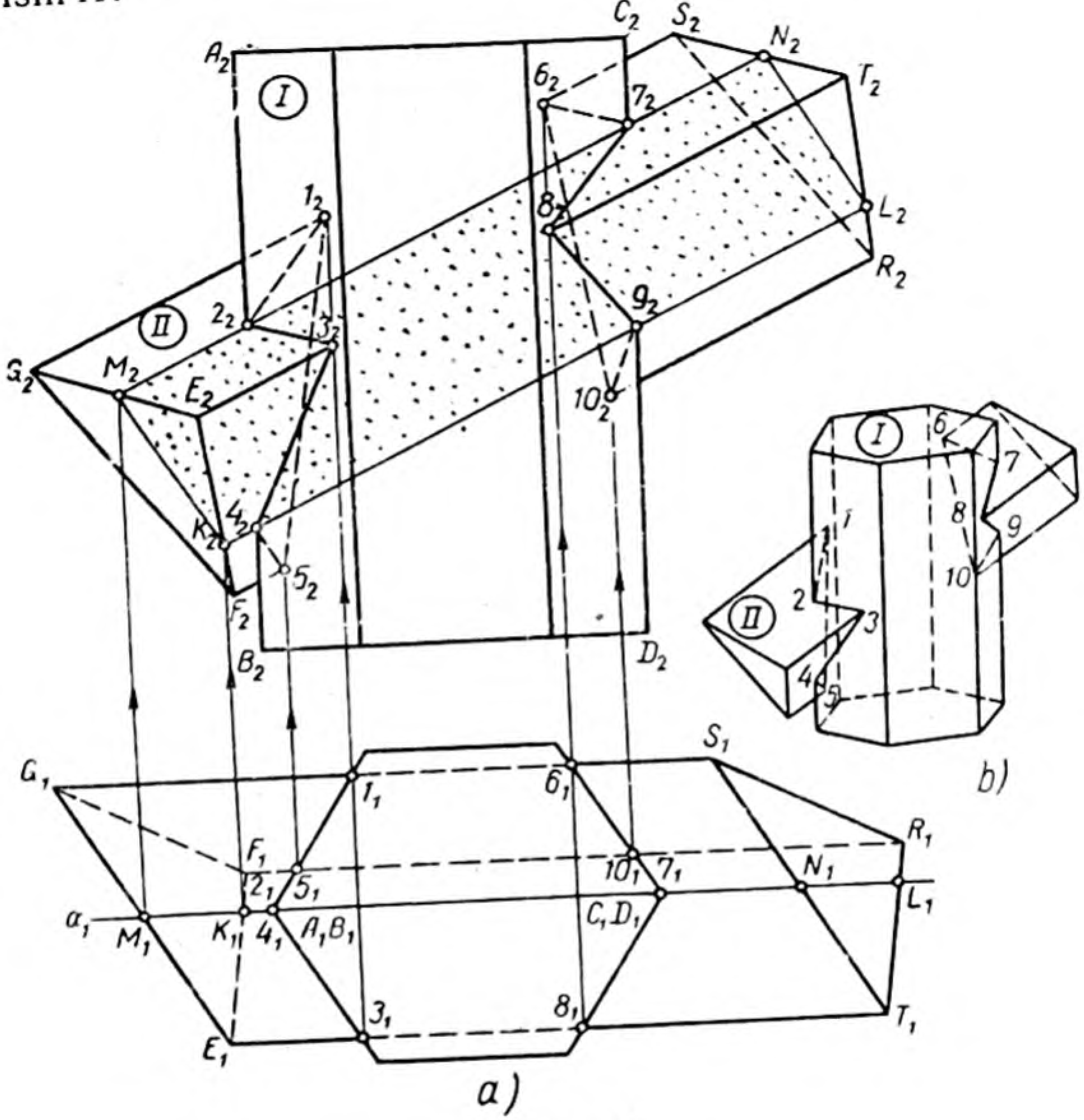
The surfaces of two geometrical bodies may intersect each other in one or two closed lines. There is one line of intersection when the surfaces intersect only partially as if one surface "bites" into another, as it were.

Two lines of intersection are obtained when the intersection of the surfaces is complete, in other words, when one surface pierces right through the other. The construction of the lines of intersection of the surfaces of two polyhedra consists first in finding the points of intersection of the edges of one polyhedron with the faces of the other. Then the points of intersection of the edges of the second polyhedron with the faces of the first are determined. Finally, the points found are joined by straight lines, taking into account the visibility of the separate sides of the polygon in which the surfaces of the polyhedra intersect. In more complex cases special procedures facilitate the task of joining up the points contained in the line of intersection of surfaces of two polyhedra. One such method is given in § 40.

When determining the visibility of the portions of the lines of intersection of two surfaces the following rule should be kept in mind. If two lines are visible when projected onto some plane of projection, then their point of intersection is also visible when projected onto that same plane. The point of intersection of two lines not visible, or of one visible line with a hidden line, is not visible.

2. We shall now examine the case when a right regular hexagonal prism *I* is intersected by a triangular oblique prism *II* (Fig. 350).

Of the six lateral edges of prism *I* only edges *AB* and *CD* intersect the faces of prism *II*, since horizontal projections A_1B_1 and C_1D_1 of those edges lie within the limits of the horizontal projection of prism *II*.



A_2
C_2
S_2
N_2
T_2
I
6_2
7_2
1_2
8_2
L_2
R_2
II
2_2
3_2
9_2
G_2
M_2
E_2
10_2
K_2
4_2
5_2
F_2
B_2
D_2
I
6
7
1
8
9
10
II
2
3
4
5
b)
S_1
G_1
1_1
6_1
R_1
F_1
10_1
7_1
N_1
2_1
5_1
L_1
a_1
M_1
K_1
4_1
A_1B_1
C_1D_1
3_1
8_1
T_1
E_1
a)

Fig. 350

First find the points of intersection of the lateral edges *AB* and *CD* of prism *I* with the faces of prism *II*. For this purpose pass an auxiliary frontal plane α through the given edges. This plane intersects the face *EGST* of prism *II* in line *MN* and the face *EFRT* of the same prism in line *KL*. Since plane α is parallel to the lateral edges of the prism *II*, these auxiliary lines *MN* and *KL* must also be parallel to those lateral edges. The section given when the surface of prism *II* is cut by plane α is the quadrilateral *KLNM*.

The frontal projections M_2 and K_2 of points *M* and *K* are found by means of the horizontal projections M_1 and K_1. Then, by drawing lines M_2N_2 and K_2L_2 through M_2 and K_2 we find their intersection with projections A_2B_2 and C_2D_2 of edges AB and CD. These points of intersection are the frontal projections 2_2, 4_2, 7_2 and 9_2 of the points in which the edges AB and CD of prism *I* pierce the faces of prism *II*. The horizontal projections 2_1, 4_1, 7_1 and 9_1 of these points coincide with the horizontal projections of edges AB and CD, respectively.

The other points contained in the lines of intersection of the prisms, i. e., points *1, 3, 5, 6, 8* and *10*, can be obtained directly, without auxiliary constructions, since they are points in which the lateral edges ET, FR and GS of prism *II* intersect the lateral faces of prism *I*, which are horizontal-projecting planes. By passing projectors from the horizontal projections 1_1, 3_1, 5_1, 6_1, 8_1 and 10_1 we construct the frontal projections 1_2, 3_2, 5_2, 6_2, 8_2 and 10_2 of points *1, 3, 5, 6, 8* and *10*.

Points 1_2, 2_2, 3_2, 4_2 and 5_2, and also points 6_2, 7_2, 8_2 and 10_2, are joined by straight lines, taking into account their visibility when projecting onto plane Π_2 the various sides of two pentagons in space in which prisms *I* and *II* intersect. The horizontal projections of the sides of these pentagons coincide with the horizontal traces of those lateral faces of prism *I* to which the sides belong.

The axonometric representation of these two intersecting prisms is seen in Fig. 350, *b*.

3. When constructing points of the lines of intersection of two curved surfaces with the help of projecting planes, both surfaces are intersected by a series of auxiliary planes. Sections of each surface given by one and the same plane are constructed. The intersection of these corresponding sections gives the required points. For instance, if a sphere and the circular cylinder intersecting it are cut by a horizontal plane α (Fig. 351), then the intersection of the two circles obtained, which are sections of the sphere and the cylinder, gives points A and B contained in the lines of intersection of these surfaces.

The construction of the intersection of two cylinders by means of auxiliary frontal surfaces is shown in Fig. 352.

Since these planes are parallel to the axes of the cylinders, they cut the surfaces of the cylinders along generatrices. The points *1* and *2* where the front generatrix of cylinder *II* intersects cylinder *I*, and also points *3* and *4*, where the rear generatrix pierces cylinder *I*, are found directly. The horizontal projections 1_1, 2_1, 3_1 and 4_1 of these points are contained in the horizontal projection of the lateral surfaces of cylinder *I*. The frontal projections 1_2, 2_2,

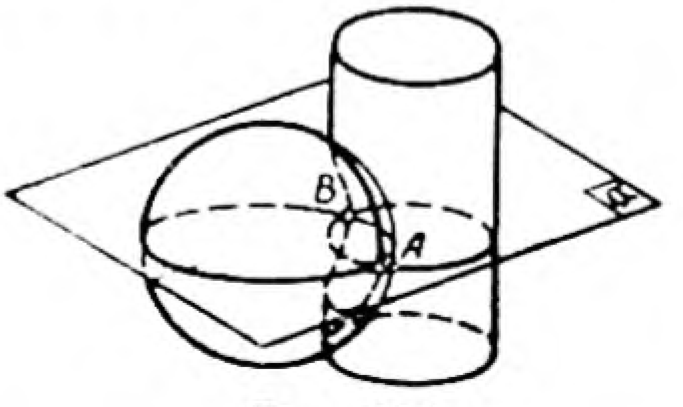


Fig. 351

3_2 and 4_2 are situated on the frontal projections of the front and rear generatrices of cylinder *II* and on the respective projectors.

Points *11* and *12* and also *9* and *10*, in which the top and bottom generatrices of cylinder *II*, respectively, intersect with the surface of cylinder *I*, are also determined directly.

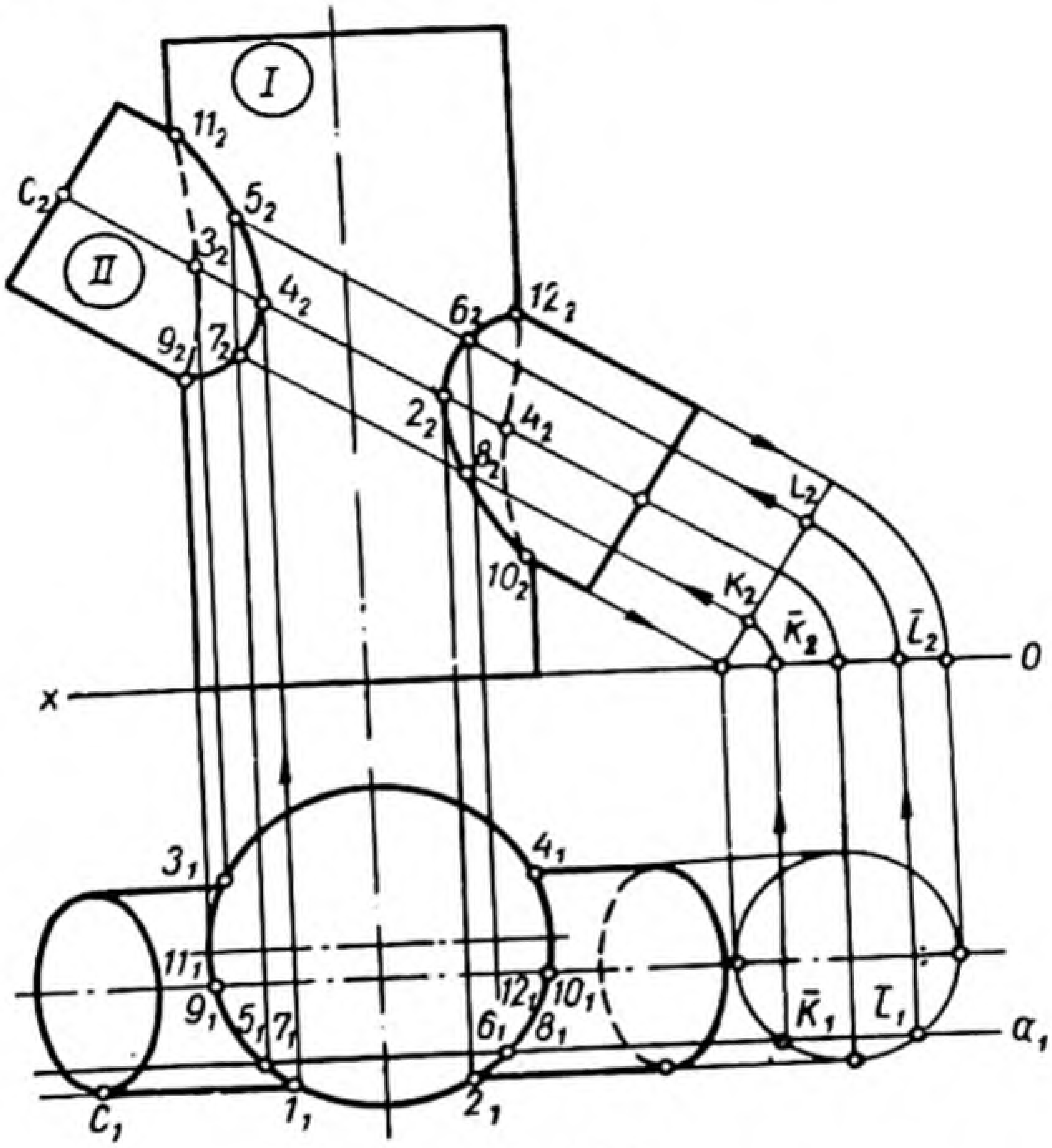


Fig. 352

In order to find points *5, 6, 7* and *8*, belonging to the lines of intersection of the surfaces of cylinders *I* and *II*, an auxiliary frontal plane α is drawn. The generatrices along which plane α cuts the surface of the two cylinders are constructed. In order to make the construction more convenient and exact the circular base of cylinder *II* is rotated to lie in plane Π_1. Having obtained points 5_1, 7_1 and 6_1, 8_1 where trace α_1 of plane α intersects the horizontal projection of the lateral surface of cylinder *I*, we construct the frontal projection of the generatrix which contains points *5* and *7* and the generatrix which contains points *6* and *8*. Then the points $\bar{K}_1$ and $\bar{L}_1$, where α_1 intersects the circular base of cylinder *II* made to coincide with plane Π_1, are found. Having done this we can obtain first points $\bar{K}_2$ and $\bar{L}_2$, and then points K_2 and L_2.

Through K_2 and L_2 the frontal projections of the generatrices along which plane α cuts the surface of cylinder *II* are drawn and extended to intersect the frontal projections of the corresponding generatrices in which this same plane cuts cylinder *I*. This gives us the frontal projections of points *5, 6, 7* and *8* contained in the line of intersections of the cylinders. A series of auxiliary frontal planes similar to plane α are drawn. This gives us further points belonging to the lines of intersection of the cylinders. Continuous curves are drawn through the points obtained. When drawing these lines the visibility of the various points projected onto plane Π_2 should be taken into account.

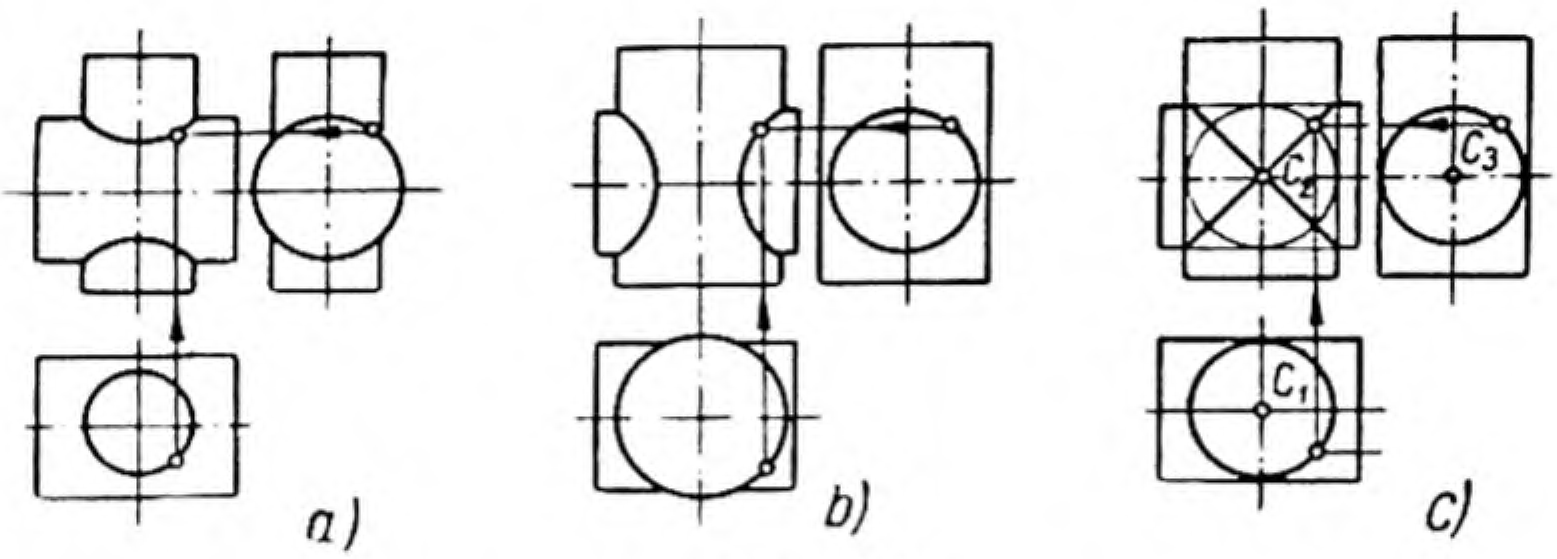


Fig. 353

4. Intersecting cylinders are shown in Fig. 353, *a, b, c*. The cylinders with horizontal generatrices in all three cases have the same diameter, but the cylinders with vertical generatrices have different diameters. Depending on the size of the diameters of cylinders with vertical generatrices the lines of intersection of the cylinders are projected onto plane Π_2 either in the form of curves — hyperbolas (Fig. 353, *a, b*) — or as straight lines (Fig. 353, *c*). In the latter case both cylinders are of the same diameter. Such cylinders may be considered as surfaces described about a sphere with its centre in point *C* and its radius equal to the distance from point *C* to the generatrices of the cylinder. The two plane curves — ellipses — obtained by the intersection of the given cylinders have two points in common with the surface of the sphere about which the two cylindrical surfaces are described.

5. An example of the construction of the intersection of the two cylindrical surfaces making up a tee-pipe connection is shown in the multiview drawing (Fig. 354). The points used in the construction of this line of intersection, or transition, were found by using auxiliary horizontal planes α, β and γ.

6. The construction of the intersection of a cylinder and a cone, both axes of which are perpendicular to plane Π_1 (Fig. 355), was carried out by using auxiliary horizontal planes α, β and γ to intersect the cone and cylinder in circles. These circles are projected

onto plane Π_2 as straight lines and onto plane Π_1 as circles. The horizontal projections of all the circles in which the planes intersect the surface of the cylinder coincide with the projection of the cylinder on plane Π_1. The diameters of the circular sections cut on the cone by the auxiliary planes are determined from the frontal projection, as indicated by an arrow for the circular section of the cone cut by plane γ.

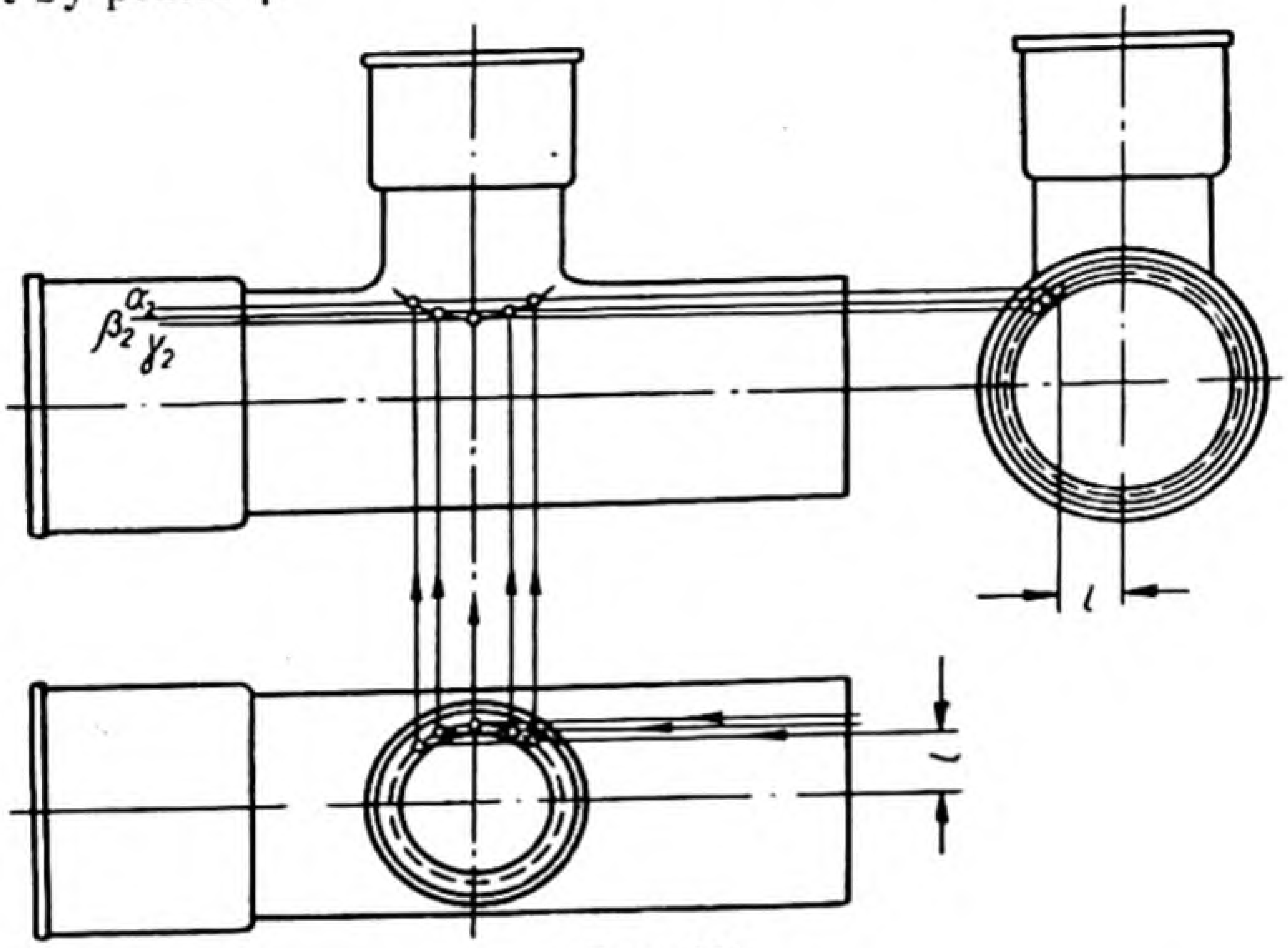


Fig. 354

The intersection of the circle, which is the projection of the cylinder on Π_1, with the horizontal projections of the circles, obtained when the cone is cut by planes α, β and γ, determine the horizontal projections A_1, B_1, C_1, D_1, E_1 and F_1 of points A, B, C, D, E and F, through which the line of intersection of the cylinder and cone pass. The frontal projections A_2, B_2, C_2, D_2, E_2 and F_2 of these points are obtained by passing projectors through $A_1, B_1, \ldots, F_1$ to the frontal traces of the corresponding planes.

The special points, the highest point M and lowest point N, are found with the help of plane δ passed through the axes of the cylinder and cone. The points of intersection of the left-hand (L. H.) and right-hand (R. H.) generatrices of the cylinder with surface of the cone are constructed by passing planes φ and ξ through the axis of the cone and through each of the two generatrices of the cylinder. Then the generatrices along which these planes intersect the cone are found. This construction is shown for point K.

7. Three cases of intersection of a cylinder with a cone are shown in Fig. 356. The three cylinders are of the same diameter.

Comparing the diameter of the cylinder with that of the circular section of the cone whose centre coincides with point C, i. e., with the point of intersection of the axes of the two bodies, it is seen that in the first case (Fig. 356, *a*) the diameter of this section of the cone is smaller than the diameter of the cylinder, in the second case (Fig. 356, *b*) these diameters are equal, and in the third case (Fig. 356, *c*) the diameter of the circle of the cone is larger than that of the cylinder. The second case (Fig. 356, *b*) differs from the first and third in that a circle may be described from point C_2 to touch both the outline generatrices of the cylinder and of the cone. Therefore, the surfaces of these two bodies are tangential to a sphere with its centre in point C_2. These two surfaces intersect each other in two plane curves, ellipses, having two points in common.

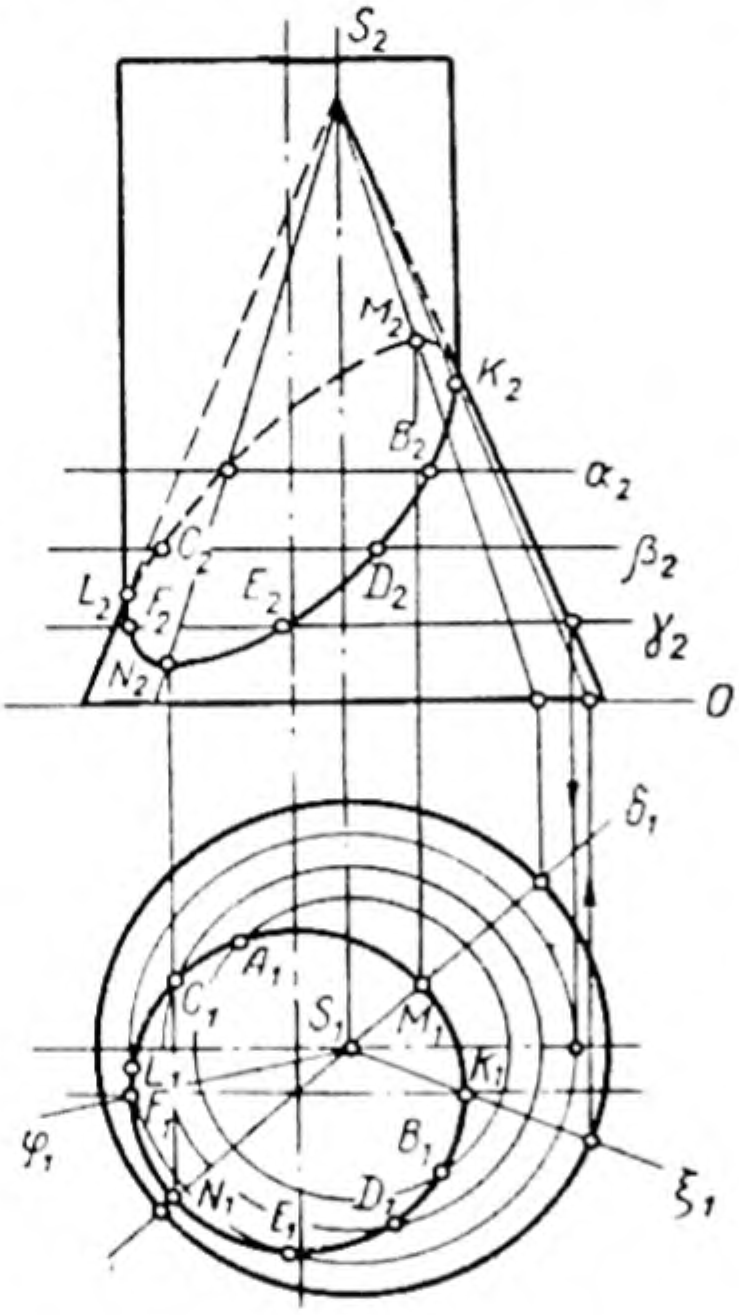


Fig. 355

8. The construction of the lines of intersection of the frustum of a cone with a sphere is shown in the multiview drawing (Fig. 357).

Since the axis of the cone and the vertical axis of the sphere are located in the same frontal plane, the points A and B of intersection of the R. H. and L. H. generatrices of the cone with the principal meridian of the sphere are determined directly. Points A_2 and B_2 are the frontal projections and A_1 and B_1 the horizontal projections of these points, A being the highest point and B the lowest point of the line of intersection of the cone and sphere.

In order to construct points D and E, in which the front and

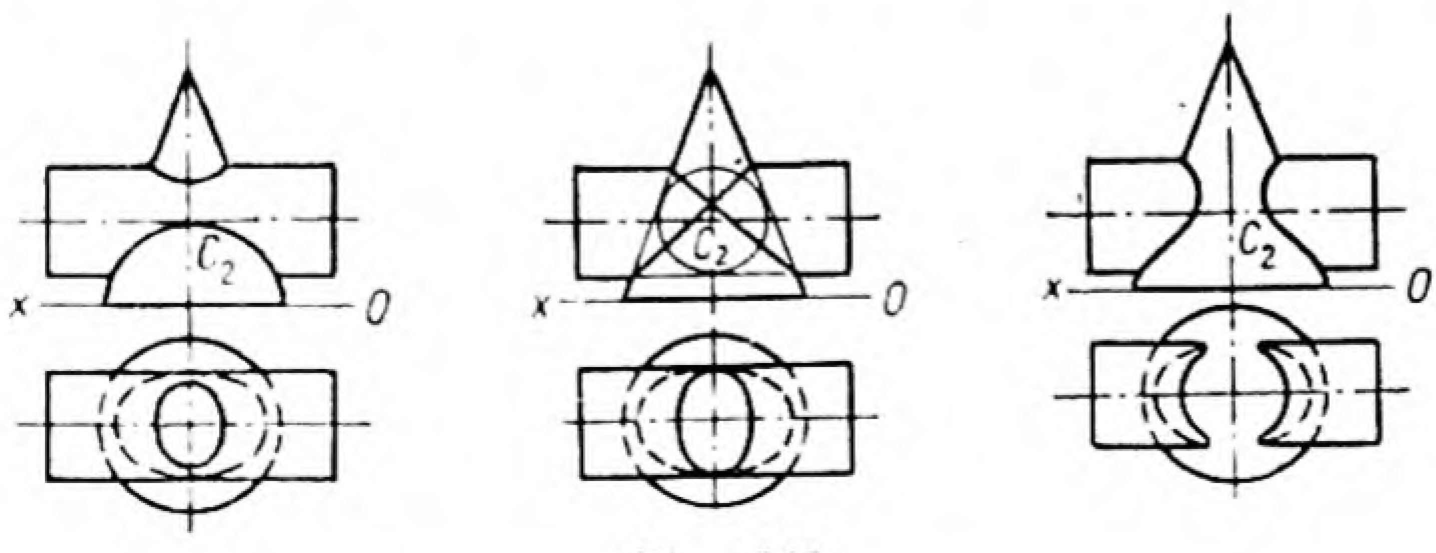


Fig. 356

rear generatrices of the cone intersect the sphere, profile plane β is drawn. This plane intersects the surface of the sphere in a circle with radius r', which is projected onto plane Π_3 in its true size. Therefore an arc of radius r' is described with point C_3 as its centre. This arc intersects the profile projections of the front and rear generatrices of the cone in points D_3 and E_3. These are the boundaries of visibility in the projection on plane Π_3 of the various parts of the line of intersection. At the same time they are the points of intersection of the profile projections of the front and rear generatrices of the cone with the profile projection of the curve of intersection of the surfaces of the cone and sphere. Using points D_3 and E_3 we find points D_1, E_1, $D_2 \equiv E_2$.

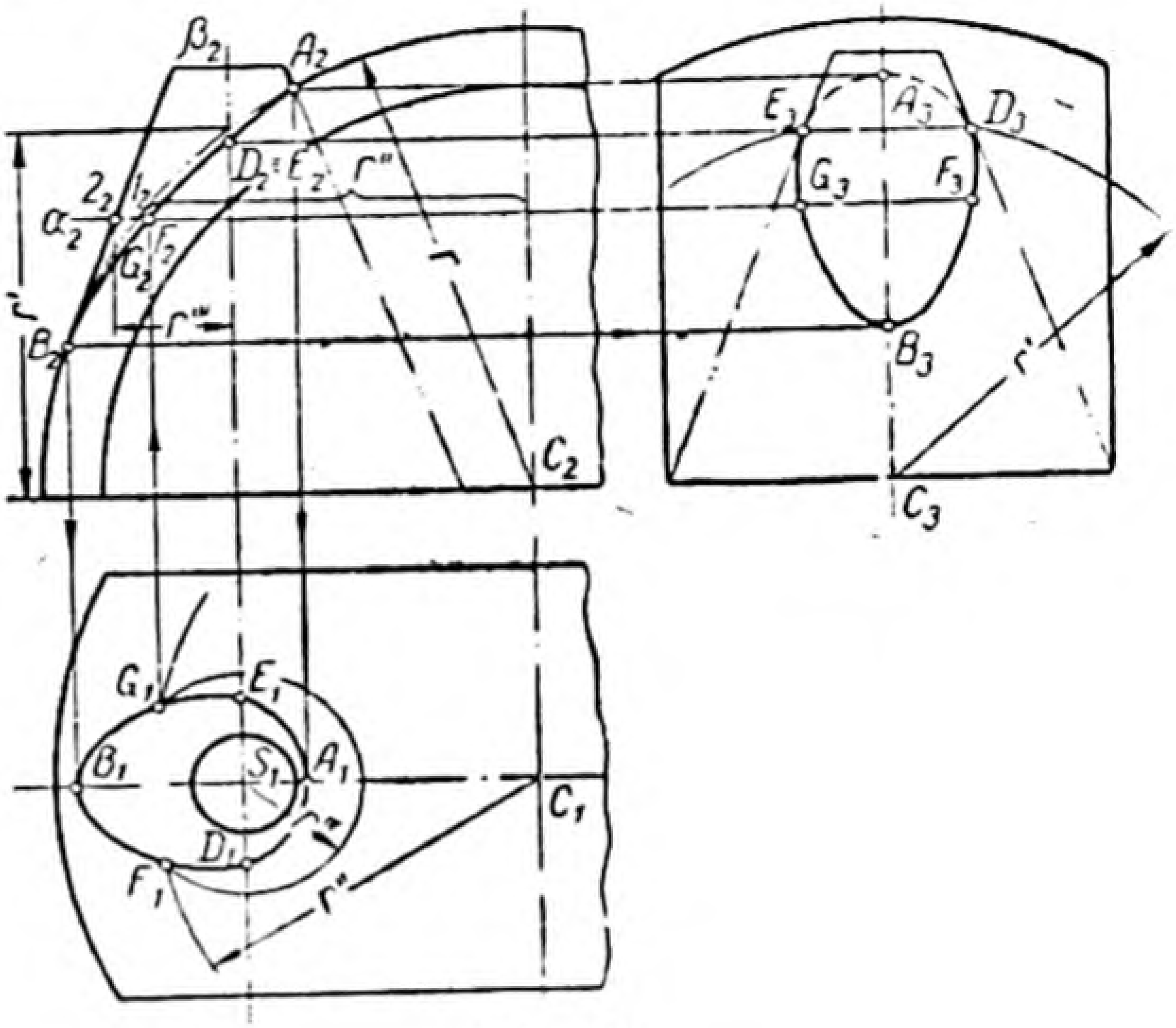


Fig. 357

The intermediate points F and G are constructed by drawing a horizontal plane α to cut the sphere in a circle of a radius r'' and to cut the cone in a circle of a radius r'''. These circles are projected onto plane Π_1 as true views, and their intersection determines the horizontal projections F_1 and G_1 of points F and G contained simultaneously in plane α and in the line of intersection of the cone and cylinder. Using points F_1 and G_1 the points F_2 and G_2 are found with the help of projectors drawn through F_1 and G_1 to intersect trace α_2 of the auxiliary horizontal plane α.

The construction of the profile projections A_3, B_3, G_3 and F_3, of points A, B, G and F is carried out as usual by constructing the third projection of the points using two of their known projections.

Using other auxiliary horizontal planes, a series of intermediate points of the curve of intersection of the given cone and sphere may be found.

When a sufficient number of points has been found the projections of the curve may be drawn through the corresponding projections of the points.

§ 40. Construction of the Intersection of Surfaces by Means of Oblique Planes

1. An oblique plane passing through a line parallel to the edges of a prism, intersects the surface of the prism along lines parallel to these edges, see Fig. 340, *b*. If we intersect two prisms by a plane α which passes through two lines, one of which is parallel to the edges of the first prism and the other parallel to the edges of the second prism, we obtain lines respectively parallel to these edges on the surface of each prism.

These basic premises when selecting the most simple oblique section planes such as α, α', α'', α''' and α'''' for the construction of points contained in the lines of intersection of the two oblique prisms *ABCDE* (*I*) and *KLMN* (*II*), the bases of which are contained in plane Π_1, see Fig. 358.

Let us determine the direction of the horizontal traces f_1, q_1, p_1, s_1 and k_1 of the auxiliary cutting planes which are parallel to the lateral edges of both prisms. Take any convenient point Q and draw two lines through it, one, QM, parallel to the lateral edges of the first prism, the other, QM', parallel to the lateral edges of the second prism. Having found the horizontal traces M and M' of these lines draw the trace MM'. This line MM' determines the direction required.

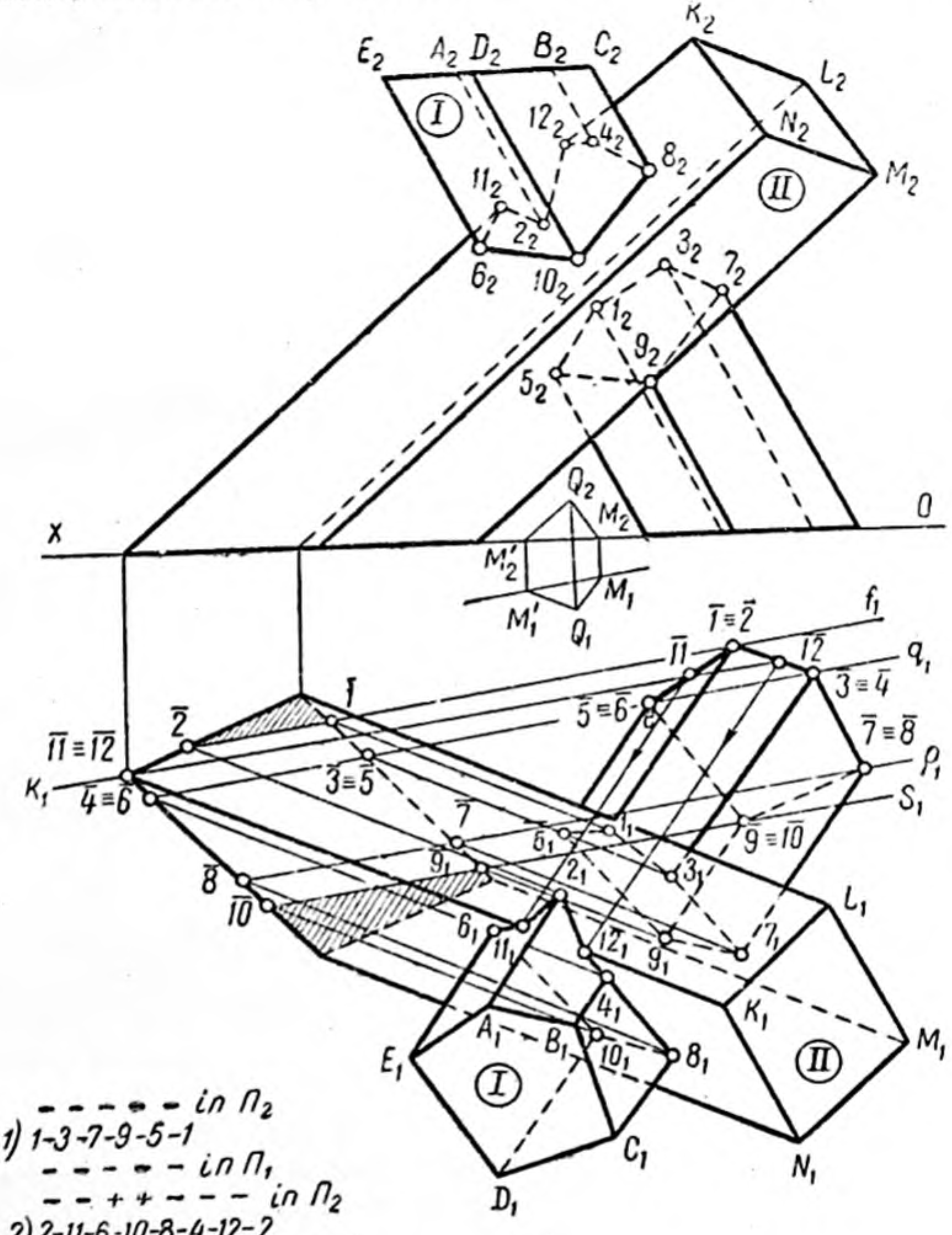
The first auxiliary plane α is passed through edge A of the first prism. The horizontal trace f_1 of plane α will pass through the horizontal trace $(\bar{1} \equiv \bar{2})$ of this edge A. The trace s_1 of the last auxiliary plane α'''' passes through the horizontal trace $(\bar{9} \equiv \overline{10})$ of edge D of the prism. The other auxiliary planes lie between these two extreme planes.

Certain parts of the second prism, which lie outside the space contained between planes α and α'''' do not intersect the surface of the first prism. The areas of the horizontal projection of the base of the second prism, which correspond to these parts, lie below trace f_1 and above trace s_1. They are indicated on the drawing by hatching.

Plane α, passed through the lateral edge A of the first prism, intersects the surface of the second prism along lines $\bar{1}$-1_1 and $\bar{2}$-2_1 parallel to its lateral edges. In Fig. 358 only the horizontal projections of these lines are shown. The horizontal projection of edge A of the first prism intersects the horizontal projections of lines $\bar{1}$-1_1 and $\bar{2}$-2_1 in points 1_1 and 2_1, respectively. These are the horizontal projections of the points of intersection *1* and *2* of edge A of the first prism with the surface of the second prism.

By drawing projectors from points 1_1 and 2_1 perpendicular to ground line Ox to intersect the frontal projection of edge A, we obtain the frontal projections 1_2 and 2_2 of points *1* and *2*. In Fig. 358 the projectors are not shown.

2. The method of similar denotations. By a similar construction, with the help of plane α'''' passed through edge *D*, we



K_2
E_2
A_2
D_2
B_2
C_2
L_2
N_2
M_2
I
II
12_2
4_2
8_2
11_2
2_2
6_2
10_2
3_2
7_2
1_2
9_2
5_2
Q_2
M_2
M_2'
M_1
M_1'
Q_1
x
0
f_1
$\bar{1} \equiv \bar{2}$
$\bar{11}$
$\bar{12}$
q_1
$\bar{3} \equiv \bar{4}$
$\bar{5} \equiv \bar{6}$
$\bar{7} \equiv \bar{8}$
p_1
S_1
$\bar{11} \equiv \bar{12}$
$\bar{2}$
$\bar{1}$
K_1
$\bar{4} \equiv \bar{6}$
$\bar{3} \equiv \bar{5}$
$\bar{7}$
$\bar{9} \equiv \bar{10}$
5_1
1_1
$\bar{9}_1$
2_1
3_1
$\bar{8}$
$\bar{10}$
6_1
11_1
12_1
9_1
7_1
L_1
4_1
A_1
B_1
10_1
8_1
K_1
E_1
I
II
M_1
C_1
D_1
N_1
— — — — — in Π_2
1) 1-3-7-9-5-1
— — — — — in Π_1
— — + + — — — in Π_2
2) 2-11-6-10-8-4-12-2
+ + — — + + + in Π_1

Fig. 358

obtain the points of intersection *9* and *10* of this edge with the surface of the second prism.

Plane α'', the horizontal trace of which is q_1, passes through edges *B* and *E* of the first prism and intersects the surface of the second prism along the lines $\bar{3}$-*3* and $\bar{4}$-*4*. The intersection of these lines $\overline{3}$-*3* and $\overline{4}$-*4* with edges *B* and *E* gives points *3* and *4* and also points *5* and *6* of the line of intersection of the two given prisms.

The planes α' and α''', passed through the edge *C* of the first prism and through the edge *K* of the second prism, determine two pairs of points, *7* and *8*, and *11* and *12*, belonging to the line of intersection of the prisms.

After obtaining twelve points, contained at one and the same time in the surfaces of both the first and second prisms, join up these points to give two space polygons. The sequence in which the points are connected by straight lines is established with the help of the method of similar notations.

In accordance with this method use projected points *1, 2, 3, ..., 12* parallel to the edges of the prism onto the base of that prism (Fig. 358). The oblique projections of these points received on plane Π_1 will coincide with the horizontal traces of the lateral edges and lines along which the auxiliary planes intersect the faces of the prism. The oblique projections of the points on plane Π_1 should be denoted in the same way as the corresponding points of the lines of intersection of the prisms. For example, the oblique projection of point *1* on the base of the second prism is denoted as $\overline{1}$ and the two coinciding projections of points *1* and *2* on the base of the first prism as $\overline{1} \equiv \overline{2}$ and so on.

The sequence in which the oblique projections of these points are situated on the bases of the prisms corresponds to the sequence of the points on the lines of intersection in space. Therefore, we can determine the sequence in which the points should be joined in space by making the round of the oblique projections. The direction will be clockwise and we can start with any one of the points, say point *1*. So, beginning with points $\overline{1}$ on the base of each prism, we note down on paper the number *1* and pass on to the nearest points. On the base of the first prism point $\overline{12}$ lies next to point $\overline{1}$ and on the base of the second prism we find the duplicate point $\overline{3} \equiv \overline{5}$. Having. established that the notations of these points differ, we bypass point $\overline{12}$ and take the next point $\overline{3} \equiv \overline{4}$ on the base of the first prism. The points $\overline{3} \equiv \overline{4}$ and $\overline{3} \equiv \overline{5}$ have one common number $\overline{3}$. We then write down *3* next to *1*. Continuing the movement round the bases of both prisms we find a common notation of points $\overline{7} \equiv \overline{8}$ and $\overline{7}$, so we write *7* next to *3*. The next points with similar denotations are $\overline{9} \equiv \overline{10}$ and $\overline{9}$. Consequently, we note down *9* next to *7*.

Still continuing this movement around the base of the second prism we come to the bottom hatched part of the base. For this base the direction should now be reversed, i. e., we must move towards point $\overline{3} \equiv \overline{5}$, which corresponds to the closed outline of the lines of intersection in space. Moving in this new direction we establish the similar notation $\overline{5}$ in points $\overline{5}\equiv\overline{6}$ and $\overline{3}\equiv\overline{5}$. Therefore, after *9* we write down *5*. Finally, by passing $\overline{11}$ we return to the initial points $\overline{1}$ and $\overline{1}\equiv\overline{2}$ and again write down the common notation *1*. The line of intersection of the prisms is now closed. We have a list of numbers *1-3-7-9-5-1*. The corresponding projections of points contained in the lower line of intersection of the prisms are joined up in this same order.

By carrying out a similar operation in a counter-clockwise direction we establish the sequence in which the points of the second, upper line of intersection of the prisms should be joined. This sequence is *2-11-6-10-8-4-12-2*. The starting point in this case was point *2*.

The visibility of the separate points, and the parts of the lines of intersection adjacent to them, is readily determined from the oblique projections of these points on the base of the prisms. If two oblique projections of a point contained in the lines of intersection, or transition line, have similar notations and are visible in the given plane of projections, then the orthographic projections of this point and the part of the transition line adjacent to it are also visible.

Similarly, if one or both oblique projections of one of the points are hidden, its orthographic projection and the projection of the parts of the line of intersection adjacent to this point on that same plane of projection are also hidden.

The visibility of separate parts of the line of intersection of the prisms can be indicated by the signs + or −, as seen in Fig. 358. The visible parts are denoted by the plus sign and the parts not visible by the minus sign.

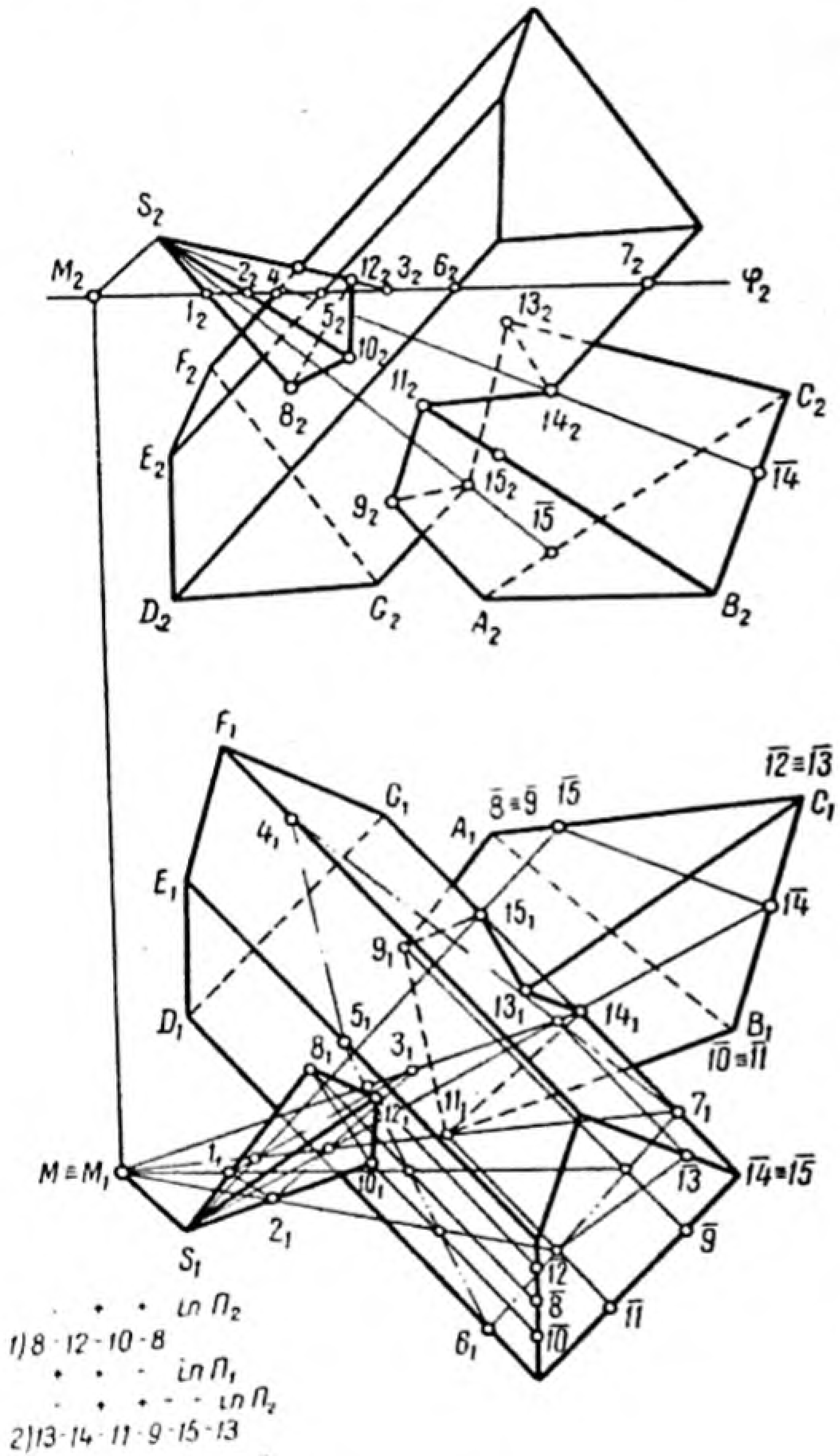
The projections of the lower line of intersection of the prisms, which is a plane pentagon *1-3-7-9-5*, are not visible. The projections of the various sides of the space heptagon *2-11-6-10-8-4-12*, which is the upper line of intersection of the prism, are drawn either as continuous or broken lines, depending on the visibility of each of the sides when projected onto planes Π_1 and Π_2.

3. The construction of the intersection of a tetragonal prism with a triangular pyramid is shown in Fig. 359. The bases of these two solid figures are not contained in any of the planes of projection.

In order to obtain points on the line of intersection of the prism and pyramid auxiliary planes are passed through line *SM* drawn through the vertex *S* of the pyramid parallel to the lateral edges of the prism. A similar plane was used when constructing points of intersection in Fig 341.

Since the bases of the prism and pyramid are not contained in the horizontal plane of projection an auxiliary horizontal plane φ is drawn to intersect the lateral surface of the pyramid in triangle *1-2-3* and the lateral surface of the prism in the tetragon *4-5-6-7*.

The horizontal projections of these lines of intersection are shown by dashes and two dots. Point $M \equiv M_1$ is the horizontal projection of the piercing point of line SM in plane φ. Through this point we draw the traces in plane α of the auxiliary cutting planes passing through line SM. This sheath of planes opens like a fan in space. We then find the horizontal projection of the lines along which these planes cut the surfaces of the prism and pyramid. On the surface of the prism these section lines are the lateral edges themselves, or lines parallel to them. On the surface of the pyramid the section lines are lateral edges or lines contained in the faces and passing through the vertex S. At the intersection of the horizontal projections of the respective section lines we find the horizontal projections $8_1, 9_1, 10_1, 11_1, 12_1, 13_1, 14_1$ and 15_1 of points contained in the line of intersection of the prism and pyramid.



S2
M2
2₂ 4₂
12₂ 3₂ 6₂
7₂
φ2
1₂
5₂
13₂
F₂
10₂
11₂
8₂
C₂
14₂
E₂
14
15₂
9₂
15
D₂
G₂
A₂
B₂
F₁
12≡13
G₁
8≡9 15
C₁
A₁
4₁
E₁
14
15₁
9₁
D₁
5₁
13₁
14₁
B₁
8₁
3₁
10≡11
12₁
11₁
7₁
M≡M₁
1₁
10₁
13
14≡15
2₁
9
S₁
12
8
11
6₁
10
. • • ın Π₂
1) 8 - 12 - 10 - 8
• • - ın Π₁
- • • - - ın Π₂
2) 13 - 14 - 11 - 9 - 15 - 13
• - - • ın Π₁

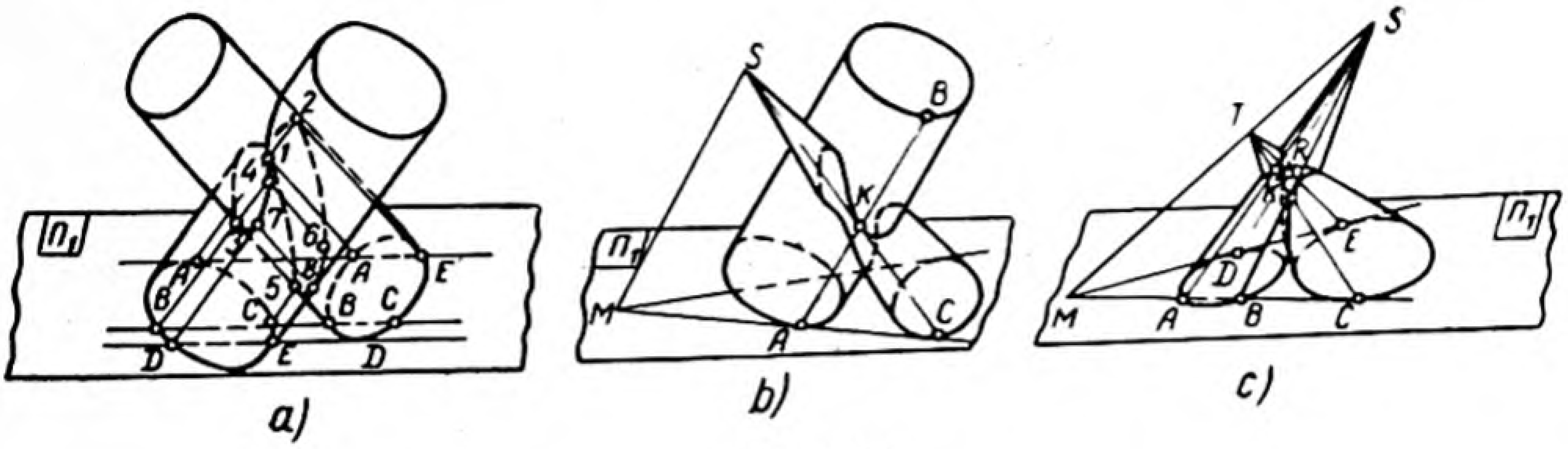
Fig. 359

The frontal projections $8_2, 9_2, 10_2, 11_2, 12_2, 13_2, 14_2$ and 15_2 of these points are found by passing projectors through their horizontal projections.

The sequence in which corresponding projections of the eight points obtained are joined up to form the two closed lines of intersection, *8-12-10* and *13-14-11-9-15*, is determined by the method of similar denotations. For this purpose the points of intersection must be projected obliquely onto the bases of the two given bodies.

4. When constructing the line of intersection of the surfaces of two oblique cylinders the most simple auxiliary planes are those drawn parallel to the generatrices of the cylinders (Fig. 360, *a*).

When the surfaces of a cylinder and cone intersect, the simple auxiliary planes are those drawn through the vertex S of the cone parallel to the generatrices of cylinders (Fig. 360, *b*). Finally, to find the intersection of the surfaces of two cones the most convenient auxiliary planes are those which pass through the vertices of the cones (Fig. 360, *c*).



a) *b)* *c)*

Fig. 360

Similar auxiliary planes are used when constructing the lines of intersections of an oblique cylinder and a prism, of a cylinder and a pyramid, of a cone and a prism, of a cone and a pyramid. In every case these auxiliary planes give us section lines on each of the bodies. The intersections of these section lines give us points contained in the lines of intersection of the two given surfaces.

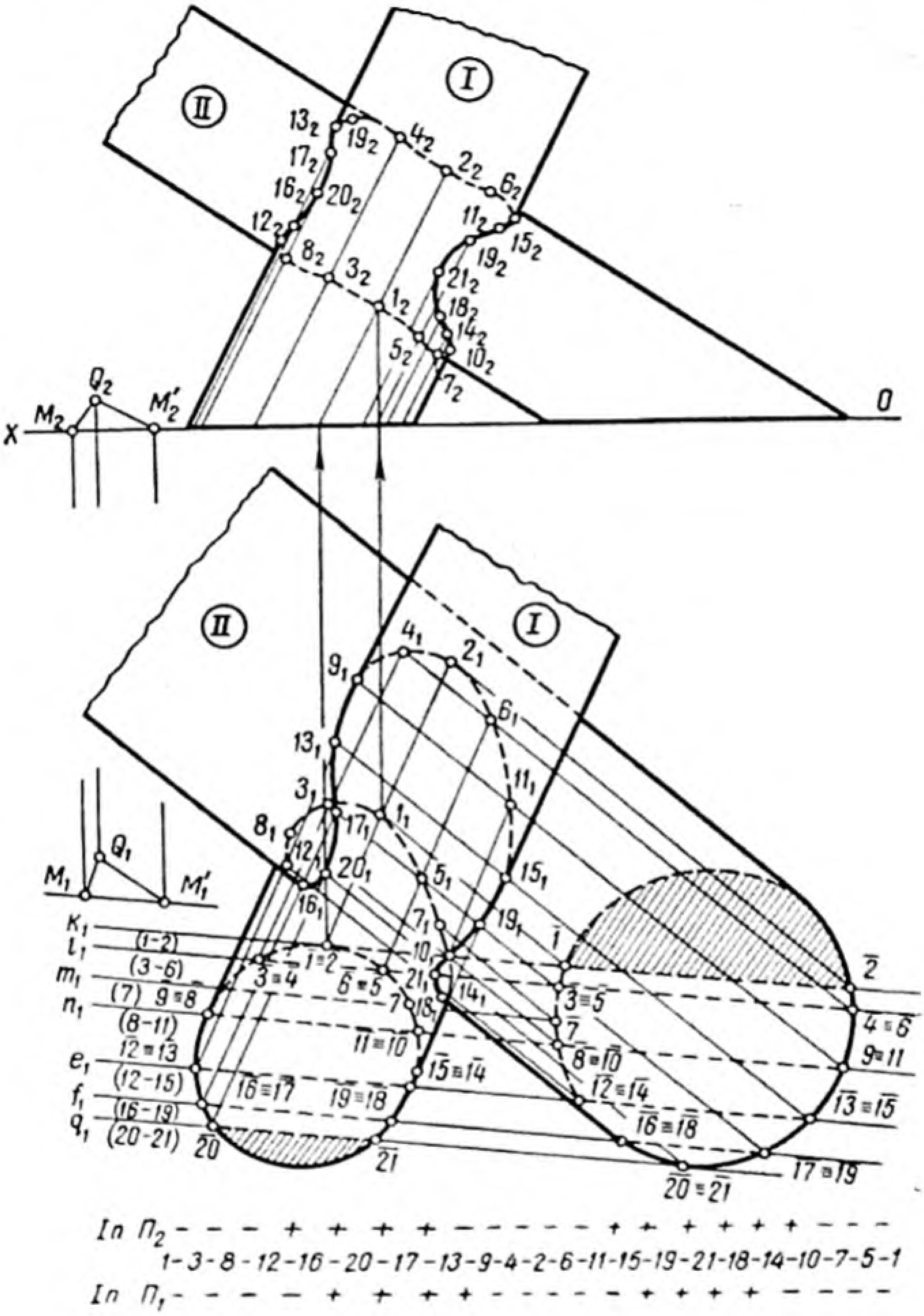
5. An example of the use of simple auxiliary planes to determine the transition line of two curved surfaces is the construction of the intersection of the surfaces of the elliptical cylinders, shown in Fig. 361.

The direction of the horizontal traces of the auxiliary planes is determined as follows: Lines QM and QM' are drawn respectively through any point Q parallel to the generatrices of the given cylinders *I* and *II*. The horizontal traces of all the auxiliary planes cutting the surfaces of cylinders along the generatrices will be parallel to the horizontal trace M_1M_1' of plane QMM'.

The trace k_1 of the first extreme auxiliary plane is drawn as a tangent to the base of the first cylinder. The trace q_1 of the other extreme auxiliary plane is drawn as a tangent to the base of the second cylinder. With the help of these two planes we obtain the pairs of points *1, 2* and *20, 21* on the line of intersection of the cylinders *I* and *II*. Other points on this line are found with the help of a series of planes situated between the first and the last planes. Each of these planes gives us four points on the required line of intersection of the cylinders.

Having the horizontal projections of points *1, 2, 3, . . . , 21* we find their frontal projections by means of projectors and the generatrices of the first cylinder, as shown in Fig. 361 by arrows for point 1_2.

Both the sequence of joining up the corresponding projections of the points of the curve of intersection and the visibility of separate points, when projected onto surfaces Π_1 and Π_2, are determined by the method of similar denotations. For this purpose



II
I
X
0
M_2
Q_2
M_2'
M_1
Q_1
M_1'
k_1
l_1
m_1
n_1
e_1
f_1
g_1
(1-2)
(3-6)
(7) 9≡8
(8-11)
12≡13
(12-15)
(16-19)
(20-21)
In Π_2 - - - + + + + - - - - - + + + + + + - - -
1-3-8-12-16-20-17-13-9-4-2-6-11-15-19-21-18-14-10-7-5-1
In Π_1 - - - - + + + + - - - - - + + + + - - - -

Fig. 361

points *1, 2, 3, ..., 21* are projected obliquely onto the bases of the cylinders parallel to their generatrices. The notations of the points are written out in one row, *1-3-8-12-16-20-17-13-9-4-2-6-11-15-19-21-18-14-10-7-5-1*, since the line of intersection of the cylinders is a single closed curve.

6. A scheme of the possible positions of the extreme traces of simple auxiliary planes relative to the bases of the intersecting bodies is shown in Fig. 362, *a*.

Trace k_1 of one extreme plane is a tangent to the base of the first body and intersects the base of the second body in two points. The trace q_1 of the other extreme plane intersects the base of the first body in two points and is a tangent to the base of the second. This is case *1*, when the two bodies intersect each other in a single closed curve. The intersection of the cylinders, shown in Fig. 361, is an example of such a case.

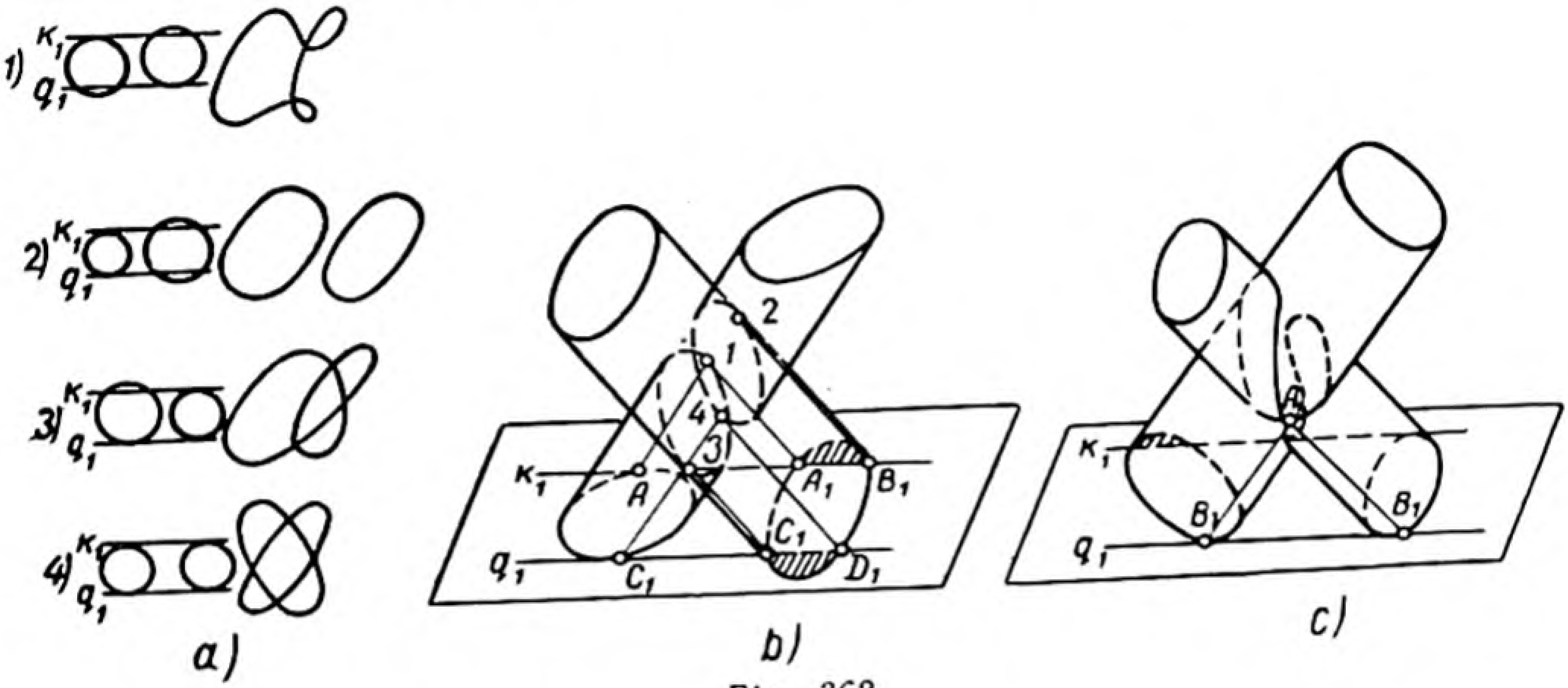


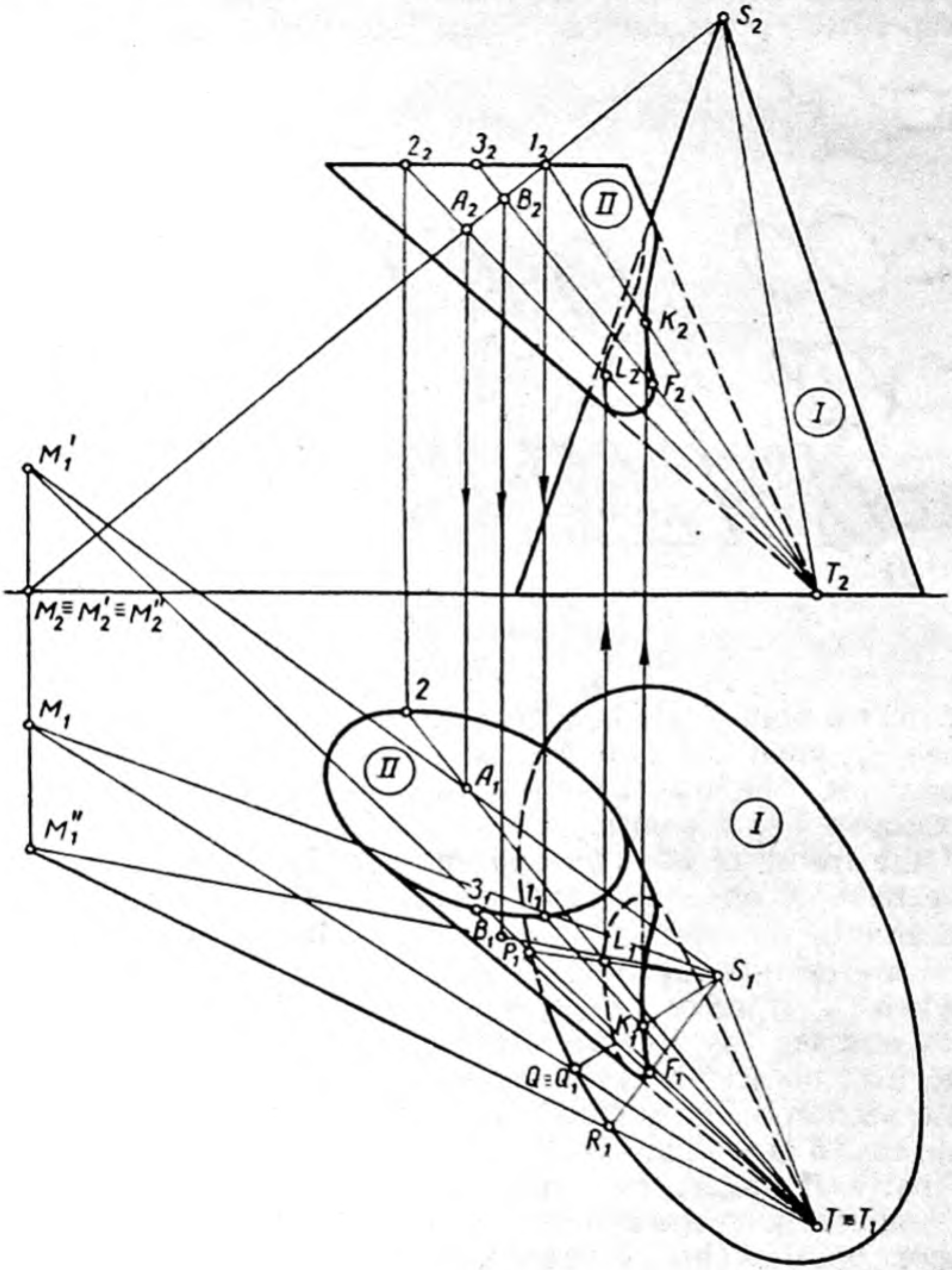
Fig. 362

If the traces of the two extreme auxiliary planes are tangents to the base of one body and cut the base of the other, the two surfaces intersect in two closed curves. This is case *2*, an example of which is seen in Fig. 362, *b*.

When the trace of one extreme plane touches the bases of both bodies and the trace of the other extreme plane touches the base of the first body, but cuts the base of the second, the two curves of intersection of the surfaces have one common point. An example of this case *3* is seen in Fig. 362, *c*.

Finally, if the traces of the extreme planes touch the bases of both bodies, the surfaces intersect in two plane curves having two common points. This is case *4*, of which the intersection of the surfaces of cylinders, shown in Fig. 353, *c*, may serve as an example.

7. In the constructions considered above, when simple oblique auxiliary planes were used to obtain points in the lines of intersection of surfaces, the bases of the latter were taken as lying in one plane. The construction of the line of intersection of two elliptical cones, the bases of which are situated in different planes, is shown in Fig. 363. The base of cone *I* is contained in the horizontal plane of projection Π_1 and the base of cone *II* in a plane parallel to Π_1. The auxiliary cutting planes are drawn through line *ST* joining



S_2
2_2
3_2
1_2
A_2
B_2
II
K_2
L_2
F_2
I
M_1'
$M_2 \equiv M_2' \equiv M_2''$
T_2
2
M_1
II
A_1
I
M_1''
3_1
1_1
B_1
P_1
L_1
S_1
K_1
$Q \equiv Q_1$
F_1
R_1
$T \equiv T_1$

Fig. 363

the cone vertices, therefore these planes intersect both conical surfaces along their generatrices.

Let us examine the construction of a point *K* belonging to the line of intersection of the cones.

Line *ST* and line *S1*, which cuts *ST* and passes through a point *1* on the base of the second cone, determine the simplest auxiliary plane α. The horizontal trace of plane α passes through the horizontal piercing point M_1 of line *S1* and through point $T \equiv T_1$ which is the horizontal projection of vertex of cone *II*. Point $Q \equiv Q_1$, at the intersection of the horizontal projection of the base of cone *I* with the trace M_1T_1, is joined to point S_1. Point 1_1 is joined to point T_1. These two straight lines are the horizontal projections of the generatrices *SQ* and *T1* contained in plane α. The point K_1, at the intersection of projections S_1Q_1 and T_11_1 of these generatrices, is the horizontal projection of point *K*. Having point K_1, with the help of a projector we find point K_2 on T_21_2.

Points *L* and *F* are found by a similar construction using auxiliary planes α' and α'' passing through line *ST* and through points *A* and *B* located on the generatrices *T2* and *T3* of cone *II* respectively. To simplify the drawing, when determining the projections A_2 and B_2, the points *A* and *B* are taken so that the lines S_2A_2 and S_2B_2 coincide with line S_21_2.

Curves are drawn through the corresponding projections of points *K, L, F* and of other points not shown in the drawing. These curves are the lines of intersection of the two given elliptic cones.

§ 41. The Method of Spherical Sections

1. If a surface of revolution, the axis of which is parallel to the plane of projection, is cut by a sphere with its centre on the axis of revolution, the circle formed by the intersection of these two surfaces will be projected onto the given plane of projection as a straight line perpendicular to the corresponding projection of the axis of revolution.

Such a case is shown in Fig. 364, where a sphere *(I)* with its centre at *C* intersects a cone *(II)*, a cylinder *(III)* and a surface of revolution *(IV)*, the axes of which are parallel to the corresponding planes of projection and pass through the centre of the sphere *C*. The axes of the cone and cylinder are parallel to planes Π_1 and Π_3. It follows that the circles, obtained by the intersection of the cylinder and the cone with the sphere, are projected onto planes Π_1 and Π_3 as straight lines perpendicular to the projections of the axes. The axis of the body of revolution *IV* is parallel to planes Π_1 and Π_2. It follows that the circle of intersection of this surface with the sphere is projected onto Π_1 and Π_2 as a straight line, in each case perpendicular to the corresponding projections of the axis of revolution.

The relationships outlined above can be used for the construction of the line of intersection of a cylinder and a cone, the axes of which are parallel to plane Π_2 and intersect in point *C*, as shown in Fig. 365. Let us assume point *C* as the centre of a sphere with radius r and construct the frontal projections of auxiliary circles in which the cylinder and cone are intersected by the spherical surface. Both circles are projected onto plane Π_2 as straight lines A_2B_2 and D_2E_2, respectively perpendicular to the frontal projections of the axes of the cylinder and cone. A_2, B_2, D_2 and E_2 are the points of intersection of the frontal projections of the limiting elements of the cylinder and cone with the circle of radius r in which the sphere is projected onto plane Π_2. Since the auxiliary circles obtained lie on the surface of one and the same sphere, point 2_2 in which lines A_2B_2 and E_2D_2 intersect, will determine the frontal projections of the two points 2 lying on the line of intersection of the cylinder and cone.

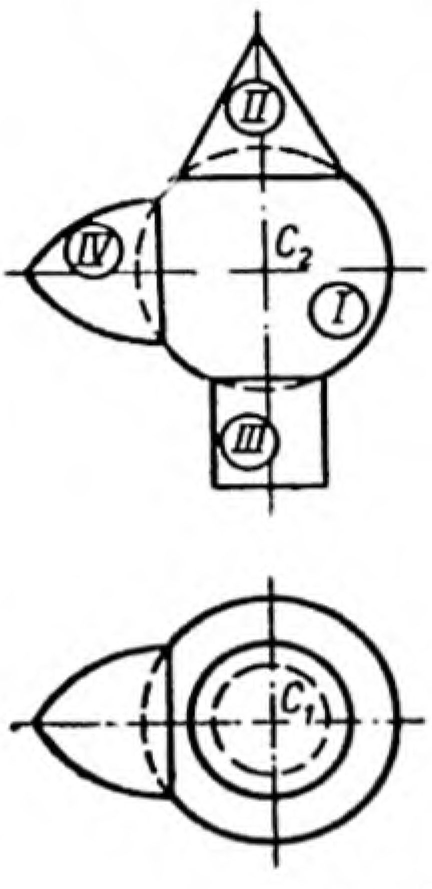


Fig. 364

Other auxiliary spheres can be described having the same centre C in order to find a series of points on the lines of intersection of the surfaces of the cylinder and cone.

The frontal projections 1_2 and 4_2 of points 1 and 4, in which the upper generatrix of the cylinder intersects the upper and lower generatrices of the cone, also lie on the frontal projection of line of transition, since these generatrices are situated in the same frontal plane passing through the axes of the cylinder and the cone. A curve is drawn through points 1_2, 2_2, 3_2 and 4_2. This is the frontal projection of the line of intersection of the cylinder

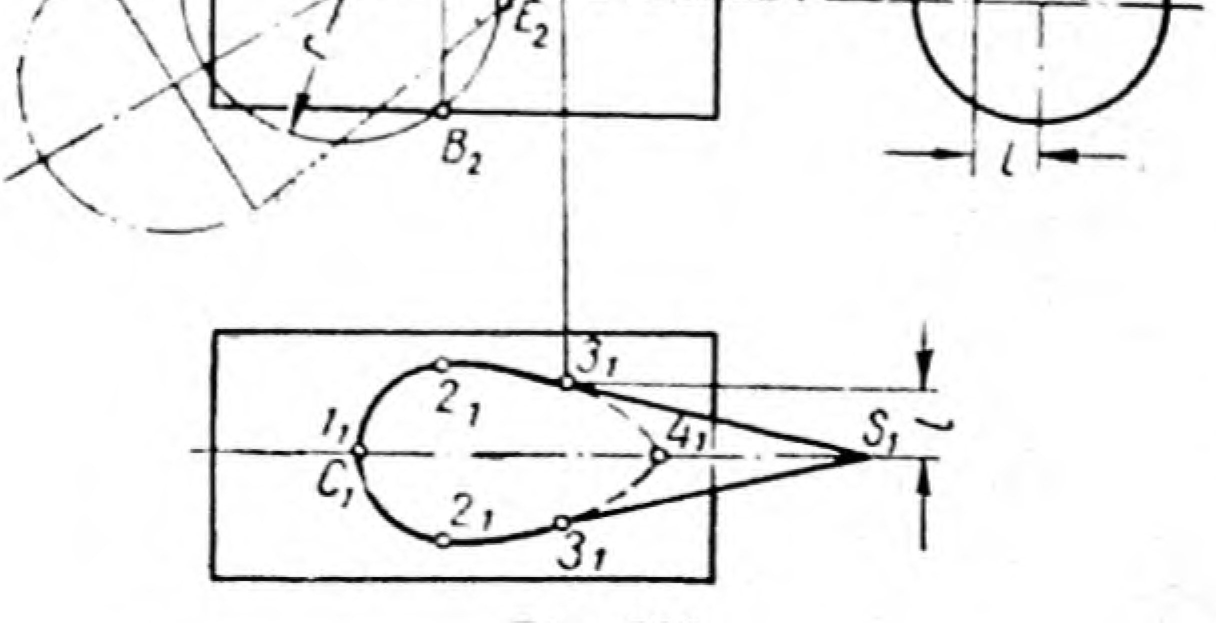


Fig. 365

and the cone. The profile projection 2_3-3_3-1_3-4_3-3_3-2_3 of the line of intersection coincides with the circle at which the surface of the cylinder is projected onto plane Π_3.

The horizontal projection of the line of intersection is constructed using the frontal and profile projections already obtained. Once points 1_2 and 4_2 are known points 1_1 and 4_1 can be found with the help of projectors.

It should be noted that the radius of the largest auxiliary sphere should be slightly less than the distance from centre C_2 to the frontal projection of the point on the transition line farthest away from C_2, in this case, 4_2. The radius of the smallest sphere should be equal to the distance from centre C_2 to the frontal projection of the most distant generatrix.

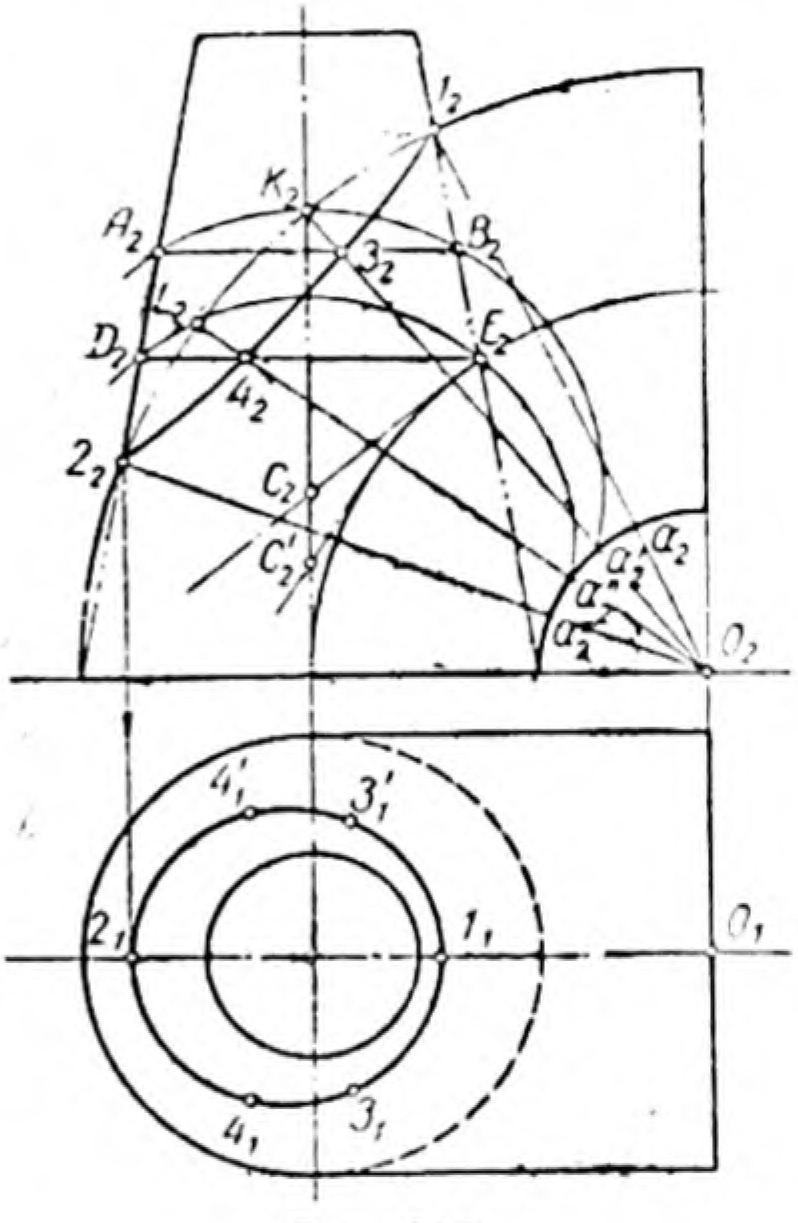


Fig. 366

2. Let it be required to determine the intersection of a cone and an annular torus (Fig. 366).

The basic points of the intersection of the given surfaces lie at the intersection of the limiting elements of the cone with the outside generatrix of the torus. The highest point is 1_2, the lowest being 2_2.

The frontal plane passing through the axis of the cone is at the same time the plane of symmetry of the torus, so it is convenient to use spherical sections to determine intermediate points belonging to the line of intersection of the cone and torus.

Let us subdivide that portion of the torus limited by planes α and α''' into separate parts with the help of frontal-projecting planes α, α', α'', α''' passing through the centre O of the torus. These parts may be called "curved cylinders". We can assume the surface of each part to be the surface of a right circular cylinder touching a curved cylinder in the circle of intersection of the torus by a plane. We are then able to construct the points of intersection of these straight cylinders with the cone.

Let us follow through the construction of a point 3_2 belonging to the intersection of the cone and the cylinder lying between planes α and α'. The axis of this cylinder intersects the axis of the cone in point C_2. With point C_2 as the centre we describe a circle, the radius of which is equal to the distance from C_2 to point K_2 in which trace α'_2 intersects the outside generatrix of the torus. This circle is the frontal projection of the auxiliary spherical surface intersecting the cone and the assumed right cylinder contained between planes α' and α''. The frontal projection of the circular section of the cone is line A_2B_2 and it is perpendicular to the frontal projection of the axis of the cone. The frontal projection of the circle of intersection of the cylinder coincides with trace α'_2. The intersection of these lines gives point 3_2 contained in the frontal projection of the line of intersection of the cone and the torus.

Point 4_2 is found in a similar manner by drawing a circle with radius C'_2L_2 having C'_2 as its centre. Point 4_2 lies at the intersection of lines D_2E_2 and L_2O_2.

The construction of the horizontal projections 3_1, $3'_1$, 4_1 and $4'_1$ of points 3, $3'$, 4 and $4'$ is carried out by drawing the circular sections of the cone given by the horizontal planes, the frontal traces of which pass through points 3_2 and 4_2 respectively. From points 3_2 and 4_2 projectors are drawn to intersect the horizontal projections of these circles. Having obtained points 3_1, $3'_1$, 4_1 and $4'_1$, we draw a curve through points 1_1, 3_1, 4_1, 2_1, $4'_1$ and $3'_1$. This curve is the horizontal projection of the intersection of the cone and torus.

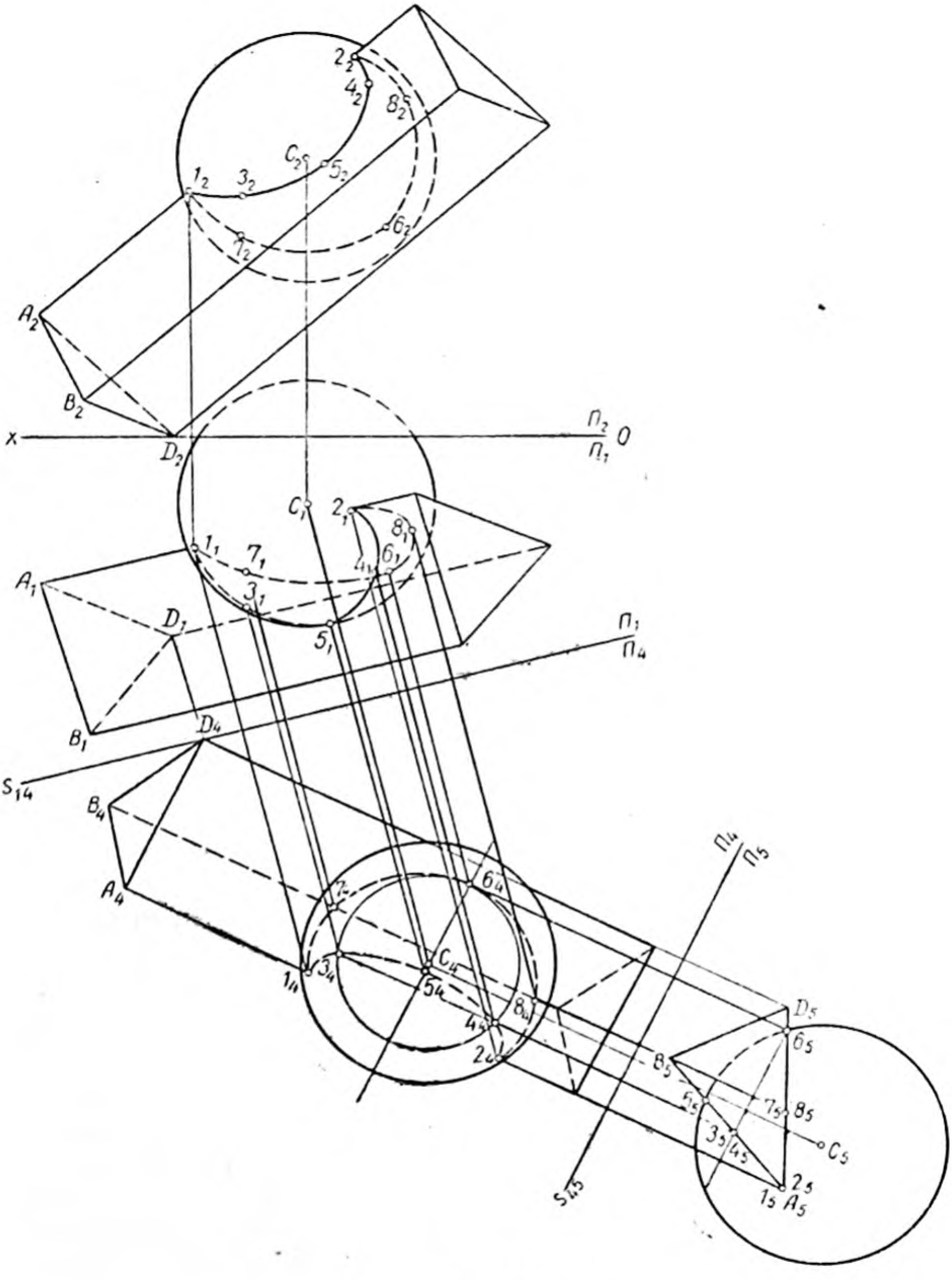
§ 42. Transformation of Projections and Oblique Projection Used to Determine the Intersection of Surfaces

1. The transformation of projections for constructing the intersection of two surfaces, say a prism and sphere (Fig. 367), consists in obtaining a projection of the two surfaces in which the line of intersection coincides with the projection of one of the given surfaces, in this case, of the prism. This may be accomplished by successive revolutions of the given bodies about the corresponding axes, or by successive substitutions of the planes of projection.

In the multiview drawing (Fig. 367) the problem is solved by substituting the planes of projection. Plane Π_2 is substituted by plane Π_4 perpendicular to plane Π_1 and parallel to the lateral edges of the prism. These edges will be projected onto plane Π_4 in their true length. The second new plane Π_5, perpendicular to plane Π_4, is also perpendicular to the lateral edges of the prism. The prism is then projected onto this plane Π_4 as the triangle $A_5B_5D_5$. The sides of this triangle represent the projection of the faces of the prism. The part A_55_5 of side A_5B_5 of the triangle is the projection on plane Π_5 of the arc of the circle in which face AB of the prism cuts the surface of the sphere.

Similarly, A_56_5 is the projection on plane Π_5 of the arc of the circle in which face AD of the prism cuts the sphere. Since side B_5D_5 of the triangle does not intersect the circle having C_5 as its centre, it follows that face BD of the prism does not cut the surface of the sphere.

Let us now take a series of points on the sides A_5B_5 and A_5D_5



2_2
4_2
8_2
C_2
5_2
1_2
3_2
7_2
6_2
A_2
B_2
D_2
x
Π_2
0
Π_1
C_1
2_1
8_1
1_1
7_1
4_1
6_1
3_1
5_1
A_1
D_1
Π_1
Π_4
B_1
D_4
S_{14}
B_4
A_4
7_4
6_4
C_4
3_4
1_4
5_4
8_4
4_4
2_4
Π_4
Π_5
D_5
6_5
B_5
5_5
7_5
8_5
3_5
4_5
C_5
2_5
1_5
A_5
S_{45}

Fig. 367

of the triangle and, using the method of inverse projection, find the projections of these points, first in the system of planes Π_1-Π_4 and then in the principal system of planes of projection Π_1-Π_2. Points *5* and *6* are special points. The edge *A* of the prism intersects the sphere in points *1* and *2*. Points *3, 4, 7* and *8* are intermediate points.

The projections of points *1, 2, 3, 4, 5, 6, 7* and *8* on plane Π_4 are obtained by cutting the sphere with planes parallel to the plane of projection Π_4. This construction is shown for points 3_4, 4_4 and 6_4. The projections of the points on planes Π_1 and Π_2 are found by means of their projections on planes Π_4 and Π_5. It should be remembered that the distances of these points from the respective planes of projection remain unchanged. The two arcs of circles in which the sphere is cut by the prism faces *AB* and *AD* are projected onto planes Π_4, Π_1 and Π_2 as arcs of ellipses.

2. The construction of the intersection of two oblique prisms using oblique projection is shown in the multiview drawing (Fig. 368). The prisms are projected obliquely onto plane Π_1 in direction *s* parallel to the lateral edges *DK, EL* and *FM* of the second prism.

The oblique projection of prism *II* on plane Π_1 coincides with the orthographic projection $K_1L_1M_1$ of its base *KLM* on that same plane Π_1. The oblique projection of prism *I* on plane Π_1 is shown in the drawing by dots and dashes. The oblique projections of its vertices are found in the same way as the horizontal traces of oblique lines are found. This construction is shown for vertex *A*.

The intersection of the oblique projections of the lateral edges of the first prism with the sides of triangle $K_1L_1M_1$, i. e., with the oblique projection of prism *II*, determines points $\overline{1}_1$, $\overline{2}_1$, $\overline{3}_1$, $\overline{4}_1$, $\overline{5}_1$, $\overline{6}_1$, $\overline{7}_1$ and $\overline{8}_1$. These points are the oblique projections of points *1, 2, 3, 4, 5, 6, 7* and *8*, contained in the line of intersection of the prisms.

By inverse projection points 1_1, 2_1, 3_1, 4_1, 5_1 and 6_1 are found at the intersection of the horizontal projections of the oblique projectors with the orthographic projections on plane Π_1 of the corresponding edges of the first prism. In order to determine points 7_1 and 8_1 two auxiliary lines *S7* and *P8* are used. These lines are first drawn on the oblique projection of the prism through the duplicate point $\overline{7}_1 \equiv \overline{8}_1$. Then they are drawn on the projection of prism *II* on plane Π_1. This construction is indicated by arrows. The frontal projections 1_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2 and 8_2 of all the eight points found are constructed by passing projectors through points 1_1, 2_1, etc., to intersect the frontal projections of the respective lateral edges of the prism.

3. The line of intersection of a cylinder and a pyramid (Fig. 369) may also be constructed by the method of oblique projection of both

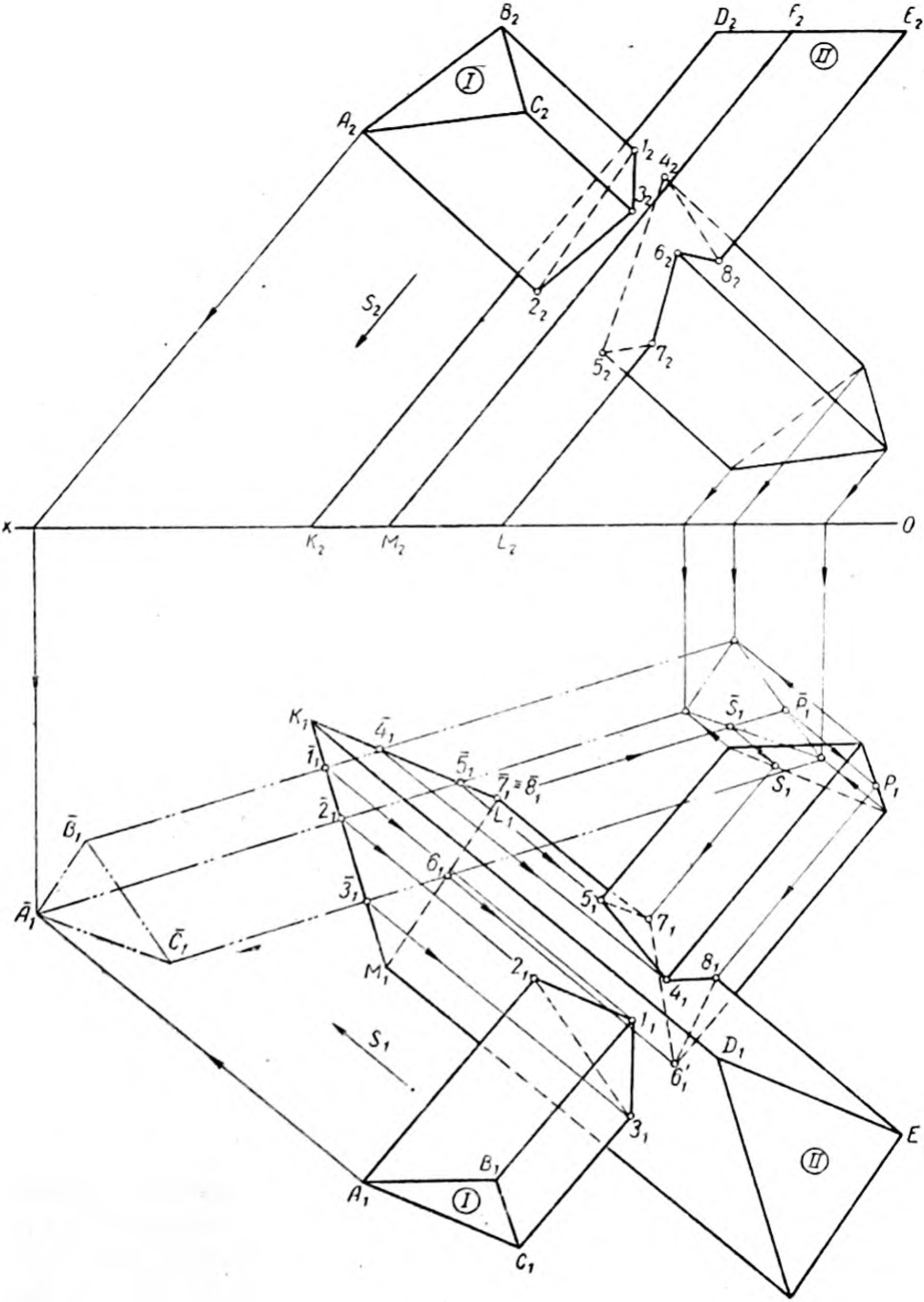

B_2
D_2
F_2
E_2
Ⓘ
ⒾⒾ
A_2
C_2
1_2
4_2
3_2
6_2
8_2
S_2
2_2
5_2
7_2
x
K_2
M_2
L_2
0
K_1
$\bar{4}_1$
$\bar{S}_1$
$\bar{P}_1$
$\bar{1}_1$
$\bar{5}_1$
$\bar{7}_1 \equiv \bar{8}_1$
S_1
P_1
$\bar{B}_1$
$\bar{2}_1$
L_1
6_1
$\bar{A}_1$
$\bar{3}_1$
5_1
7_1
$\bar{C}_1$
M_1
2_1
8_1
4_1
1_1
D_1
S_1
$6'_1$
E_1
3_1
B_1
A_1
Ⓘ
ⒾⒾ
C_1
F_1

Fig. 368

bodies onto plane Π_1 in direction l parallel to the generatrices of the cylinder. The oblique projection of the cylindrical surface coincides with the orthographic projection on plane Π_1 of the base of the cylinder itself. The oblique projection of the base of the pyramid coincides with the orthographic projection of its base on plane Π_1. Next point $\bar{S}_1$, the oblique projection on plane Π_1 of the vertex of the pyramid S, is found. Point $\bar{S}_1$ is joined by straight lines with points $A_1 \equiv \bar{A}_1$, $B_1 \equiv \bar{B}_1$ and $C_1 \equiv \bar{C}_1$. This gives us the oblique projection $\bar{S}_1\bar{A}_1\bar{B}_1\bar{C}_1$ of the pyramid on plane Π_1. The intersection of the projections $\bar{S}_1\bar{A}_1$, $\bar{S}_1\bar{B}_1$ and $\bar{S}_1\bar{C}_1$ of the edges of the pyramid with the ellipse, which is the oblique projection of the cylinder and coincides with the orthographic projection of the base

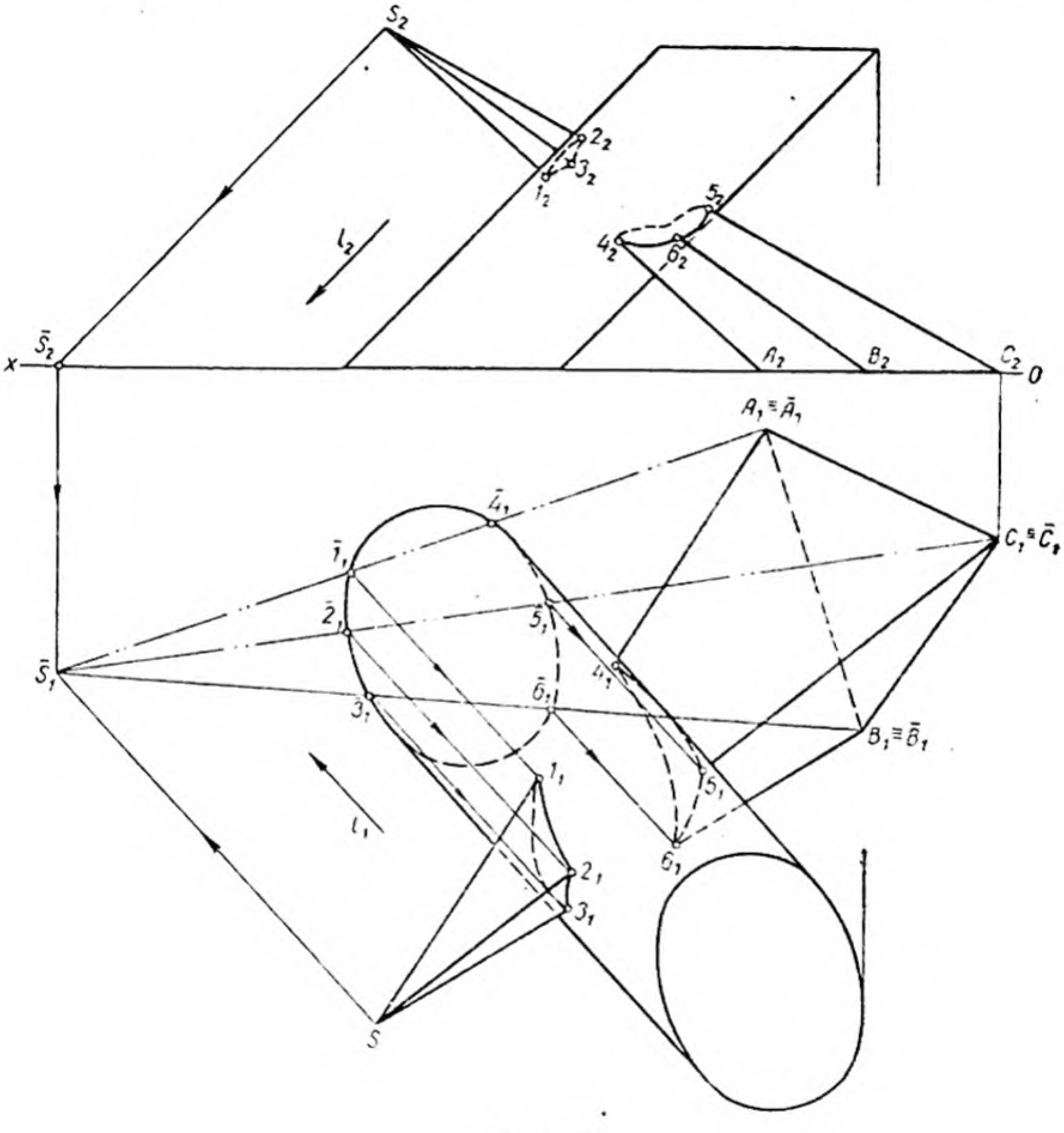

S_2
2_2
3_2
1_2
l_2
5_2
4_2
6_2
x
$\bar{S}_2$
A_2
B_2
C_2
0
$A_1 \equiv \bar{A}_1$
$\bar{4}_1$
$\bar{1}_1$
$C_1 \equiv \bar{C}_1$
$\bar{2}_1$
$\bar{5}_1$
$\bar{S}_1$
4_1
$\bar{6}_1$
$\bar{3}_1$
$B_1 \equiv \bar{B}_1$
1_1
5_1
l_1
6_1
2_1
3_1
S

Fig. 369

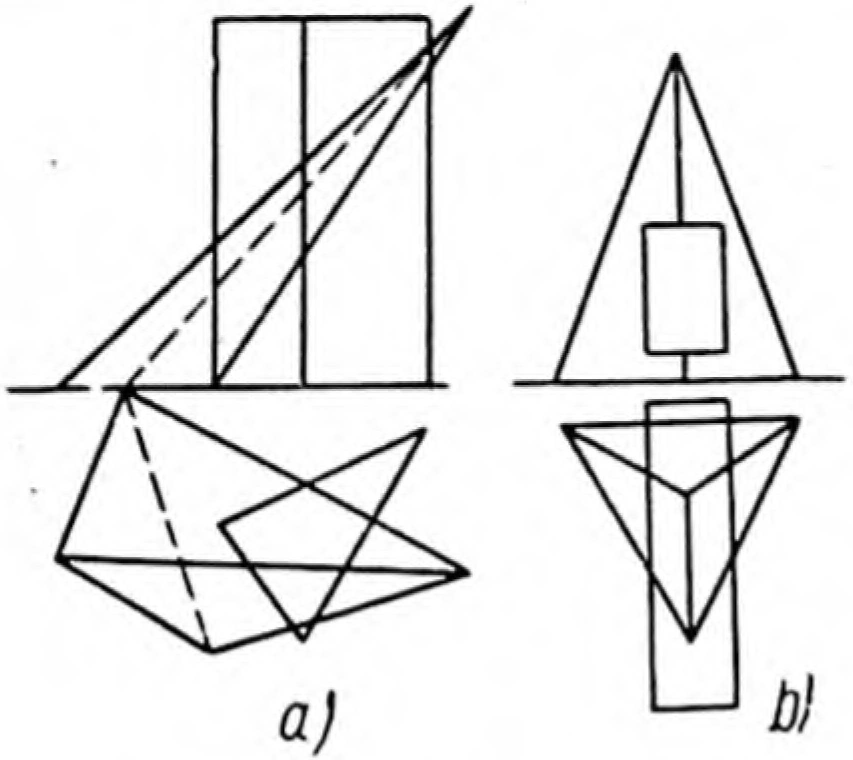


Fig. 370

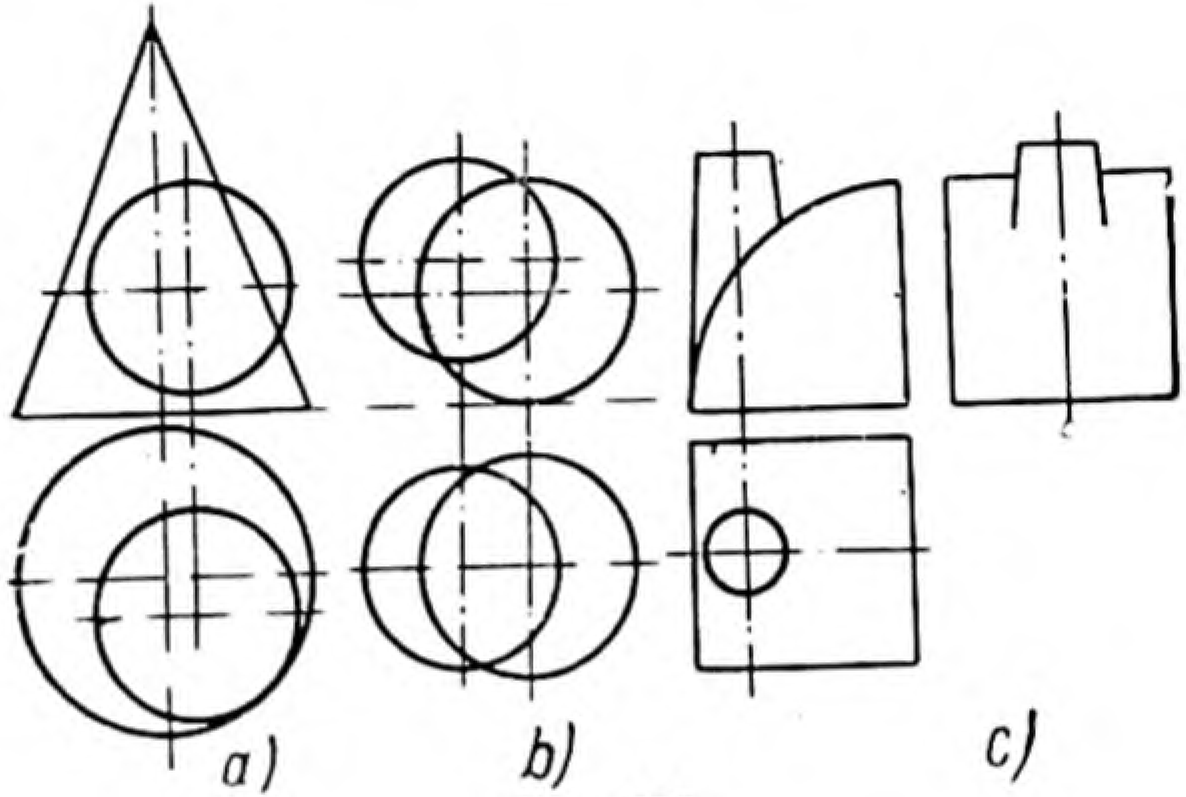


Fig. 372

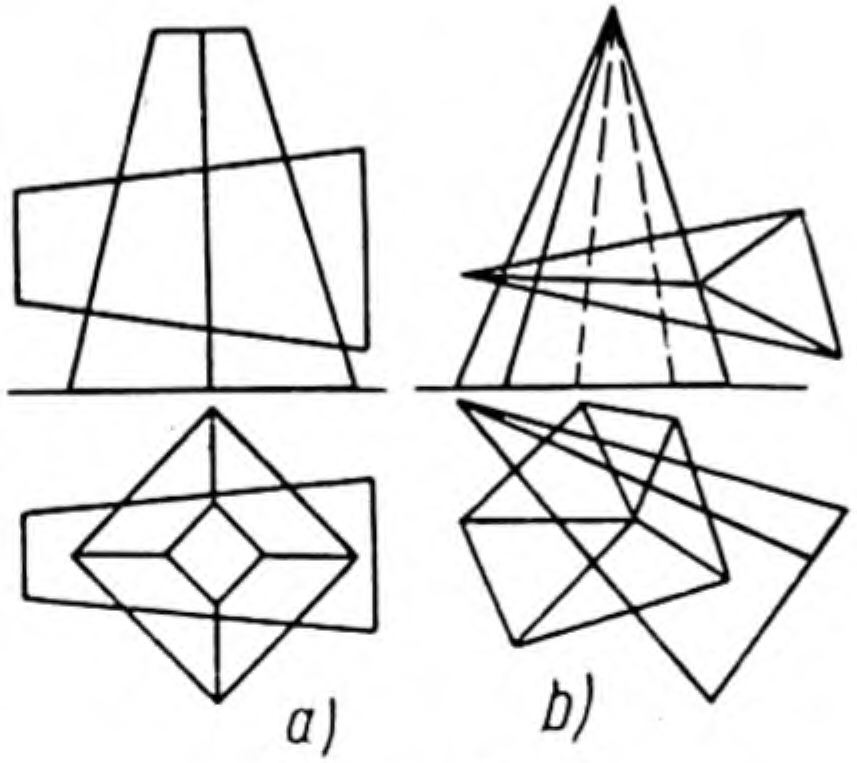


Fig. 371

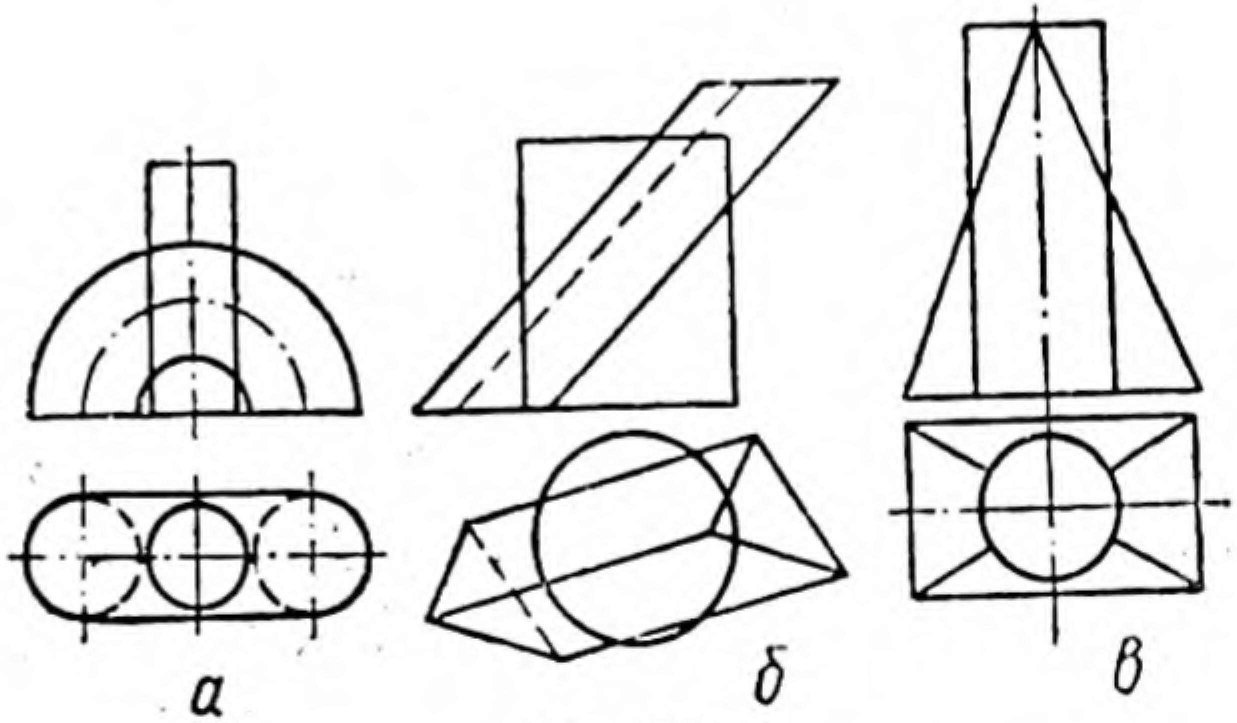


Fig. 373

of the cylinder on Π_1, determines the oblique projections $\overline{1}_1$, $\overline{2}_1$, $\overline{3}_1$, $\overline{4}_1$, $\overline{5}_1$ and $\overline{6}_1$ of the points in which the edges of the pyramid intersect the cylinder. The inverse projecting of these points contained in the intersection of the cylinder and pyramid is carried out the same way as in the previous example (see Fig. 368).

Problems and Exercises

1. How is the line of transition, or intersection of surfaces of two polyhedra constructed?

2. Describe how the method of similar notations is used to determine the intersection of two surfaces?

3. What types of auxiliary planes are used when constructing points contained in the line of intersection of two surfaces?

4. What is the procedure for constructing the intersection of two surfaces by transforming their projections?

What is the procedure when using oblique projection for this purpose?

5. Construct the intersection of the pyramid and the prism, shown in Fig. 370.

6. Construct the intersection of the pyramids, shown in Fig. 371.

Hint. In the first example (Fig. 371, *a*) the section planes used should be projecting planes. In the second case (Fig. 371, *b*) simple oblique section planes passing through the vertices of the pyramid should be used (Fig. 360, *c*).

7. Construct the intersection of the following surfaces: a) a cone and sphere, b) two spheres, c) a cone and a cylinder (Fig. 372).

8. Construct the intersection of the following surfaces: a) a torus and a cylinder, b) a cylinder and a prism, c) a cylinder and a pyramid (Fig. 373).

Chapter XI

AXONOMETRIC PROJECTION

§ 43. General

The orthographic projections of different objects completely determine their form, their dimensions, and their relative position in space. Such projections are convenient to use since two dimensions, parallel to the respective axes of projection, the ground lines, are shown without distortion while the third dimension is represented as a point. However, orthographic projections are not completely descriptive since they depict the objects not as one whole, but by means of their separate projections. From these drawings it is necessary to imagine the reconstruction of the form of the objects in space. That is why not only orthographic projections, but also axonometric projections, are widely used.

The main advantage of axonometric projections consists in their clarity. Because of this, and also due to the inherent advantage of axonometric projections, i. e., the ease with which measurements are made on the projections, they are gaining popularity in engineering. In Fig. 374 a detail, the body of a valve, is depicted both in orthographic (Fig. 374, *a*) and in axonometric projection (Fig. 374, *b*). The axonometric representation of any detail is obviously more descriptive.

When constructing axonometric projections the objects are considered as being situated in a system of rectangular coordinate axes. The objects, together with the axes, are projected onto a selected plane of projection. The latter is called the *plane of axonometric projection*. The projections obtained on this plane are called *axonometric projections*.

The projections of the coordinate axes on the plane of axonometric projection are called *axonometric axes*.

Fig. 375 shows a plane of axonometric projection Π' arbitrarily located in respect to the trihedral formed by the coordinate planes Oxz, Oxy and Oyz. The axonometric axes $O'x'$, $O'y'$ and $O'z'$ obtained in plane Π' by projecting the rectangular coordinate axes in the direction s, and the axonometric projection A' of point A, are also shown.

The projector s may be either perpendicular or inclined to plane Π'. In the first case the axonometric projections are called *rectangular* and in the latter — ***oblique***. The former are also referred to as right or right-angle projections.

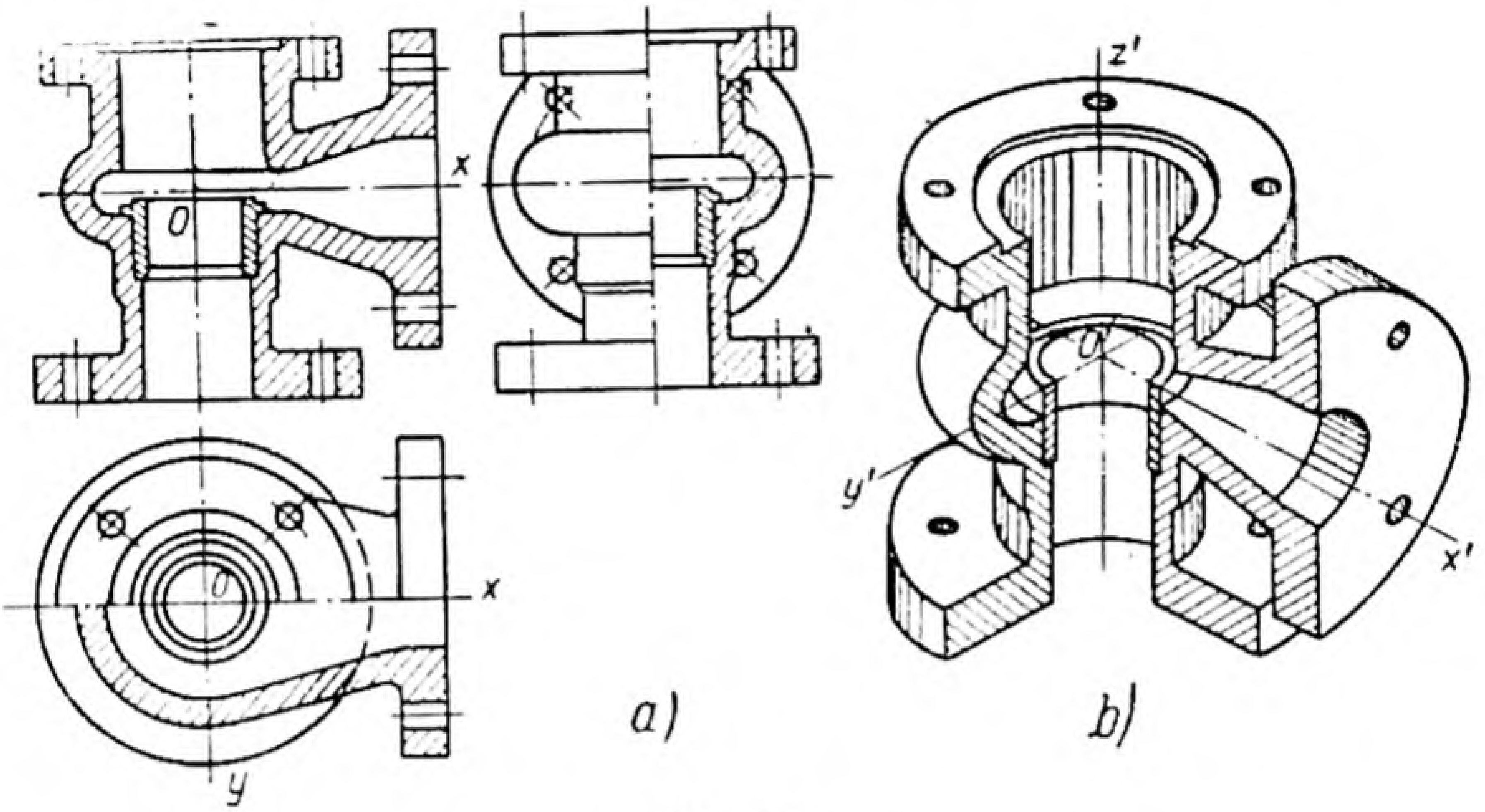


Fig. 374

The rectangular and oblique axonometric projections also differ in the degree of distortion of dimensions. The *factor*, or *index*, of *distortion* or *foreshortening* is the ratio of the length of the axonometric projection of a line to the length of the line itself.

Let the coordinate angle $Oxyz$ (Fig. 376) be projected onto the plane of axonometric projection Π' in the direction s, and let point O' be the axonometric projection of the point of origin O. Then line $O'x'$ will be the axonometric projection of Ox. The lines $O'y'$ and

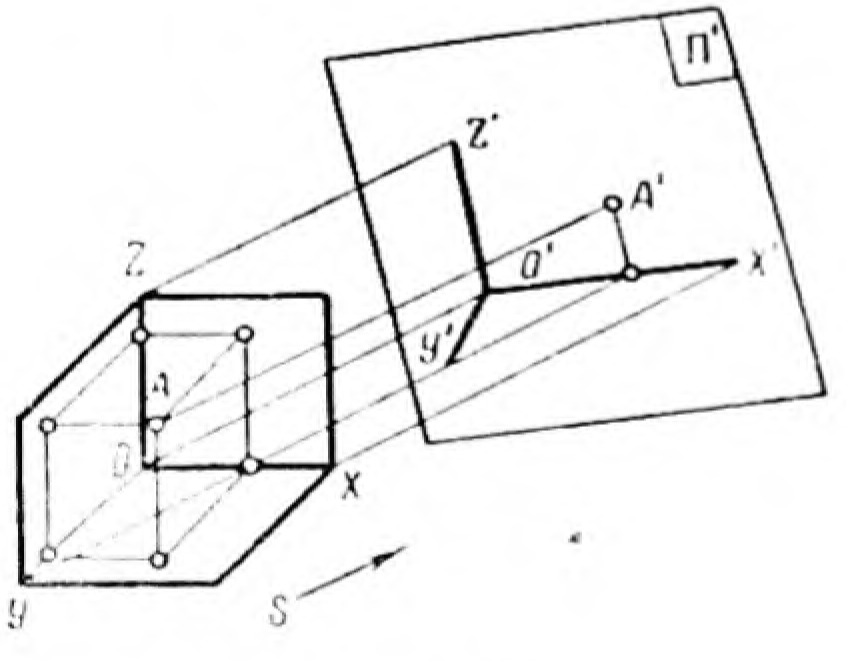


Fig. 375

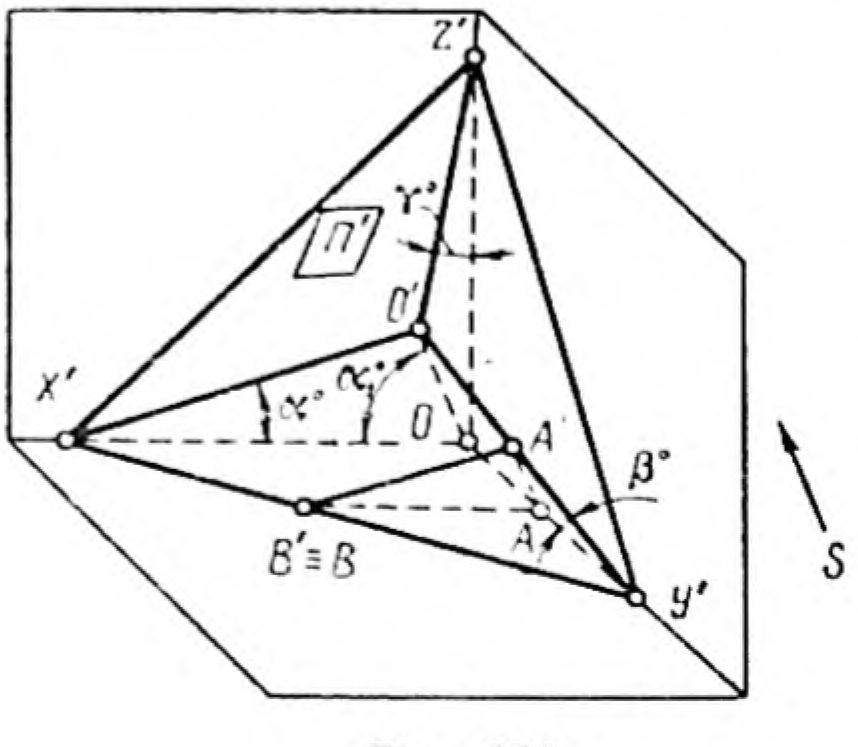


Fig. 376

$O'z'$ will be the axonometric projections of Oy and Oz. The ratios $\frac{O'x'}{Ox}=p$, $\frac{O'y'}{Oy}=q$ and $\frac{O'z'}{Oz}=r$ are the factors of distortion determining the degree of foreshortening in the directions of the axes Ox, Oy and Oz.

Let us take a line AB in the plane Oxy, parallel to axis Ox. Since parallel projection is used, the axonometric projection $A'B'$ is parallel to the axonometric axis $O'x'$. It follows that $\frac{A'B'}{AB}=$ $=\frac{O'x'}{Ox}=p$.

When all three factors of distortion are equal, i. e., $p=q=r$, the axonometric projection is called *isometric* projection and the scale used is an isometric scale.

Axonometric projection with two equal factors of distortion and the third differing is called *dimetric* projection, i. e., two measurements are equal. The scale used is a dimetric scale. If there are no equal factors axonometric projection is called *trimetric* projection, i. e., possessing three different measurements. The scale then used is called a trimetric scale.

The direction of the projection for the construction of axonometric projections is of considerable importance. This direction should be chosen so that one axonometric projection gives a complete idea of the form of the object represented. The direction of projection should not be perpendicular to any of the coordinate planes, otherwise one of the coordinate axes will be projected onto plane Π' as a point and the axonometric representation will lack clarity.

The plane of axonometric projection Π' should also be given a suitable position in space with respect to the coordinate planes Oxz, Oyx and Oyz. In rectangular axonometric projection plane Π' must cut all three coordinate axes, otherwise the projectors perpendicular to Π' may cause one of the axes x, y or z to be projected onto Π' as a point. They may also project two of the axes as a single line. For example, if plane Π' is perpendicular to plane Oxz and to axis Oz the axis Ox will be projected onto Π' as a point and the axonometric representation will not be sufficiently clear.

In oblique axonometric projection, when the position of the plane of projection does not depend on the choice of direction of the projection and, conversely, the selection of the direction of projection does not depend on the position of the plane of axonometric projection, plane Π' may be parallel to one of the coordinate axes or coordinate planes, or it may not intersect all three coordinate axes. In oblique axono-

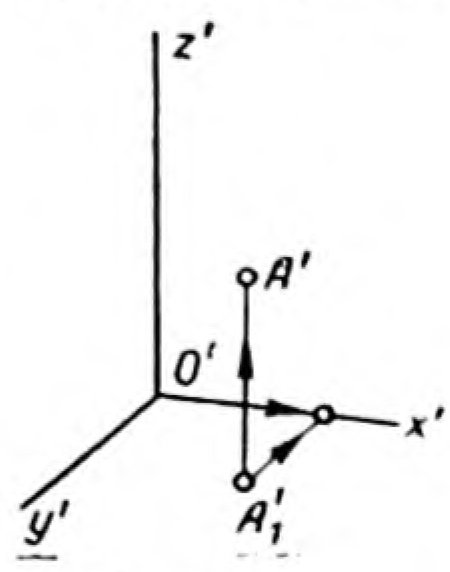


Fig. 377

metric plane Π' is usually assumed as parallel to one of the coordinate planes.

The relative positions of the plane of axonometric projection and the direction of projection in space determine the factors of distortion and the angles between the axonometric axes.

The construction of a dimetric rectangular projection A' of point A with respect to the axes $O'x'$, $O'y'$ and $O'z'$ inclined at angles 7°, 41° and 90° to the horizontal lines is shown in Fig. 377. The sequence of construction is shown by arrows. First, from point O' we lay off the value x along axis $O'x'$. Then parallel to the axis $O'y'$ we lay off the value y. This gives us point A_1'*. Finally, along the vertical, upwards from point A_1', the value z is laid off. This gives us the rectangular dimetric projection A' of point A.

§ 44. The Elements of Rectangular Axonometric Projection

1. The determination of the factors of distortion. If the projections are perpendicular to the plane of axonometric projection Π' (Fig. 376) and point O' is the axonometric projection of point O, then line $O'O$ is perpendicular to plane Π' and the lines $O'x'$, $O'y'$ and $O'z'$ will be the sides of the right triangles $O'x'O$, $O'y'O$ and $O'z'O$. Therefore: $\frac{O'x'}{Ox} = \cos\alpha°$; $\frac{O'y'}{Oy} = \cos\beta°$; $\frac{O'z'}{Oz} = \cos\gamma°$.

The left parts of the equations represent the factors of distortion:

$$\frac{O'x'}{Ox} = p; \qquad \frac{O'y'}{Oy} = q; \qquad \frac{O'z'}{Oz} = r.$$

Therefore: $\cos\alpha° = p$; $\cos\beta° = q$; $\cos\gamma° = r$.

Let the coordinates of point O' be x, y, z (Fig. 378). Let the length of $O'O$ be k, then

$$x^2 + y^2 + z^2 = k^2. \tag{1}$$

But $x = k\cos\delta°$; $y = k\cos\lambda°$; $z = k\cos\varphi°$.

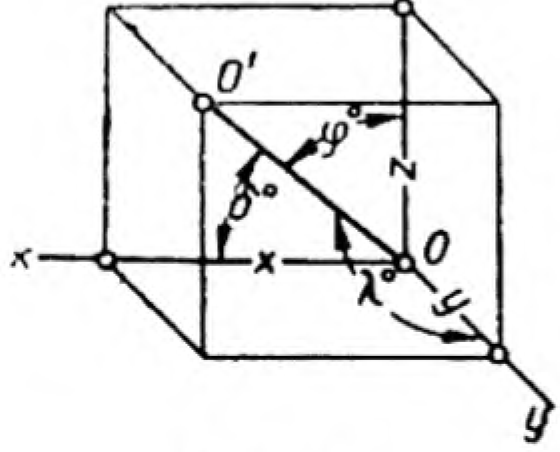


Fig. 378

Substituting these values of the coordinates of point O' in equation (1), we obtain

$$k^2\cos^2\delta° + k^2\cos^2\lambda° + k^2\cos^2\varphi° = k^2 \text{ and}$$
$$\cos^2\delta° + \cos^2\lambda° + \cos^2\varphi° = 1. \tag{2}$$

So the sum of the squares of the cosines of the angles made with the coordinate axes by the line $O'O$ passing through the origin of the coordinates is equal to unity.

The line $O'O$ (Fig. 376) forms angles $\alpha_1°$, $\beta_1°$ and $\gamma_1°$ with axes Ox, Oy and Oz.

---

* This point A_1' is called the secondary projection.

These angles are complementary to angles α°, β°, γ°. In Fig. 376 the only notation of these angles shown is that of the angle α_1° between the ground line Ox and the line $O'O$. From what has been said it follows:

$$\cos^2 \alpha_1^\circ + \cos^2 \beta_1^\circ + \cos^2 \gamma_1^\circ = 1 \qquad (3)$$

since

$$\alpha^\circ = (90^\circ - \alpha_1^\circ),\ \beta^\circ = (90^\circ - \beta_1^\circ),\ \gamma^\circ = (90^\circ - \gamma_1^\circ),$$

we have

$$\cos \alpha_1^\circ = \sin \alpha^\circ;\ \cos \beta_1^\circ = \sin \beta^\circ;\ \cos \gamma_1^\circ = \sin \gamma^\circ.$$

Hence,

$$\sin^2 \alpha^\circ + \sin^2 \beta^\circ + \sin^2 \gamma^\circ = 1. \qquad (4)$$

But

$$\sin^2 \alpha^\circ = 1 - \cos^2 \alpha^\circ;\ \sin^2 \beta^\circ = 1 - \cos^2 \beta^\circ;\ \sin^2 \gamma^\circ = 1 - \cos^2 \gamma^\circ. \qquad (5)$$

Substituting (5) in (4) we obtain

$$\cos^2 \alpha^\circ + \cos^2 \beta^\circ + \cos^2 \gamma^\circ = 2.$$

It has already been demonstrated that $\cos \alpha^\circ = p$; $\cos \beta^\circ = q$; $\cos \gamma^\circ = r$.

Therefore: $p^2 + q^2 + r^2 = 2$, i. e., the sum of the squares of the factors of distortion in a rectangular axonometric system is equal to two.

In a right isometric system the distortion factors along the three axes are equal to each other: $p = q = r$. Therefore, $3p^2 = 3q^2 = 3r^2 = 2$. It follows: $p = q = r = \sqrt{\frac{2}{3}} = 0.8165 \approx 0.82$.

Because of this, in a right isometric system, the lengths of projections of parts of lines, i. e., of segments, located along axes x, y and z or parallel to them, are equal to 0.82 of the real length of the lines which have been projected.

In a right dimetrical system of projections the distortion factors p and r along axes x and z are taken as equal while the factor q along axis y is assumed to be equal to $\frac{p}{2} = \frac{r}{2}$.

Therefore, we have:

$$p^2 + \frac{1}{4} p^2 + p^2 = 2, \text{ or } p^2 = \frac{8}{9};$$

hence,

$$p = r = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} = 0.9428 \approx 0.94$$

and

$$q = \frac{p}{2} = \frac{0.9428}{2} = 0.4714 \approx 0.47.$$

2. The determination of the angles between axonometric axes. The triangle $x'y'z'$ given by the intersection of the plane of axonometric projection Π' with the coordinate planes is called the *trace triangle* (Fig. 376). It is easy to determine the angles between the axonometric axes with the help of the trace triangle.

In right isometric projection $p=q=r$. Therefore, $\cos\alpha^\circ=\cos\beta^\circ=\cos\gamma^\circ$ which means that when the angles are acute they themselves are equal, i. e., $\alpha^\circ=\beta^\circ=\gamma^\circ$.

The line segment $O'O$ is one of the sides of the right triangles $O'x'O$, $O'y'O$ and $O'z'O$. Therefore:

$O'O=Ox'\cdot\sin\alpha^\circ$; $O'O=Oy'\cdot\sin\beta^\circ$; $O'O=Oz'\cdot\sin\gamma^\circ$. It follows that when angles α°, β° and γ° are equal, the line segments Ox', Oy' and Oz' are also equal in length.

Since Ox', Oy' and Oz' are the sides of the other three right triangles xOz, xOy, zOy, the hypotenuses xy, xz and zy of these triangles are also equal to each other, so the trace triangle is an equilateral triangle.

Angles $x'O'z'$, $x'O'y'$ and $z'O'y'$ are also equal, since their sides O_1x', O_1y' and O_1z' coincide with the altitudes of the equilateral triangle. Each of these angles is 120°.

It follows that in isometric right projection the angles between the axes are 120°. When these axes are being constructed it is usual to make axis $O'z'$ a vertical (Fig. 379).

In dimetric right projection the distortion factors p and r are equal and we have $\cos\alpha^\circ=\cos\gamma^\circ$. This means that the angles α° and γ° are also equal (Fig. 376).

But $O'O=Ox\cdot\sin\alpha^\circ$; $O'O=Oz\cdot\sin\gamma^\circ$.

Hence $O'x'=O'z'$. With $O'x'=1$ and $O'z'=1$ we obtain $x'z'=\sqrt{2}$.

Once the distortion factors for dimetric right projections have been determined it can be established that $p=r=\frac{2\sqrt{2}}{3}$. Therefore, $O'x'=O'z'=\frac{2\sqrt{2}}{3}$.

Since segments $O'x'$ and $O'z'$ are equal, it follows that the sides

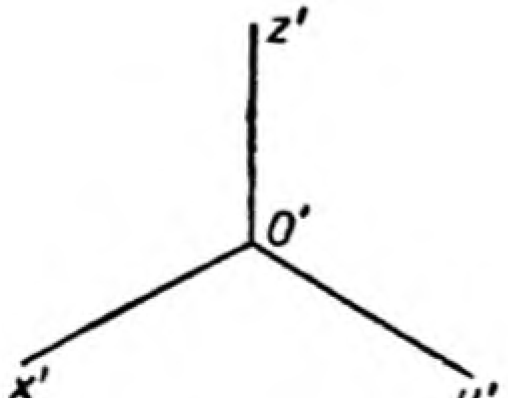


Fig. 379

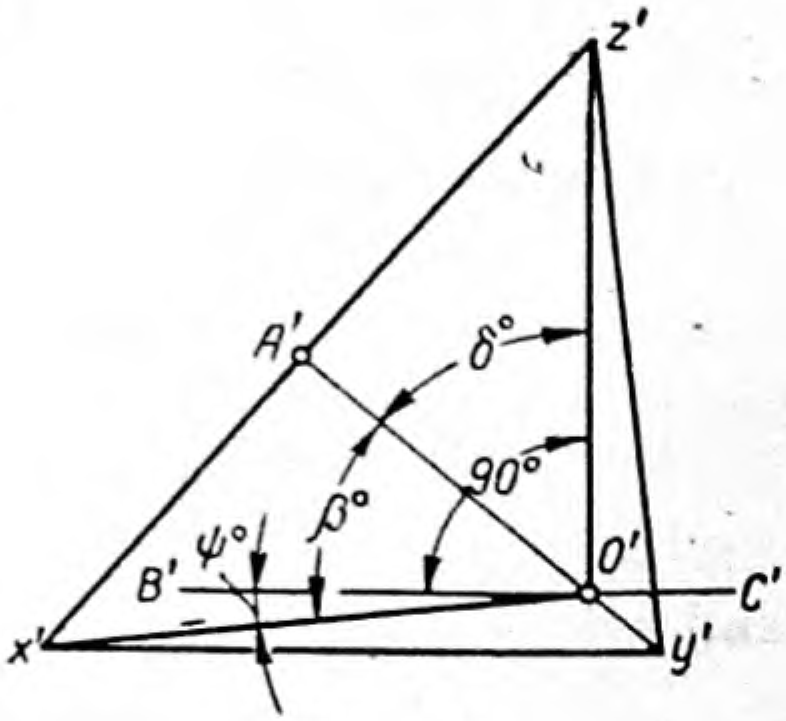


Fig. 380

$x'y'$ and $y'z'$ of the trace triangle $x'y'z'$ are also equal, i. e., this triangle is an isosceles triangle (Fig. 380).

The altitude $O'A'$ divides the side $x'z'$ of the triangle $x'y'z'$ into two equal parts, $A'x'=A'z'$; but $x'z'=\sqrt{2}$, therefore $A'x'=\frac{\sqrt{2}}{2}$ and $A'z'=\frac{\sqrt{2}}{2}$.

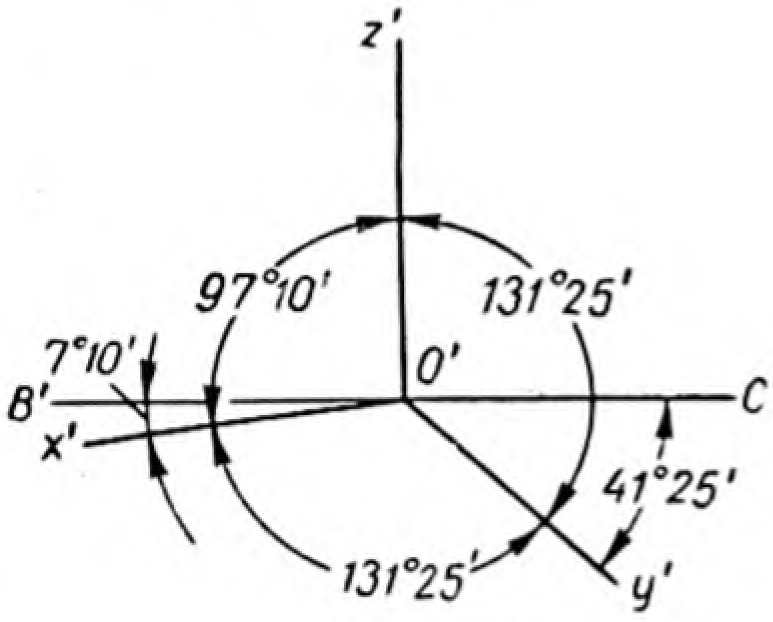


Fig. 381

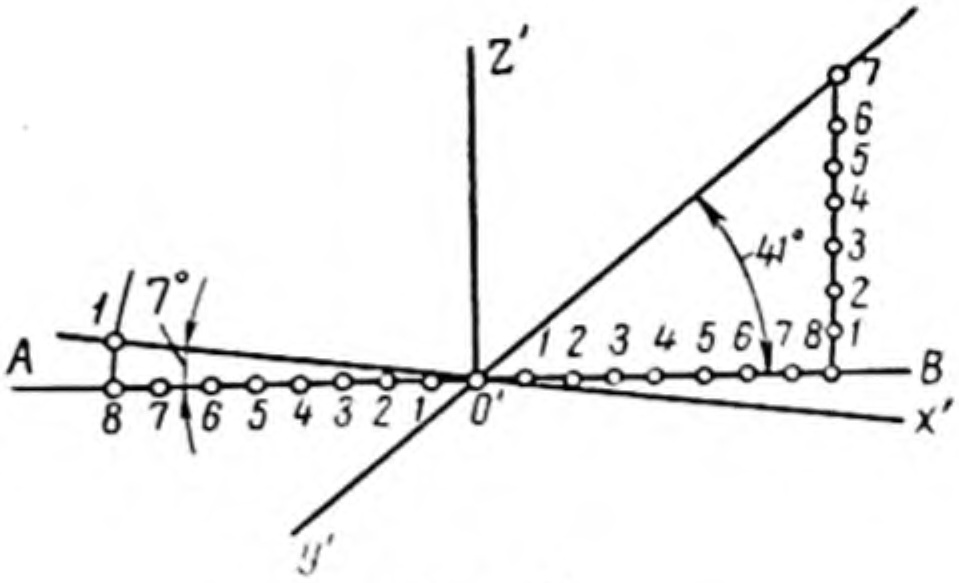


Fig. 382

Triangle $A'z'O'$ is a right triangle, therefore we have:

$$\sin\delta^\circ=A'z':O'z'=\frac{\sqrt{2}}{2}:\frac{2\sqrt{2}}{3}=0.75.$$

The angle δ° whose $\sin\delta^\circ=0.75$ is 48°35′. The angle $x'O'z'=$ $=2\,\delta^\circ=48^\circ35'\times2=97^\circ10'$.

Angle $A'O'B'$ is complementary to angle δ. But angle $A'O'B'$ is equal to angle $C'O'y'$. Since $\delta^\circ=48^\circ35'$, it follows that angle $A'O'B'=$ angle $C'O'y'=90^\circ-48^\circ35'=41^\circ25'$.

Summing up, the angle between axes $O'z'$ and $O'x'$ (Fig. 381) is 97°10′, the angle between axes $O'z'$ and $O'y'$ is 131°25′, and the angle between axes $O'x'$ and $O'y'$ is also 131°25′. Since the axis $O'z'$ is usually taken as vertical, it follows that in relation to the horizontal line $B'C'$, axis $O'x'$ makes an angle of $97^\circ10'-90^\circ=7^\circ10'$ (roughly 7°) with $B'C'$, and axis $O'y'$ makes an angle of $131^\circ25'-90^\circ=41^\circ25'$ (roughly 41°) with $B'C'$.

The construction of the dimetric axes is carried out as follows: First, the line AB is drawn (Fig. 382) and on it point O' is marked. This will be the axonometric projection of the coordinate axis. From this point axis $O'z'$ is erected perpendicular to AB. Then on the straight line AB, to each side of point O', eight equal segments of any suitable length are laid off. From the left point *8* a perpendicular to line AB is erected. On this line one of the segments is stepped off. From the right-hand point *8* a perpendicular to AB is erected and seven segments are stepped off on it. The points *1* and *7* obtained in this way are joined to point O' by straight lines. These are the axes $O'x'$ and $O'y'$ in the rectangular dimetric projection. This method is based on the construction of an angle by its tangent ($\tan 7°10' = \frac{1}{8}$; $\tan 41°25' = \frac{7}{8}$).

Eight different ways in which the axonometric semi-axes can be seen in the various octants, depending on the position of the observer's eye, are shown in Fig. 383. If the eye of the observer is in the *1* or *4* or in the *6* or *7* octants, the axis $O'x'$ will be directed to the left of point O' and the axis $O'y'$ to the right of it. When the observer's eye, i. e., the point of sight, is in one of the rest octants, *2, 3, 5* and *8*, the axis $O'x'$ falls to the right of point O' and axis $O'y'$ to the left.

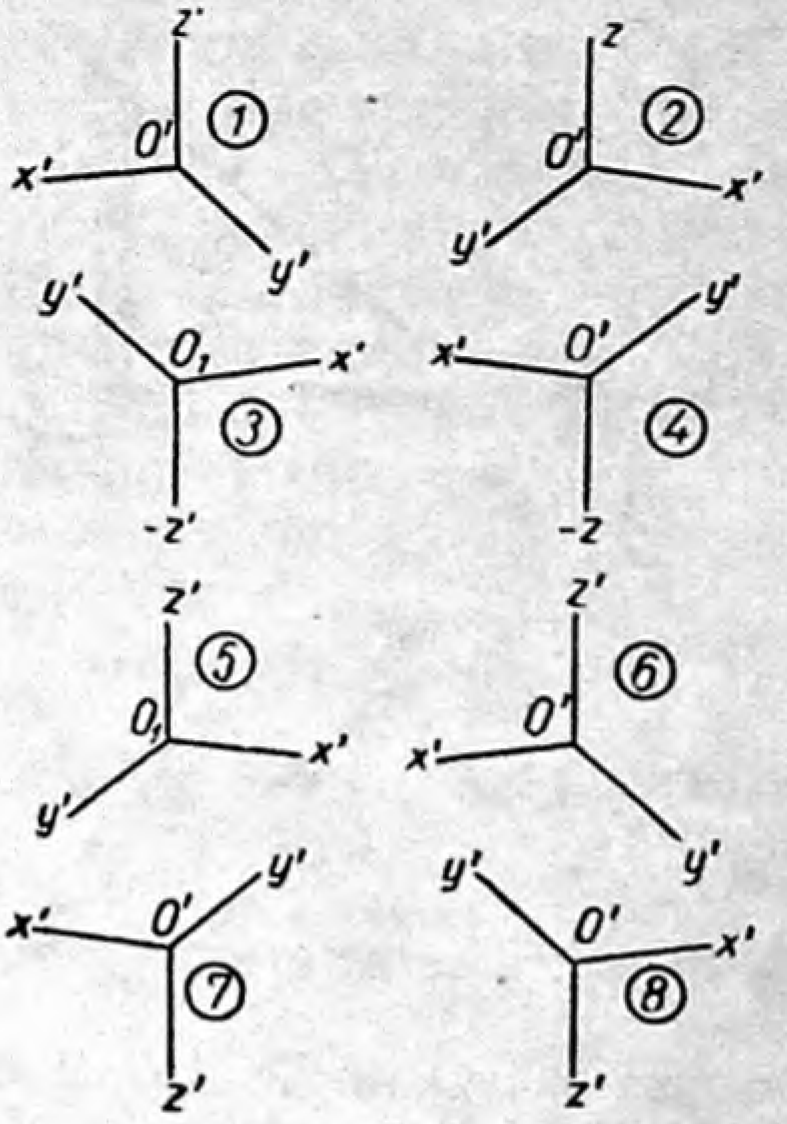


Fig. 383

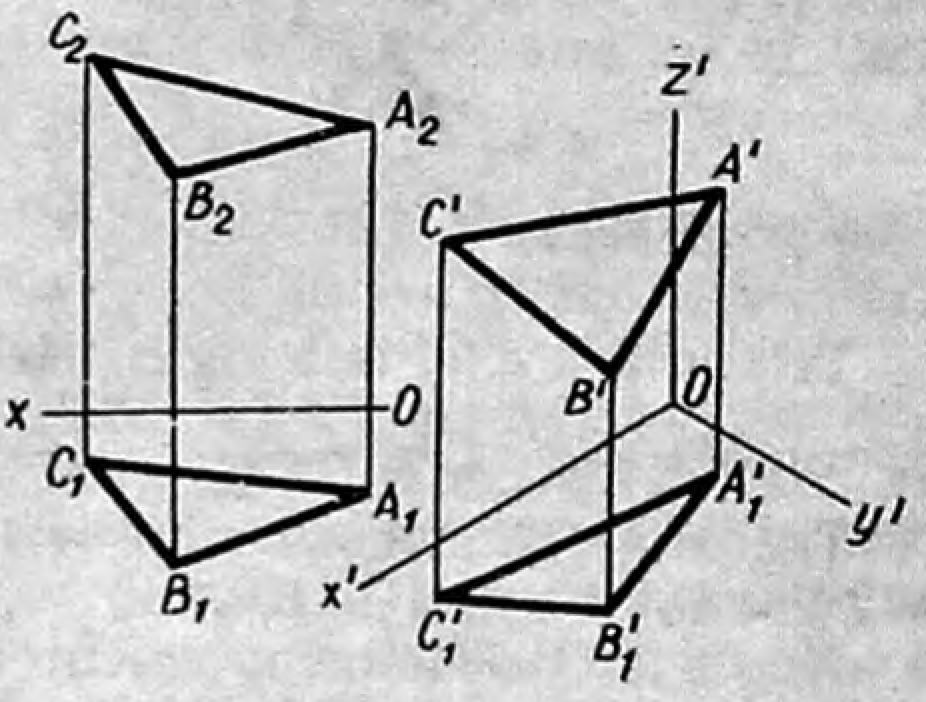


Fig. 384

In Figs. 379 and 381 the direction of the axes $O'x'$ and $O'y'$ corresponds to the case when the point of sight is in the *1* octant. In Fig. 382 the point of sight is in the *5* octant.

§ 45. The Construction of Rectangular Axonometric Projections

When constructing axonometric projections it is necessary to remember that the foreshortening or distortion dealt with in the previous paragraph concerns only those lines which lie on axes $O'x'$, $O'y'$ and $O'z'$, or lines which are parallel to these axes. The length of the axonometric projections of lines which are not parallel to the axes may be determined by constructing their extreme points by means of their coordinates.

The construction of the rectangular dimetric projection A' of point A by its given coordinates x, y, z is shown in Fig. 377. Using this technique it is always possible to construct the axonometric projection of any object from its multiview drawing, assuming that the distortion of factors for the axonometric axes are given.

The representation in rectangular isometric projection of triangle ABC, given in the multiview drawing by projections $A_1B_1C_1$ and $A_2B_2C_2$, is shown in Fig. 384. The construction is as follows: first, the axonometric coordinates x', y', z' of the vertices of the triangle are determined by multiplying the true coordinates x, y, z by 0.82. Then, using the coordinates x' and y' of the vertices A, B and C, their secondary horizontal projections A_1', B_1' and C_1' are constructed. Finally, from these points the values zA, zB, zC are laid off. The points A', B' and C' obtained are the axonometric projections of the vertices of the given triangle ABC. The triangles $A'B'C'$ and $A_1'B_1'C_1'$ are respectively its axonometric and secondary projections in the rectangular isometric projection.

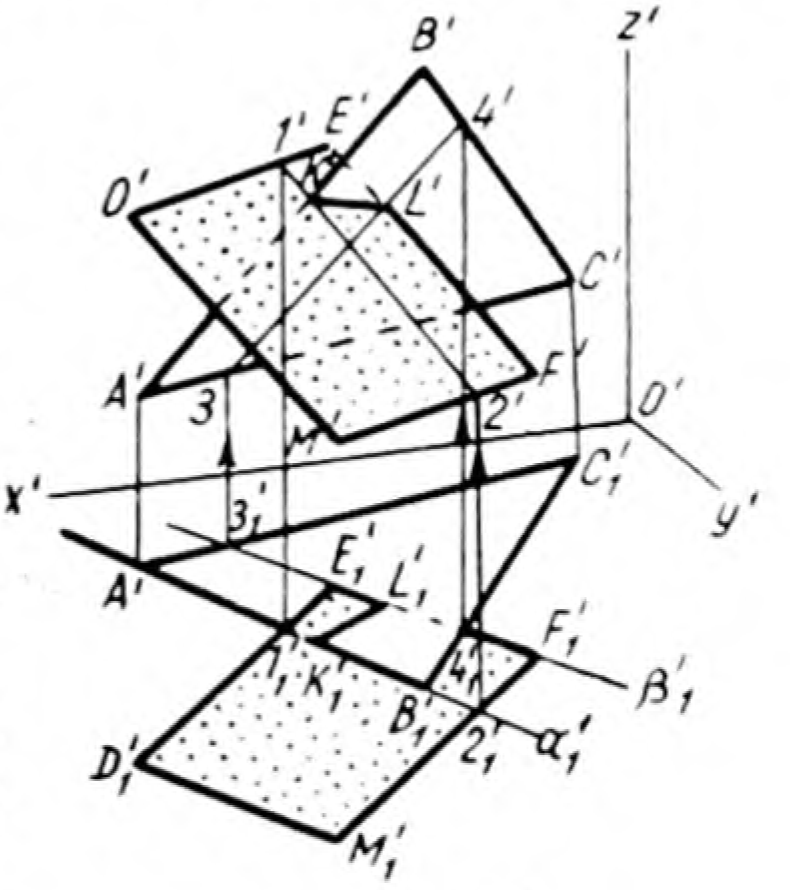


Fig. 385

The construction of the line of intersection $K'L'$ of triangle $A'B'C'$ with the parallelogram $D'E'F'M'$ in the rectangular dimetric projection is shown in Fig. 385. Just as in the multiview drawing the construction is carried out with the help of auxiliary section planes. The point of intersection K' of the side $A'B'$ of the triangle with the plane of the parallelogram is constructed by using the horizontal-projecting plane α, drawn through side $A'B'$. Trace α_1' coincides with the secondary projection $A_1'B_1'$. Point K' lies at the intersection of line $A'B'$ and line $1'2'$ obtained as the result of intersection of plane α with the plane of the parallelogram.

Point L' of line $K'L'$ is obtained with the help of a similar horizontal-projecting plane β passing through one side $E'F'$ of the parallelogram. Point L' is given by the intersection of the section line $3'4'$ and line $E'F'$.

The visibility of portions of the intersecting figures in the axonometric representation is established, as in the multiview drawing, by comparing the coordinates of separate points (see § 23, Fig. 193).

Let us consider some examples of the construction of axonometric projections of various geometrical figures and details such as encountered in engineering drawing.

1. The construction of the isometric projection of a cube (Fig. 386) is started by drawing the axis $O'z'$. At an angle of 120°

to $O'z'$, or at an angle of 30° to the horizontal, the axes $O'x'$ and $O'y'$ are then drawn, this is stage *a* seen in Fig. 386.

Then we construct the projection $A'B'C'D'$ of the top face of the cube $ABCD$ (stage *b*). Along axis $O'x'$ from point O' we lay off $O'B'=A'B'$ equal to 0.82 of the length of the edge AB. Along axis $O'y'$ from point O' we lay off $O'D'=A'D'=A'B'$.

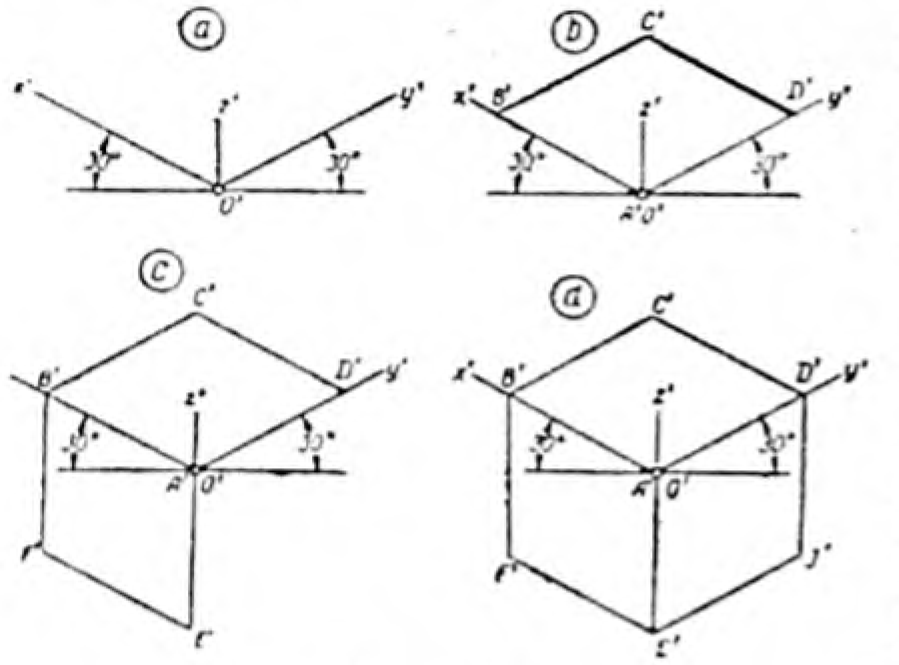


Fig. 386

Through the points B' and D' obtained we draw lines parallel to axes $O'y'$ and $O'x'$. These lines intersect in point C'. The rhomb $A'B'C'D'$ is the isometric projection of the top face $ABCD$ of the cube.

Let us construct the projection $A'B'F'E'$ of the face $ABFE$ lying in the plane $z'O'x'$ (stage *c*). For this purpose we lay off downwards from point A', along axis $O'z'$ the foreshortened length of the edge of the cube. The factor is 0.82. From point B' parallel to $O'z'$ we lay off $B'F'=A'E'$. We then join point E' with points A' and F', and point F' with point B'. The rhomb $A'B'F'E'$ is the isometric projection of the lateral face $ABFE$ of the cube.

In a similar manner we construct the projection $A'D'J'E'$ of the second lateral face $ADJE$ of the cube (stage *d*).

The hidden edge of the cube is not shown. In axonometric representations hidden outlines are generally excluded in order not to make the drawing too complicated. The faces of the cube in its isometric projection appear as rhombs, which are all the same in form and size. The outline of the complete projection of the cube is a regular hexagon. The axonometric axes and factors of distortion along these axes may be shown apart from the axonometric projection as, for example, in Figs. 388 and 389.

2. In order to avoid the large number of arithmetical computations involved in measuring the length of the various segments of straight lines and multiplying these numbers by the distortion factor (in this case 0.82), an angular scale may be used (Fig. 387). Such a scale is based on the proportional division of lines of different lengths, which form an angle with each other.

To construct the required scale a horizontal line 100 mm long is drawn. Another line 82 mm long is drawn to form any convenient angle with the first line. The ratio of the length of these lines is 82 : 100 = 0.82. Having divided the horizontal line into 10 equal segments or parts and denoting the points of division as 10, 20, etc., we join the extreme points 100 and 82 by a straight line. We then pass a series of lines parallel to line 100-82 through the points of division. These parallels divide the second line into 10 equal parts. The length of each segment received will be 00.82 of the length of each segment of the horizontal line.

In order to determine the length of the axonometric projection of any line a segment of the same length is first laid off from point F on the horizontal line. Then through the point K obtained a line KL is drawn parallel to lines of the scale. The segment FL gives the required length of the isometric projection of the original line. Angular scales may be constructed to give any degree of foreshortening, including, of course the dimetric factors of distortion 0.94 and 0.47.

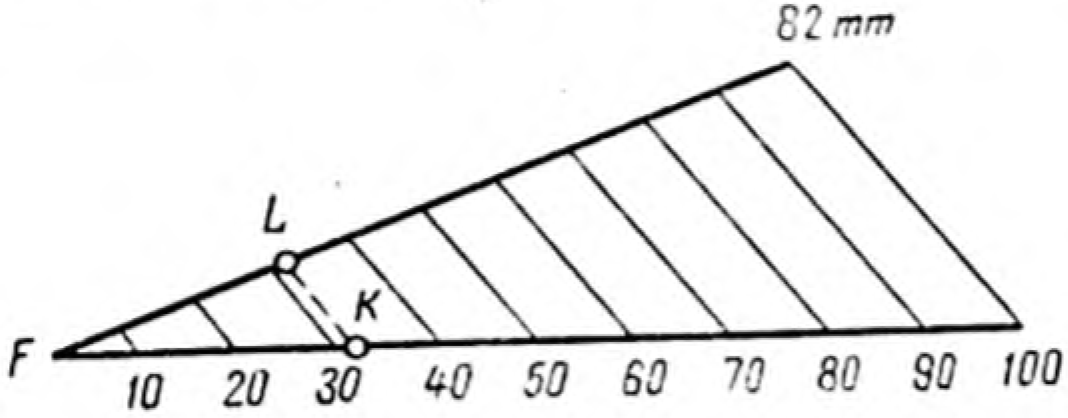


Fig. 387

3. The dimetric projections of a cube are built in the same sequence as isometric projections. In Fig. 388, *a* a cube in the dimetric projection is shown. The factor of distortion or foreshortening along axis y' is 0.47.

Distortion factors are often rounded off by using a factor equal to unity instead of 0.94 along axes $O'x'$ and $O'z'$ and along axis $O'y'$ 0.5 instead of 0.47.

When rounded off distortion factors are used the representation is called a *reduced pictorial* representation (Fig. 388, *b*). If the lengths of the edges of the cube, seen in Fig. 388, *a*, are compared with those of the edges of the cube in Fig. 388, *b*, we find that the dimetric representation of the cube seen in Fig. 388, *b* is drawn to a larger scale, i. e., 1 : 0.94, or 1.06 : 1.

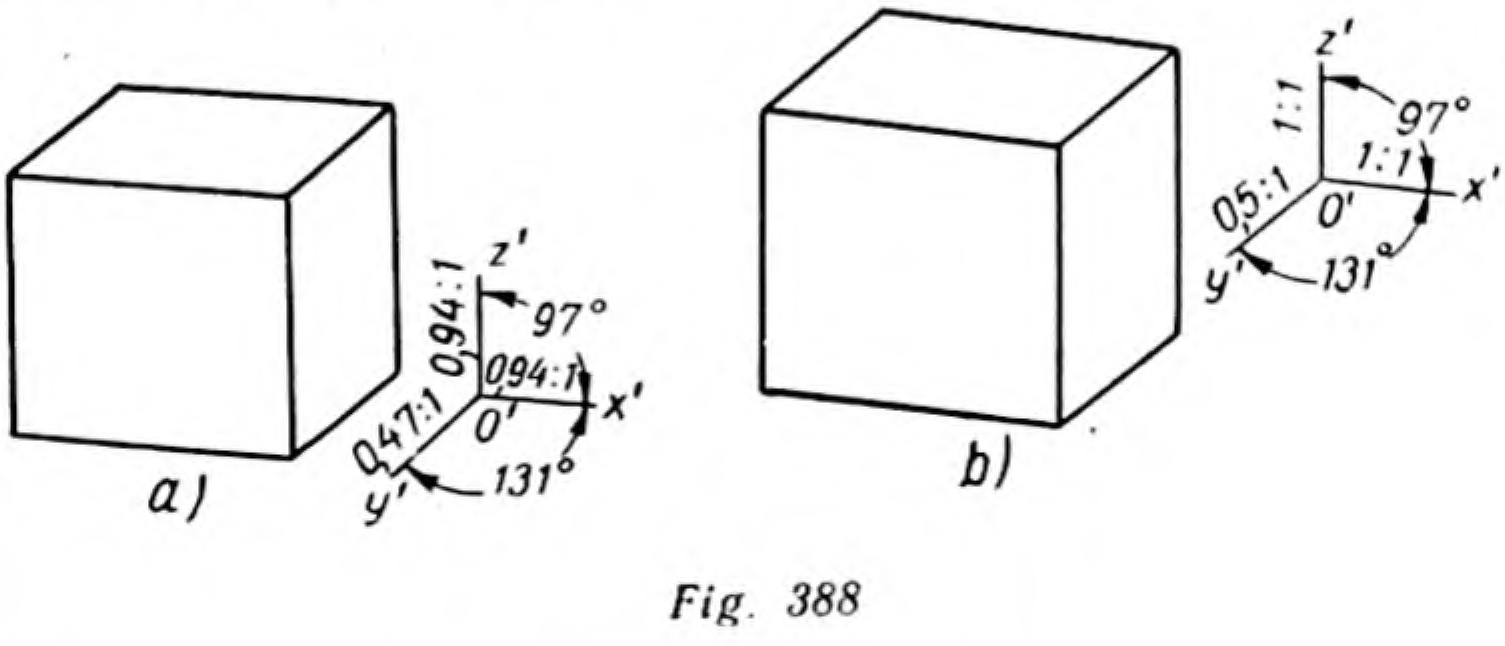


Fig. 388

In the dimetric projection of a cube two of its faces, the front and rear, parallel to the coordinate plane *xOz* are rhombs with angles 83° and 97°. All other faces are parallelograms with angles of 49° and 131°.

4. When the face of an object shown in axonometric projection is an irregular polygon, it is advisable to inscribe this polygon within a rectangle, the sides of which are parallel to the coordinate axes. This method is used when constructing the representations of a prism in isometric and dimetric projection (Fig. 389). After constructing the projection $A'B'C'F'$ of the rectangle $ABCF$ we find the axonometric projections E' and D' of points E and D by means of their coordinates x and y. Then from points A', B', C', D' and E' we drop straight lines parallel to axis $O'z'$. On these lines we lay off the length of the lateral edges taking into consideration the distortion factor along axis $O'z'$. The points thus obtained are joined by straight lines.

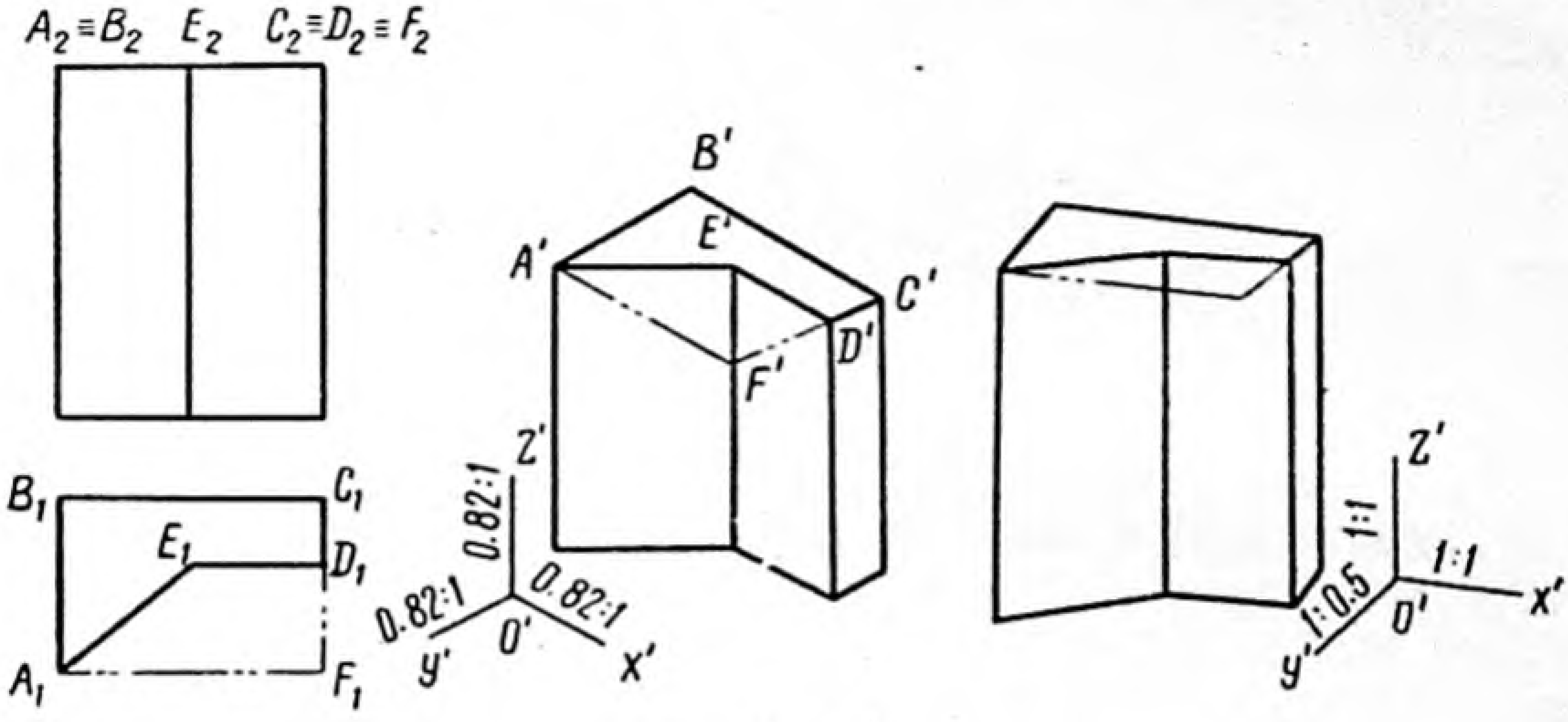


Fig. 389.

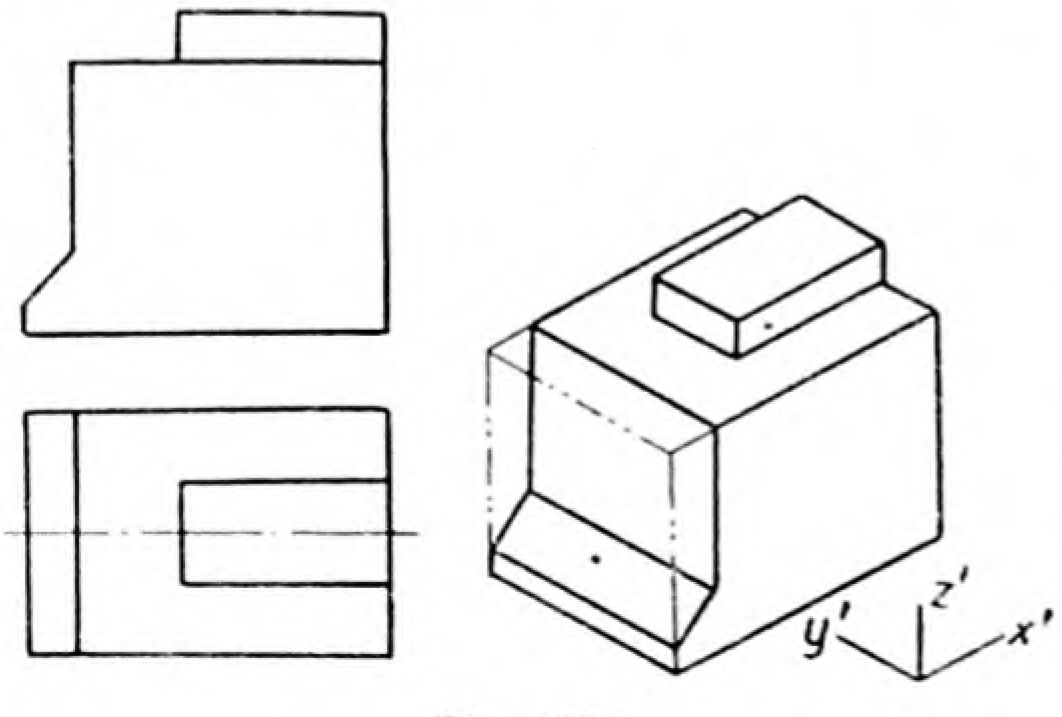


Fig. 390

5. The isometric projection of the object shown in Fig. 390 is constructed in the following sequence. First, a parallelepiped is drawn, then its left part is eliminated, and finally the top shoulder is drawn in.

6. The axonometric projection of a polyhedron, such as the irregular pyramid, shown in Fig. 391, should be built by using the coordinates of its vertices. The construction is simplified due to the fact that the vertices *A*, *B* and *C* of the triangular base of the pyramid and the foot, or lower point *D*, of the line of altitude, are all contained in the coordinate plane xOy. Having constructed the isometric axes $O'x'$, $O'y'$ and $O'z'$, we then find the axonometric

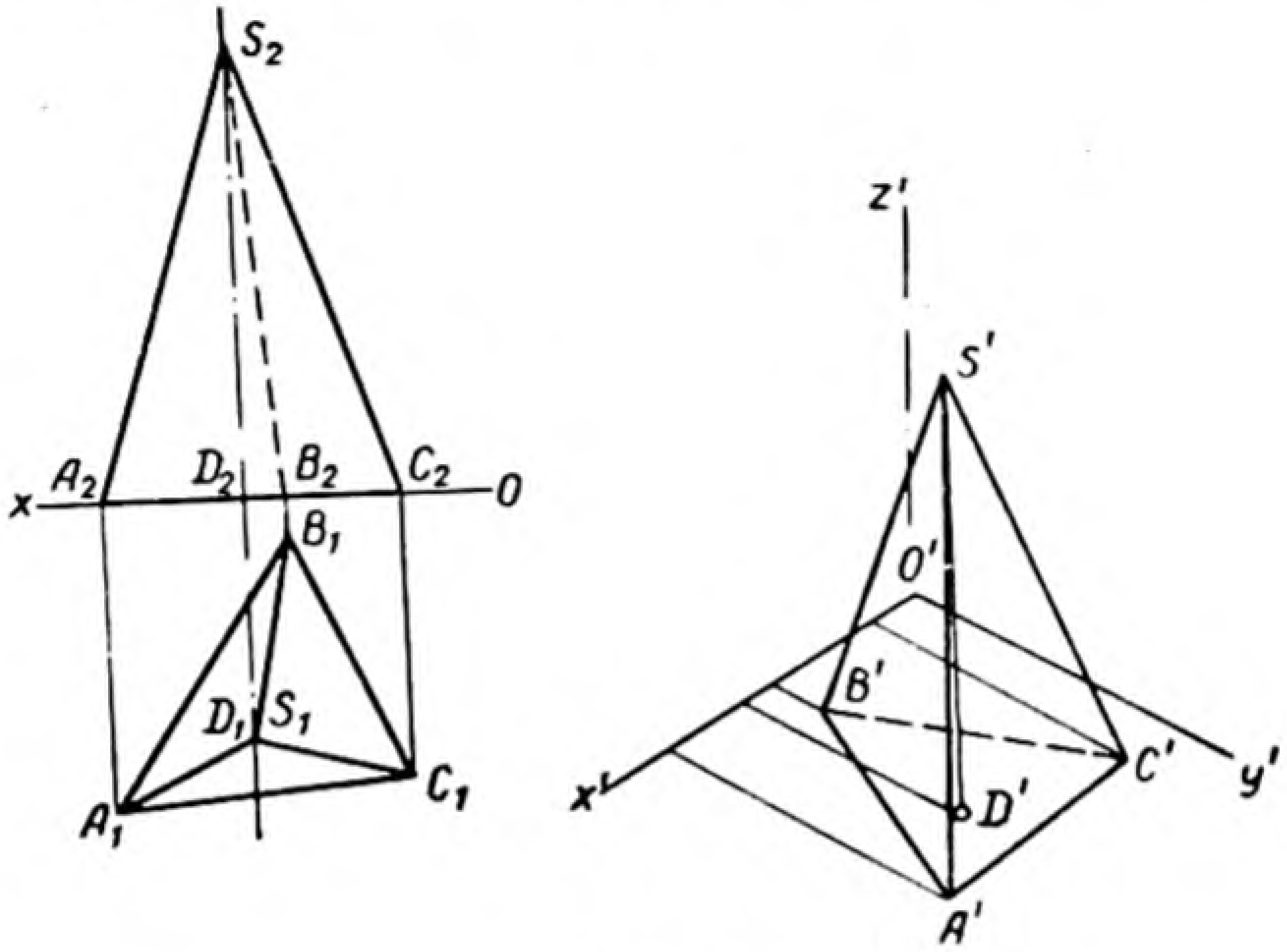


Fig. 391

projections A', B' and C' of the points of the triangular base. For this purpose, using the multiview drawing, we determine the coordinates of the points *A*, *B* and *C*. Then, using the angular scale (Fig. 387), and by means of the values of *x* for points *A*, *B* and *C*, we find the axonometric values x' of these points. These segments are then marked off from point O' along axis $O'x'$. From the points obtained on axis $O'x'$ we lay off on lines parallel to axis $O'y'$ the values y' for the vertices. The latter are determined by measuring the values *y* for points *A*, *B* and *C* in the multiview drawing and reducing them by means of the corresponding angular scale. Points A', B' and C' are joined by straight lines. The triangle $A'B'C'$ is the isometric projection of the base of the pyramid *ABC*.

Point D', the isometric projection of the foot of the line of altitude *SD*, is constructed in a similar way. Through point D' upwards we draw a vertical and lay off on it the foreshortened altitude of the pyramid (the factor is 0.82). This gives us point S', i. e., the isometric projection of the pyramid vertex *S*. We finish the construction of the isometric representation of the pyramid by joining S' with points A', B' and C' by straight lines.

7. When the outline of the lower part of the object is more complicated than its upper part, it is advisable to begin the axonometric by choosing a point of sight so that the object is viewed from below. This corresponds to the case when the eye of the observer is in one of the octants *3, 4, 7* or *8*. In Fig. 392 an example of such a construction in dimetric projection is shown. The direction of the axes x', y' and z' indicate that the observer's eye is in the *8* octant.

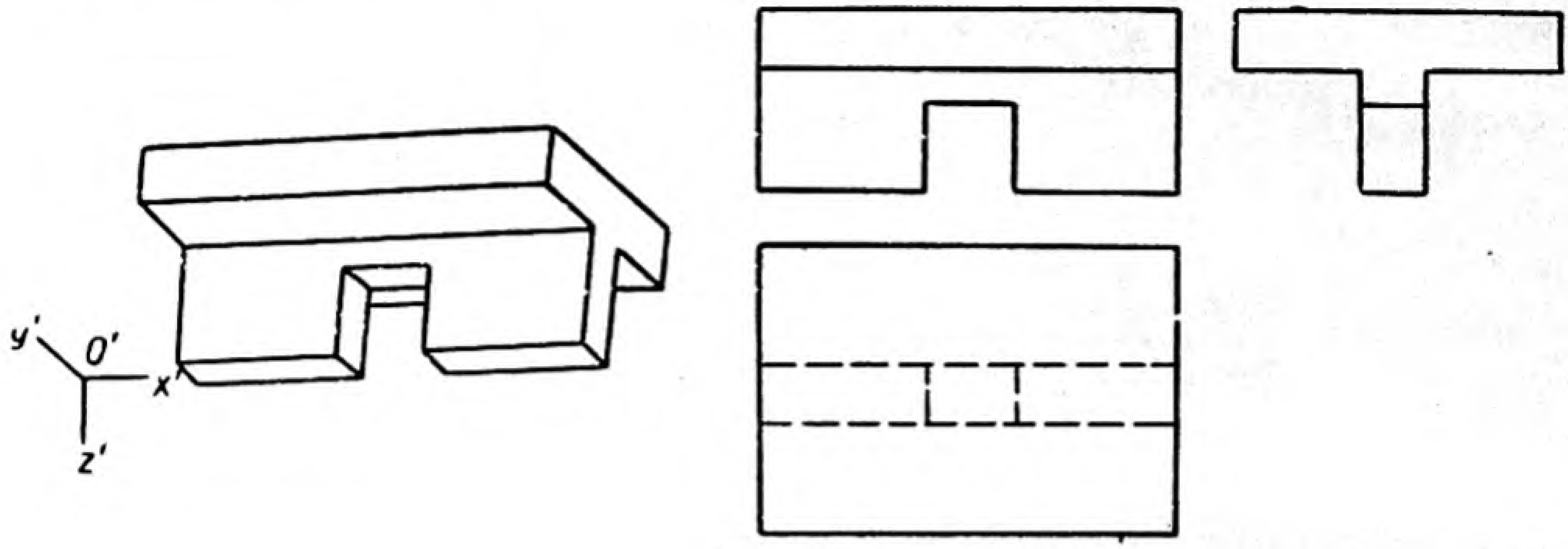


Fig. 392

8. It should be noted that the axonometric projection of an object may be also constructed directly in the multiview drawing by rotating the object successively about two axes or by projecting it onto new planes of projection.

The construction of the isometric and the dimetric projections of a cube is shown in Figs. 393 and 394. The isometric projection of the cube was obtained by substituting the planes of projection Π_1 and Π_2 by new planes Π_4 and Π_5. The dimetric projection of a cube was obtained by successive rotating of the cube about two axes. The first axis coincides with one of the lateral edges of the cube and the second passes through the left vertex of the base and is perpendicular to plane Π_2.

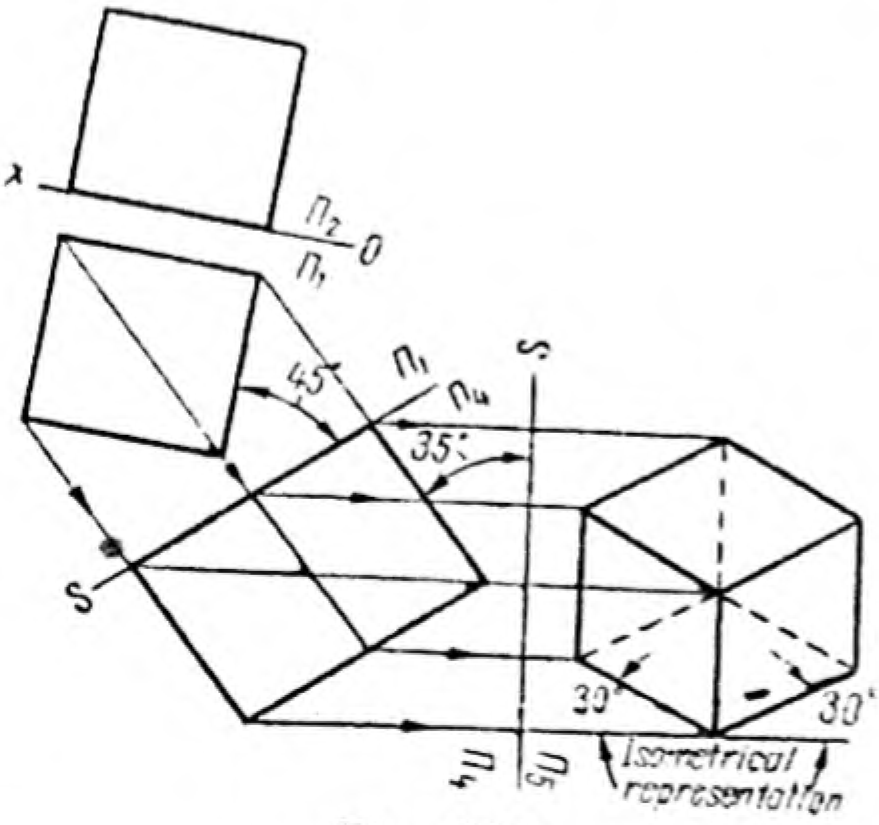


Fig. 393

9. The circles inscribed into the cube faces in isometric projection become ellipses inscribed in rhombs which are themselves the isometric projections of the faces of the cube (Fig. 395).

The centres of these ellipses lie at the intersection of the diagonals of the rhombs. The major diagonals $B'D'$, $B'E'$ and $E'D'$ of the rhombs are respectively perpendicular to axes $O'z'$, $O'x'$ and $O'y'$ and are equal to the true lengths of the diagonals of the cube (see Fig. 385). It follows that the major axes of the ellipses $1'$-$2'$, $5'$-$6'$ and $9'$-$10'$, which coincide with these diagonals, are of the same length as the diameters of the circles inscribed in the faces of the cube.

The minor axes $3'$-$4'$, $7'$-$8'$ and $11'$-$12'$ of the ellipses coincide with the minor diagonals of the rhombs and are perpendicular to

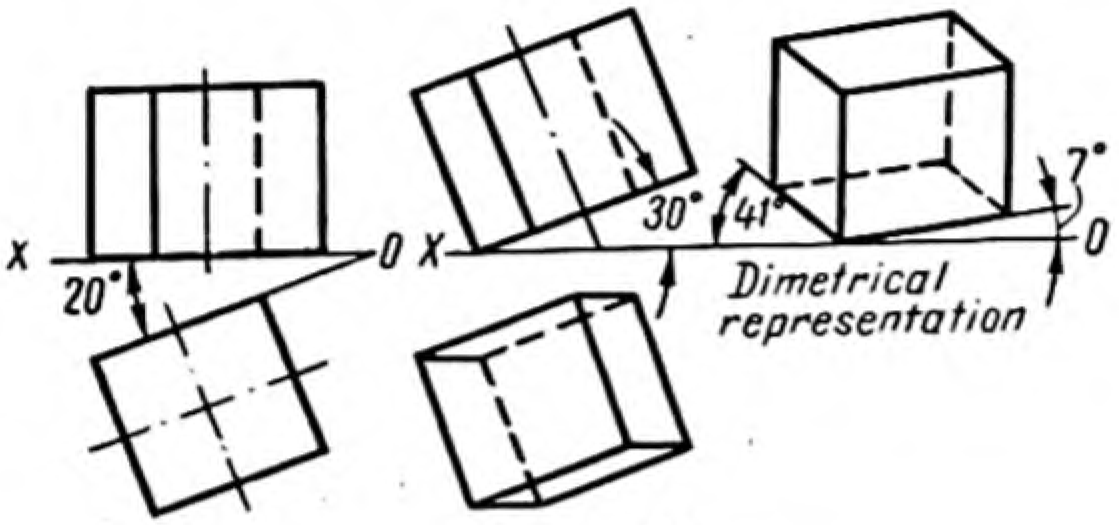


Fig. 394

the major axes of the ellipses. The minor axes of the ellipses are the isometric projections of lines, each of which is perpendicular to the two edges of the cube, which it intersects. In the isometric projection each axis lies on the same straight line as do these two edges. So the direction of the minor axis of any of the ellipses is parallel to the axonometric axis, that is perpendicular to the plane in which the given circle lies.

The minor axes of the ellipses are equal in length to 0.57 of the diameters of the circles inscribed in the faces of the cube. This may be readily seen if we consider the right triangle $S'B'C'$ (Fig. 395), where $C'S' = B'S' \cdot \tan 30°$. It follows that $2C'S' = 2B'S' \cdot \tan 30°$, or $A'C' = B'D' \cdot \tan\ 30°$ and $A'C' = 0.57B'D'$. Since axis $3'$-$4'$ is perpendicular to $B'D'$, then the same ratio 0.57 exists between the minor axis of the ellipse and diameter of the circle inscribed in the face of the cube.

It may be seen from Fig. 395 that in each of the three ellipses, besides four points which are the extreme points of the axes of the ellipse, there are also four points where the sides of the rhomb are tangential to the ellipse. These points K', L', M', N', P', Q', T', W' and R' are located on lines parallel to the isometric axes and are projections of the points of contact

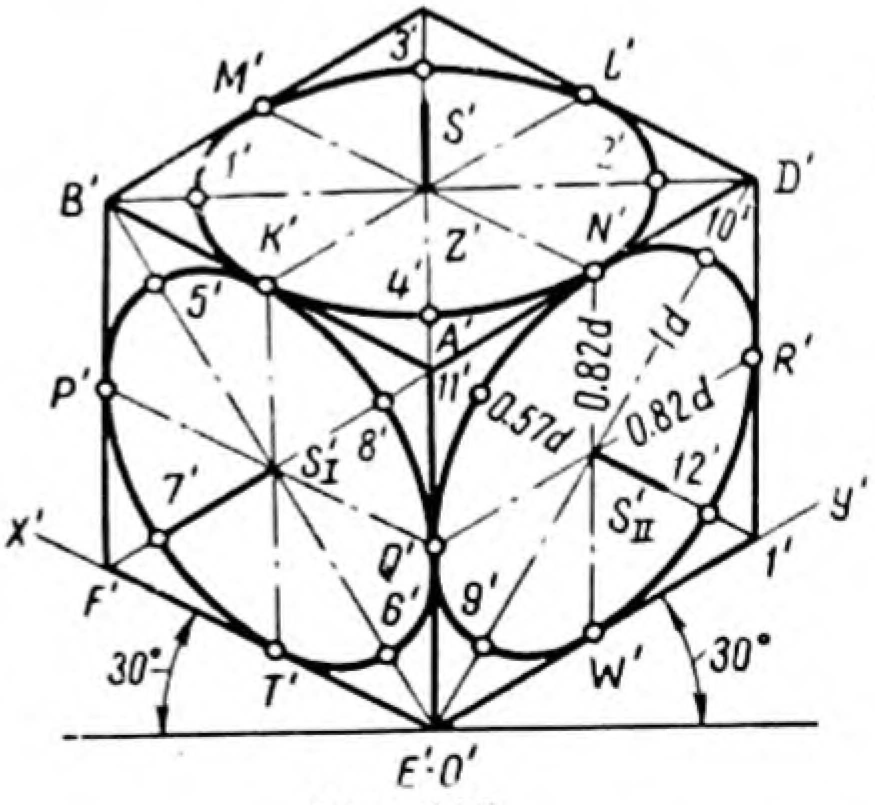


Fig. 395

of the edges of the cube which are tangential to the circles inscribed in the various faces.

10. The construction of the ellipse having its centre in point S', which is the isometric projection of the circle inscribed in the top face of the cube, is shown in Fig. 396. First a line $S'A'$ is drawn parallel to the axis $O'z'$. This line determines the direction of the minor axis of the ellipse *1'-2'*. The length of this minor axis is equal to 0.57 d, where d is the diameter of the circle. The major axis *3'-4'* of the ellipse is perpendicular to the axis *1'-2'* and the length of this major axis is equal to d. Further, through centre S' of the ellipse lines *5'-6'* and *7'-8'* are drawn respectively parallel to axes $O'x'$ and $O'y'$. The length of each of these lines is equal to 0.82 d. The ellipse is then drawn through the eight points obtained.

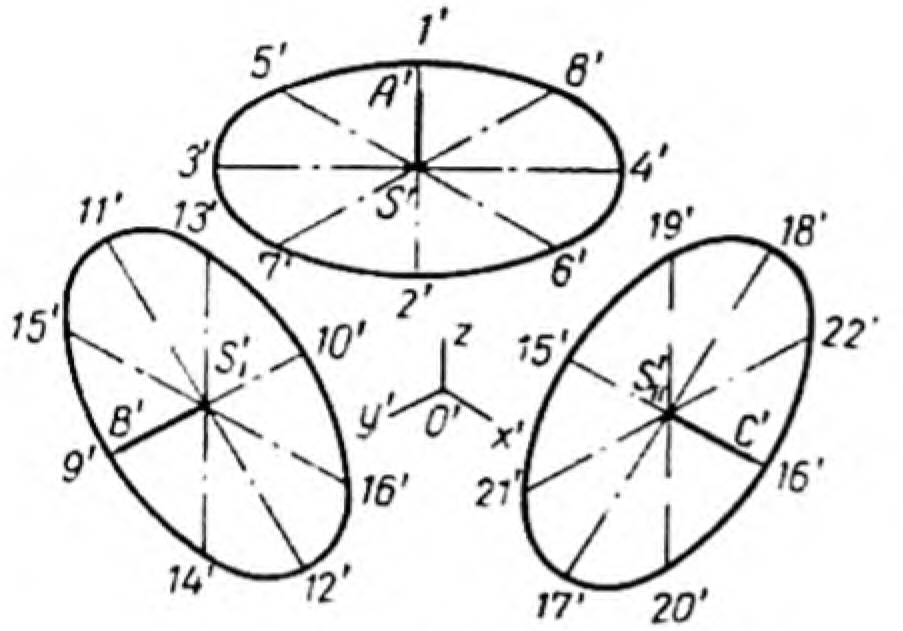


Fig. 396

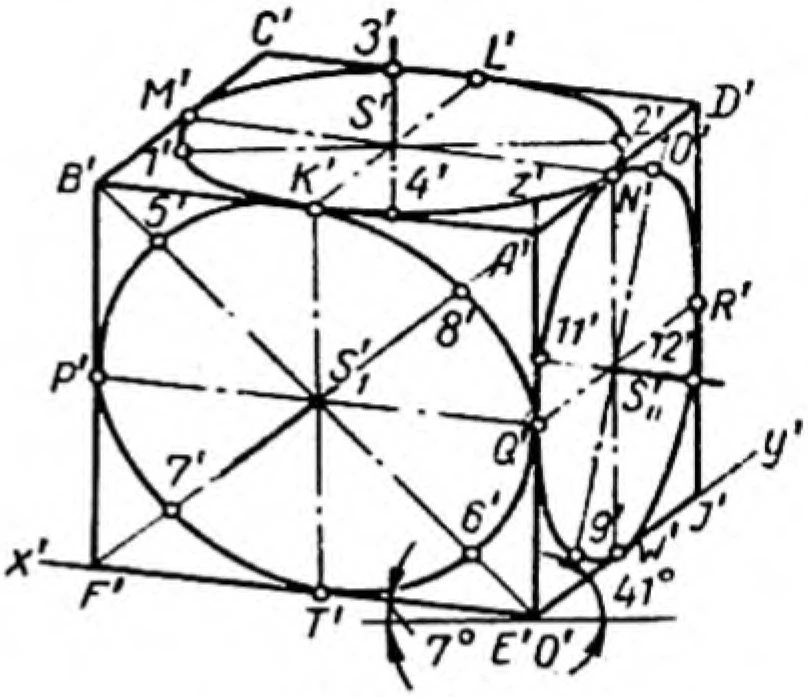


Fig. 397

The other two ellipses, with centres in points S'_I and S'_{II} which are the isometric projections of the circles in the coordinate planes zOx and zOy, are constructed in a similar way. The major axis of the left ellipse is perpendicular to axis $O'y'$ and the major axis of the right ellipse is perpendicular to axis $O'x'$.

When a circle does not lie in a coordinate plane or in a plane parallel to a coordinate plane, the ellipse may be constructed by using the coordinates of a series of points taken on this circle.

11. Isometric projections are often constructed without using the factor of distortion 0.82 to foreshorten dimensions along the axes Ox', Oy' and Oz'. In this case the lengths of the axes of the ellipses — the isometric representation of the circles lying in the coordinate planes, or in planes parallel to them, — are determined as follows, for the major axis, $1:0.82=1.22$ of the diameter of the circle and for the minor axis, $0.57:0.82=0.7$ of the diameter of the circle.

12. The dimetric projection of a cube with circles inscribed in its faces is shown in Fig. 397. The front and rear faces of the cube are seen as rhombs and the other faces as parallelograms. The minor axes of the ellipses are parallel to the respective dimetric axes, whereas the major axes are perpendicular to the minor axes.

The lengths of the major and minor axes of the ellipse depend on whether the circle lies in a top, lateral or front face, and also on the scale to which the dimetric projection is drawn. If the dimetric projection of the cube is foreshortened using the factor 0.94 in the direction of the axes $O'x'$ and $O'z'$ and 0.47 in the direction of axis $O'y'$, then the major axis of each of the ellipses is equal to the diameter of the circle itself. If the dimetric projection is constructed without foreshortening along axes $O'x'$ and $O'z'$, but is foreshortened along axis $O'y'$, using the factor 0.5, then the major axis of each of the three ellipses is equal to $1 : 0.94 = 1.06$ of the diameter. The minor axis of an ellipse, located in the coordinate plane $z'O'x'$, is equal to 0.88 of the length of the major axis. The minor axis of ellipses located in planes $x'O'y'$ and $z'O'y'$ is equal to $^1/_3$ of the length of the major axis.

The ratio of the lengths of the major and minor axes of the ellipses, $1 : 0.88$, may be easily obtained if we consider the right triangle $F'S'_1E'$, where $S'_1F' = S'_1E' \cdot \tan 41°25' = 0.88\ S'_1E'$. It follows that $A'F' = 0.88\ B'E'$. This ratio also exists between axes $7'$-$8'$ and $5'$-$6'$ of the ellipse in this coordinate plane $x'O'z'$. Therefore, if the length of the major axis $5'$-$6'$ of the ellipse is assumed equal to diameter d of the circle represented, then the length of the minor axis $7'$-$8'$ is equal to $0.88\ d$. If the length of the major axis $5'$-$6'$ is assumed as equal to $1.06\ d$, then the length of the minor axis is equal to $0.88 \times 1.06\ d = 0.93\ d$.

The relationship between $O'K$ and OK, i. e., between the axonometric projection $O'K$ of the line and the line OK itself (Fig. 398), may be considered as the relationship between the minor and major axes of the ellipse located in the coordinate plane $x'O'y'$ (Fig. 397). As can be seen from the drawing in Fig. 398, $\frac{O'K}{OK} = \sin \gamma° = \sqrt{1 - \cos^2 \gamma°}$. The value $\cos \gamma°$ is the factor of distortion r along axis $O'z'$, but since $r = \frac{2\sqrt{2}}{3}$, it follows that $\frac{O'K}{OK} = \sqrt{1 - \frac{8}{9}} = \frac{1}{3}$.

Besides the extreme points of the axes for each ellipse in the dimetric projection of a cube (Fig. 397), we have another four points: K', L', M' and N' for the upper ellipse, N', W', Q' and R' for the lateral ellipse, and K', T', P' and Q' for the front ellipse. Each of these points is the mid-point of the corresponding side of the rhomb or parallelogram and represents the dimetric projection of the point in which the circle touches the edges of the cube.

13. The construction of the ellipses, which are the dimetric projections of circles lying in the coordinate planes *zOx*, *xOy* and *zOy*, is shown in Fig. 399. To construct the top ellipse we take a point S' as its centre and then draw a line $S'A'$ parallel to axis $O'z'$. Assuming this as the line of the minor axis we draw, at right angle to it, the line of the major axis of the ellipse. The length of the major axis is taken as equal to 1.06 of the diameter of the circle, the scale being 1.06 : 1. By laying off along the major axis, to both sides of point S', 0.53 of the diameter of the circle we obtain points *3'* and *4'*. Next, by laying off from point S' upwards and downwards along axis $O'z'$ $^{1}/_{6}$ of the length of the major axis we obtain the extreme points of the minor axis of the ellipse, i. e., points *1'* and *2'*. Then, through that same point S', we draw lines respectively parallel to axes $O'x'$ and $O'y'$. On the first line, at a distance from point S' equal to the radius of the circle, we obtain points *5'* and *6'*. On the second line at a distance from point S' equal to one half of the radius of the circle we obtain points *7'*, *8'*. The upper ellipse is drawn through the eight points found in this way.

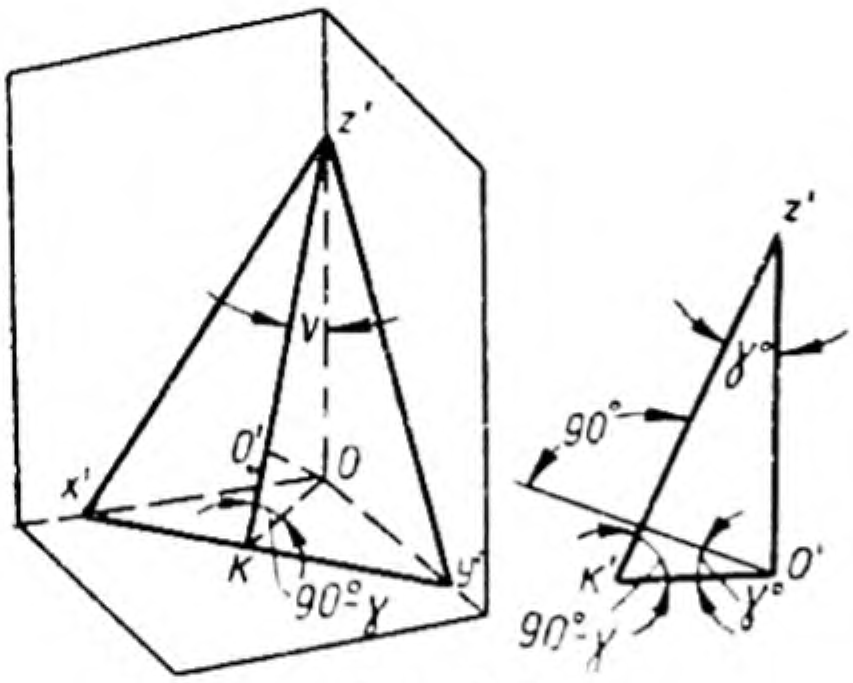


Fig. 398

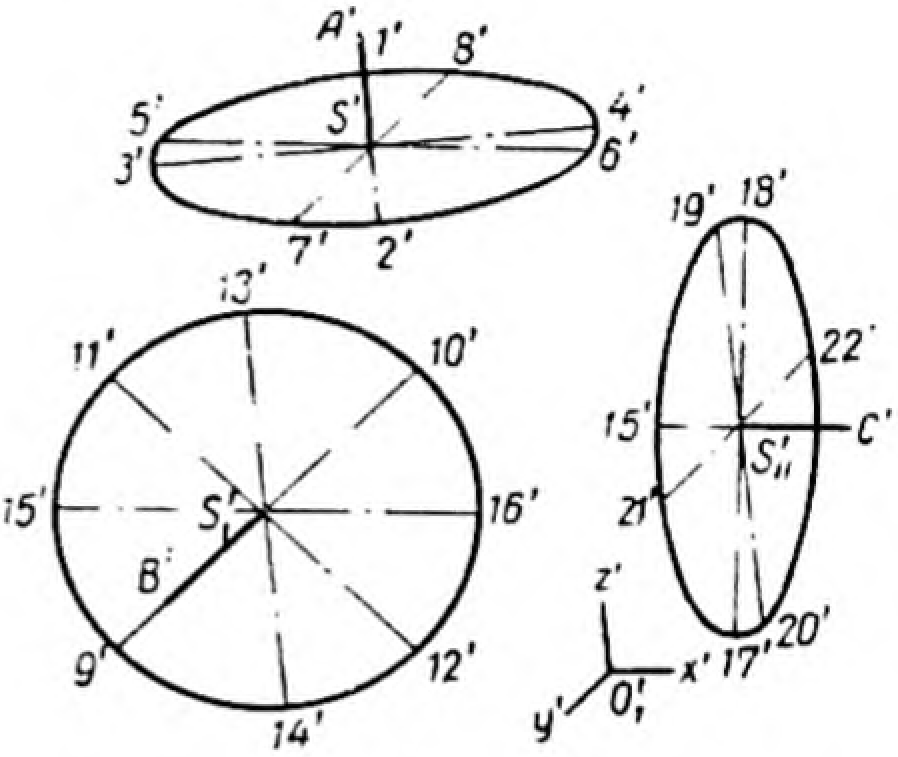


Fig. 399

The ellipse with its centre in point S'_{II} is constructed in a similar manner.

When constructing the ellipse with its centre S'_I on the line $S'B'$, which determines the direction of the minor axis, we lay off not $^{1}/_{3}$, but 0.88 of the length of the major axis of the ellipse, and both *13'-14'* and *15'-16'* are taken as being equal to the diameter of the circle.

14. Now let us consider the construction of the isometric and dimetric projections of a cylinder, a cone and a sphere.

After constructing the isometric projection of the top of the cylinder (Fig. 400), from the extreme points of the major axis of the ellipse we draw two vertical lines and lay off on them downwards segments equal to 0.82 of the altitude of the cylinder. After this we draw one half of the lower ellipse which is the isometric projection of the base of the cylinder. To construct the half of the lower ellipse on the front part of the top ellipse we take a series of points and drop verticals from them. On these verticals we mark off 0.82 of the height of the cylinder. The half of the lower ellipse is obtained by drawing a curve through the points obtained.

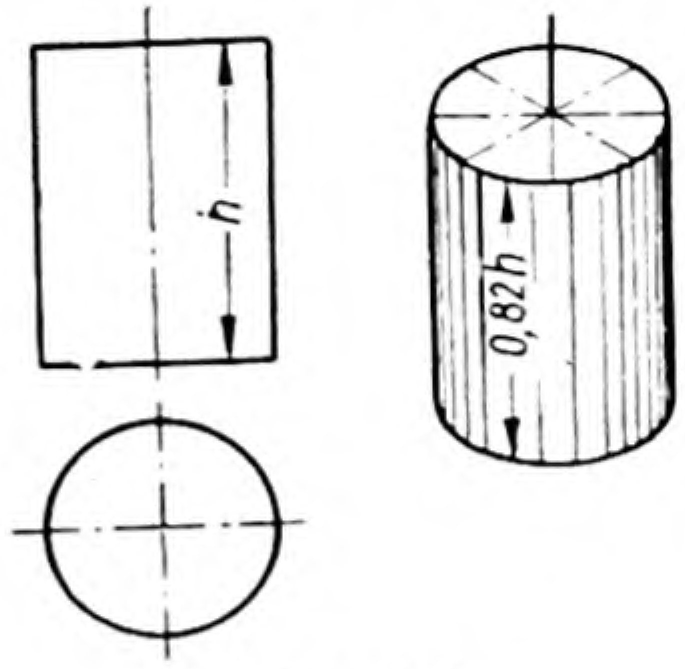


Fig. 400

The isometric projection of a cylinder with its axis in a horizontal position (Fig. 401) is begun by constructing the right-hand end. In this case the ellipse is drawn with the help of chords of the circle. One of the diameters parallel to one of the coordinate axes, say *Oy*, is divided into a number of parts and through each of the points obtained chords perpendicular to the axis are drawn. Having constructed the axonometric projection *A′B′* of this diameter *AB*, we divide it into the same number of parts and draw through the points of division a series of straight lines in the direction of the chord of the circle, in this case parallel to axis *O′z′*. On these lines, to both sides of the isometric projection of the diameter, we lay off 0.82 of the length of the corresponding semi-chords of the circle. The ellipse is then drawn through the points obtained.

The outline generatrices of the cylinder and the visible part of the left end of the cylinder should be drawn as shown in Fig. 400.

15. An example of the construction of a dimetric projection of a detail, consisting of two vertical cylinders with a through cylindrical bore, is given in Fig. 402. In the multiview drawing in accordance with the rules of working sections in engineering drawings, half of the view is shown alongside half of the vertical section.

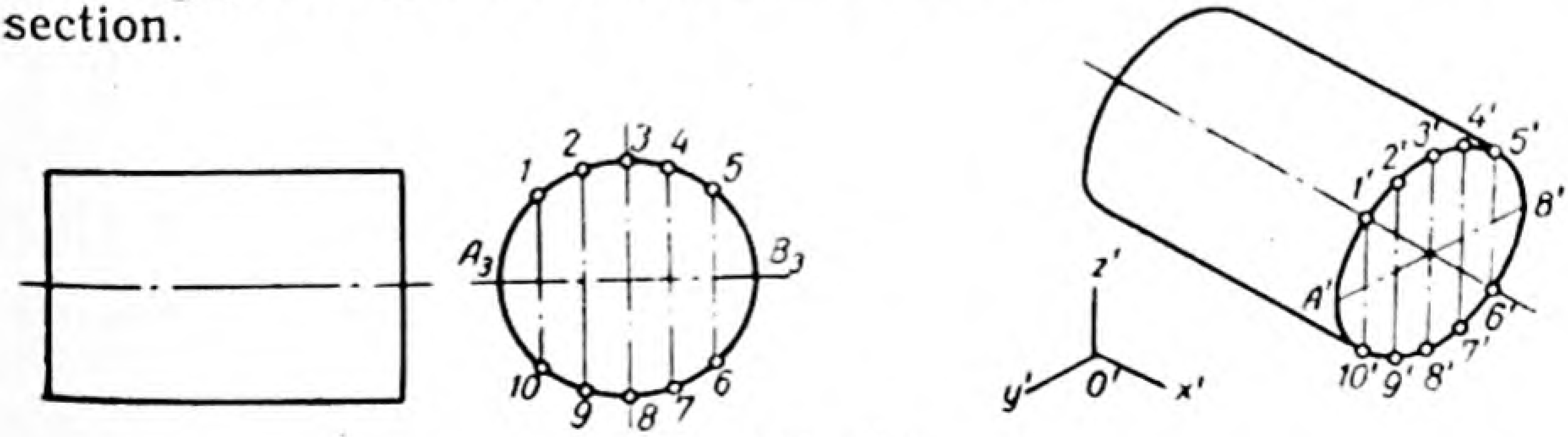
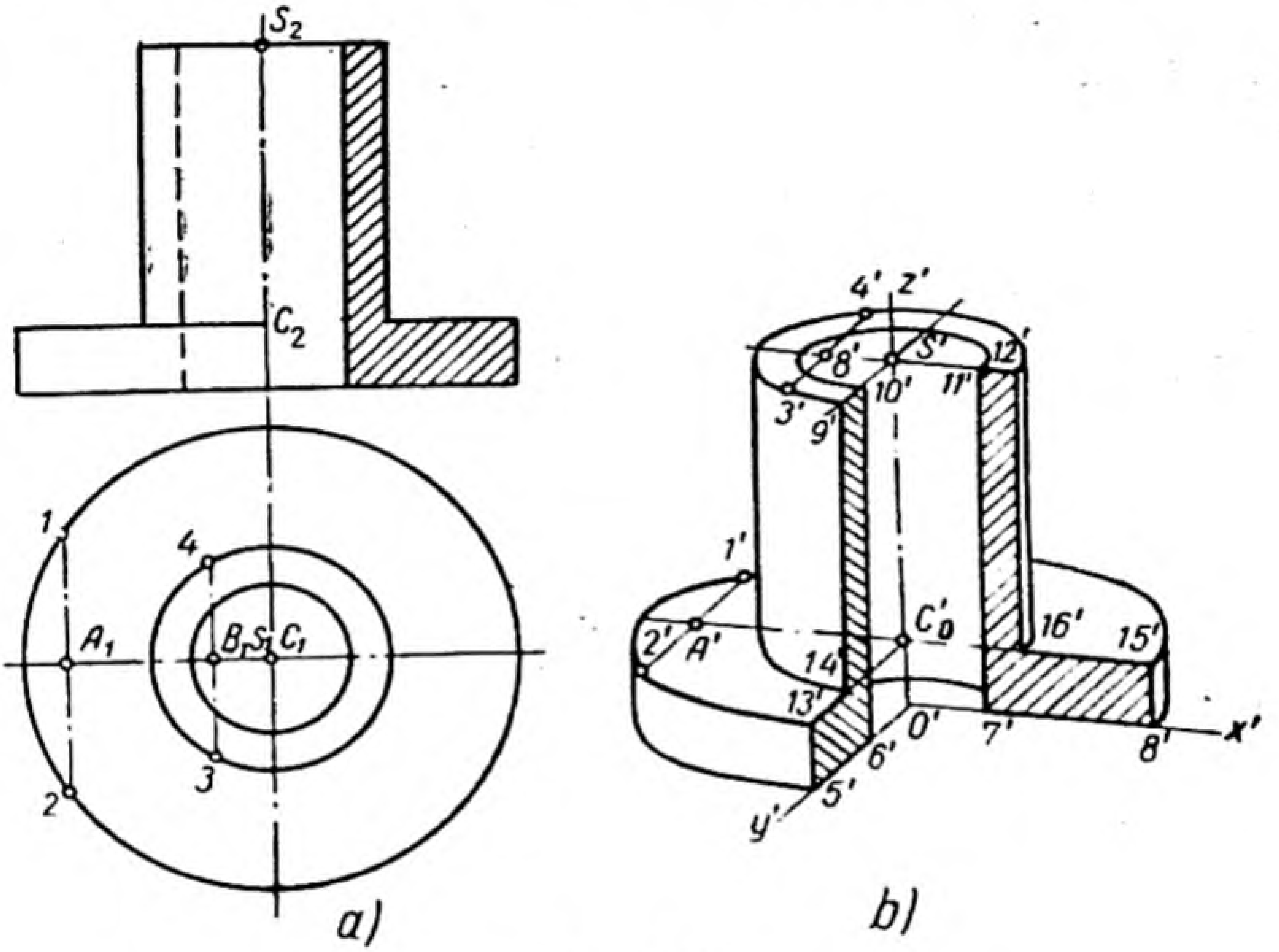


Fig. 401

In order to show the complete form of the detail in the dimetric projection part of it has been removed.

We first construct the dimetric projection of the top of the lower cylinder by means of chords. In Fig. 402 only the construction of points *1* and *2* is shown, these being the extreme points of the chord passing through point *A*. The dimetric projection *1′-2′* of this chord, parallel to axis *O′y′*, has been reduced to half the size of



a) b)

Fig. 402

the chord *1-2*. Having constructed the ellipse, which is the dimetric projection of the top of the lower cylinder, we draw in the outline generatrices of this cylinder. Next we draw in the visible part of the ellipse which is the dimetric projection of the base of cylinder.

The construction of the dimetric projection of the top cylinder is begun by constructing the upper face. To find the dimetric projection of the centre *S′* of this ellipse, from point *C′*, which is the centre of the ellipse which was constructed first, we lay off the length of the generatrix of the upper cylinder. The ellipse with its centre at point *S′* is also constructed with the help of chords. In Fig. 402 the construction of points *3′* and *4′* is shown. The ellipse representing the upper circle of the bore is constructed in the same manner. The centre of this ellipse is again point *S′*.

Having constructed the dimetric projection of the detail, we eliminate the right-hand front part cut by the coordinate planes *x′O′z′* and *y′O′z′*. Points *5′* and *8′* are given by the intersection of the lower ellipse with axes *O′y′* and *O′x′*. Points *9′*, *10′*, *11′* and *12′*

are given by the intersection of the upper ellipses with lines drawn through their common centre S' parallel to axes $O'y'$ and $O'x'$ respectively. Lines parallel to the corresponding dimetric axes should be drawn through the points obtained and through points $13'$ and $14'$, $15'$ and $16'$ which were found in the same way. This gives us the outlines of the sections cut by the coordinate planes $x'O'z'$ and $y'O'z'$.

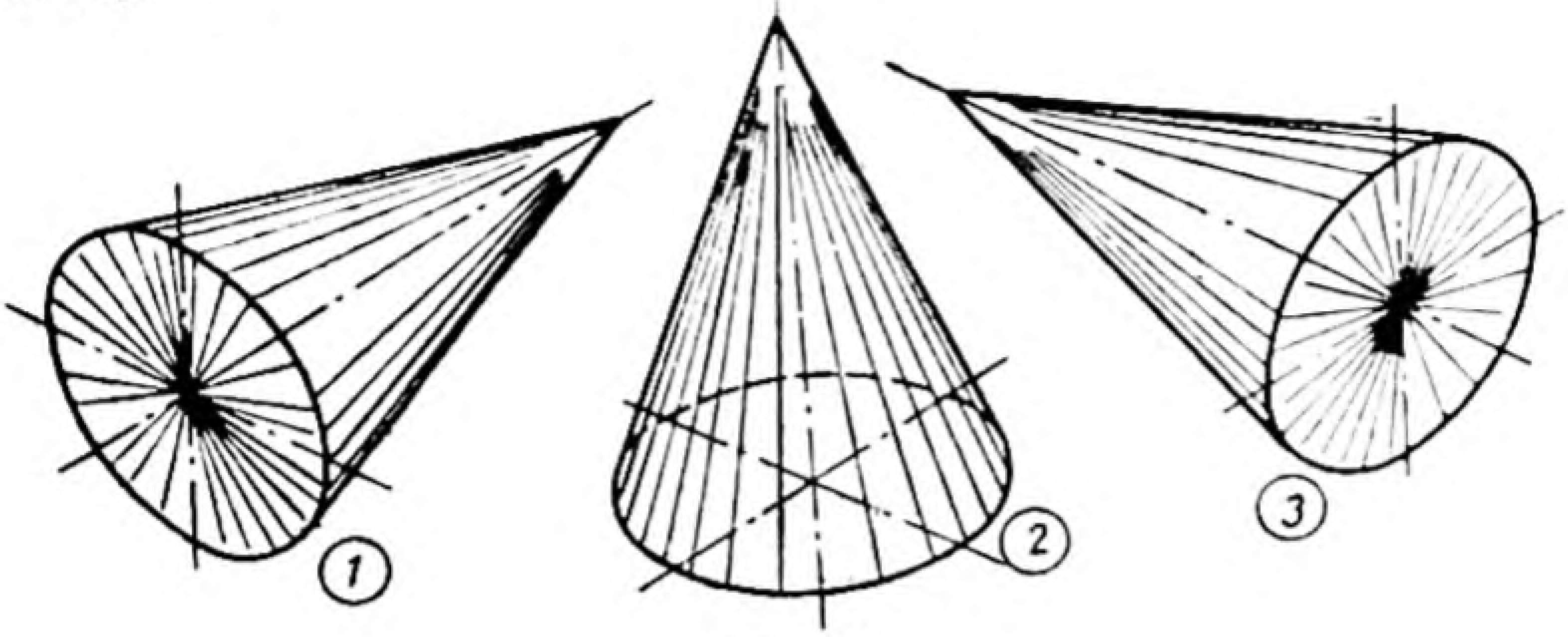


Fig. 403

After the removal of a portion of the detail, part of the base of the cylindrical bore becomes visible. To construct the visible arc of this ellipse several points are taken on the corresponding part of the upper inside ellipse. Verticals are dropped from these points. On them we mark off the altitude of the cylindrical bore. The ellipse is then drawn through the points obtained.

16. Three isometric projections of a right circular cone (its axis in each case parallel to one of the coordinate axes) are given in

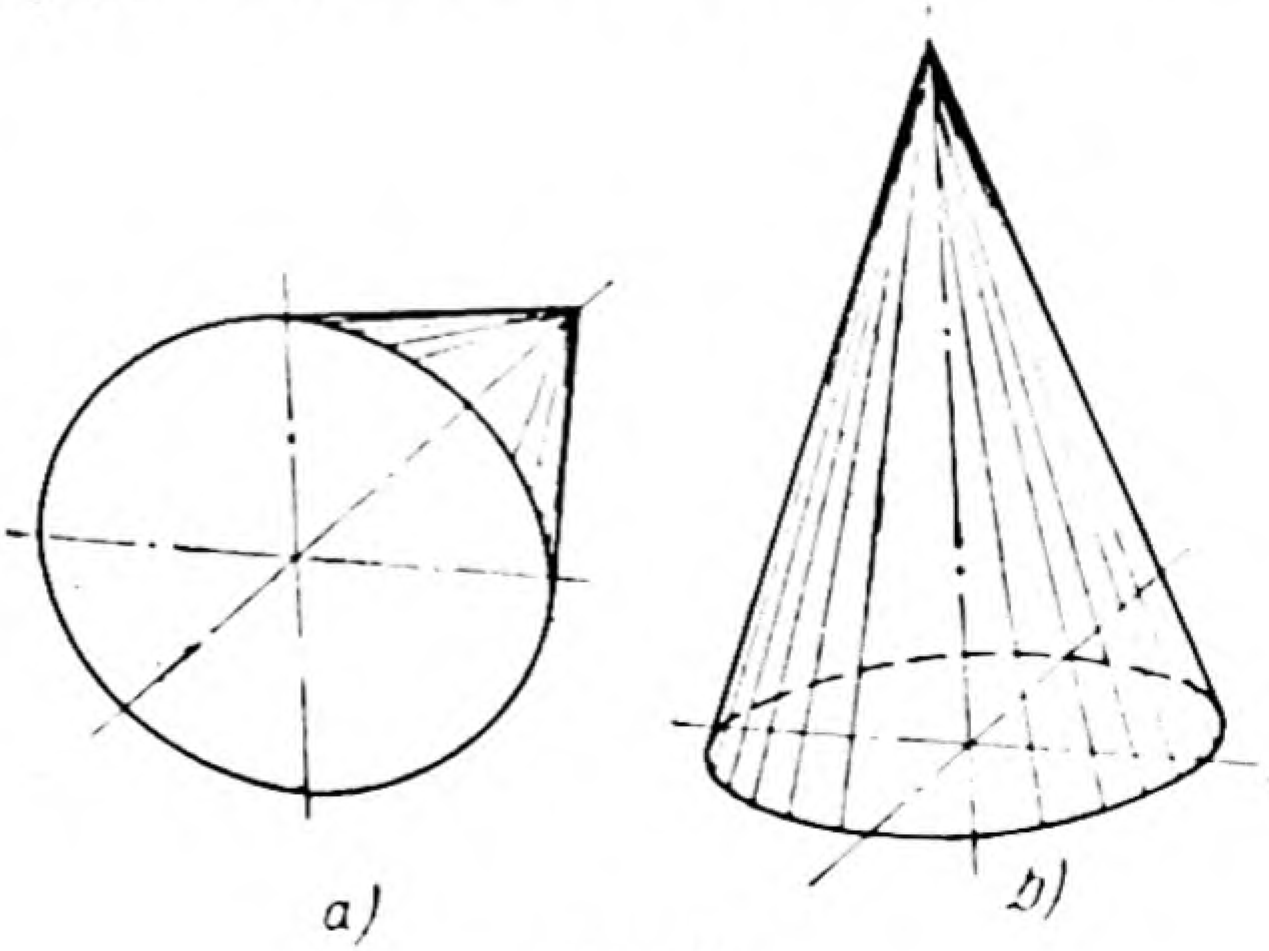


Fig. 404

Fig. 403. In the first case the axis of the cone lies in the direction of the axis Oy. In the second case its direction is that of axis Oz and in the third, that of axis Ox.

The dimetric projections of a cone are shown in Fig. 404. In the first case (Fig. 404, *a*) the base of the cone lies in a plane perpendicular to the axis Oy, and in the second case (Fig. 404, *b*) it is perpendicular to the axis Oz. In both cases the drawing of the cone is fulfilled without foreshortening along axes Ox and Oz, but it is foreshortened along axis Oy, the factor being 0.5. Therefore the major axis of the ellipses, i. e., the dimetric projections of the base, is equal to 1.06 of the diameter of the circle. The minor axes of these ellipses are respectively equal in length to 0.88 and $^1/_3$ of the major axis. The ellipses themselves are drawn through eight points.

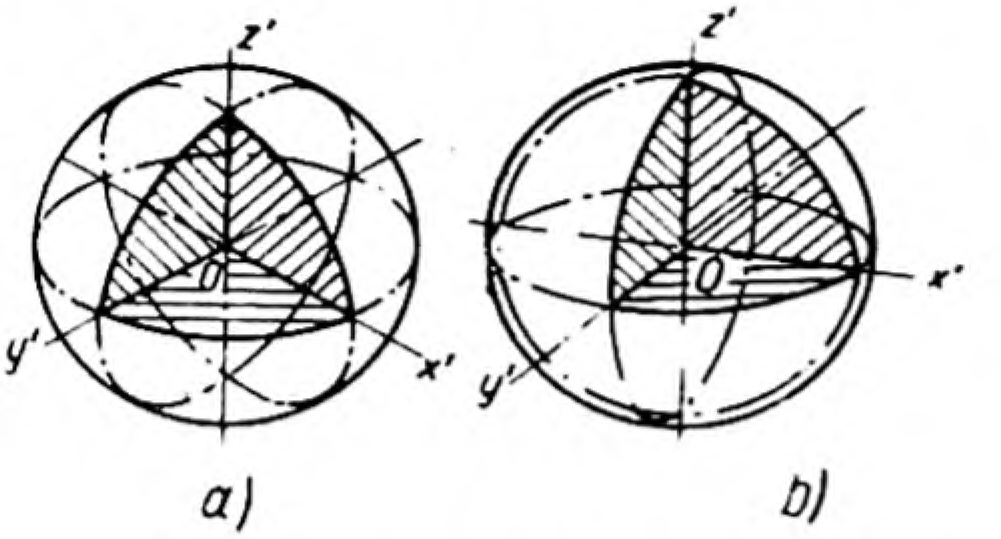


Fig. 405

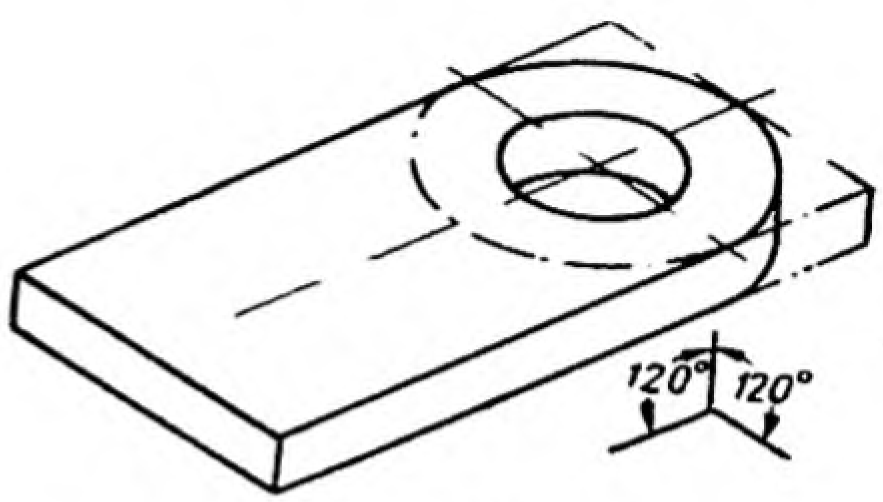


Fig. 406

17. In rectangular isometric projections a sphere appears as a circle whose diameter is equal to the diameter of the sphere. In Fig. 405, *a* the isometric projection of a sphere 22 mm in diameter is constructed with foreshortening along axes $O'x'$, $O'y'$ and $O'z'$, the factor being 0.82. On the isometric projection of the sphere three ellipses are shown. These are the isometric projections of the three

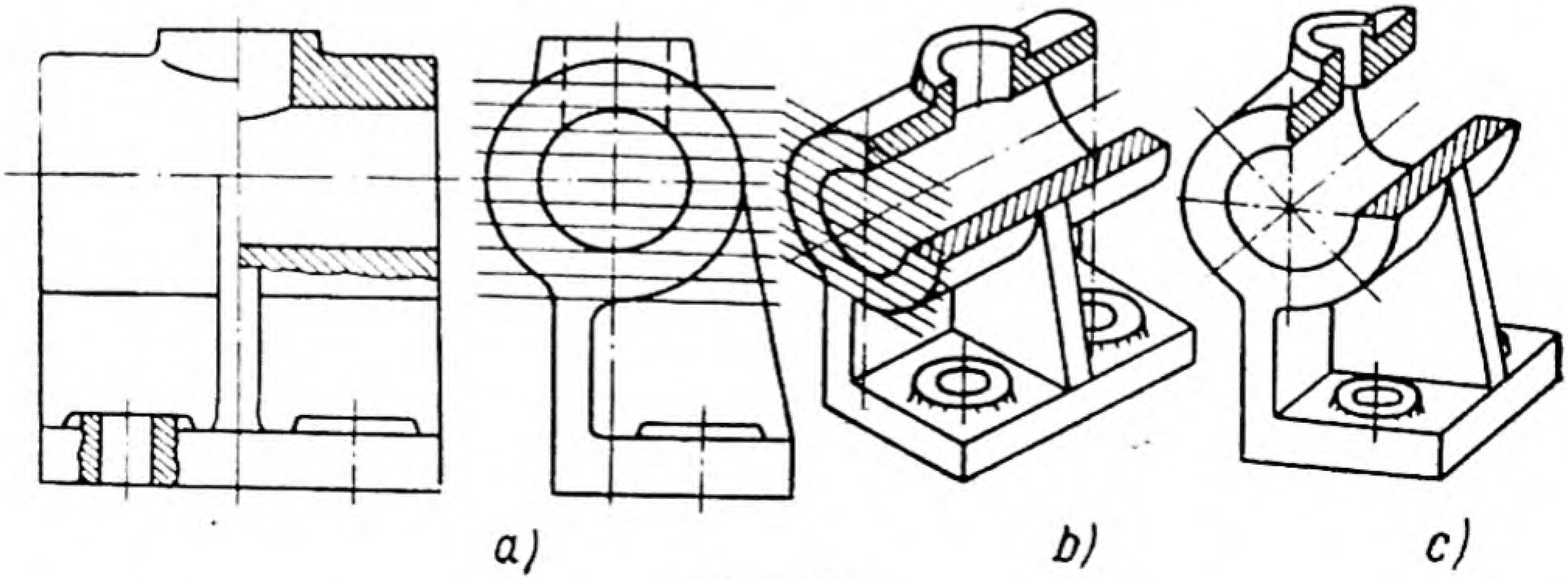


Fig. 407

major circles of a sphere located in the coordinate planes. In the drawing one eighth of the sphere is removed.

In rectangular or right dimetric projections the sphere also appears as a circle (Fig. 405, *b*). The diameter of this circle will be 1.06 of the diameter of the sphere, if the drawing is constructed

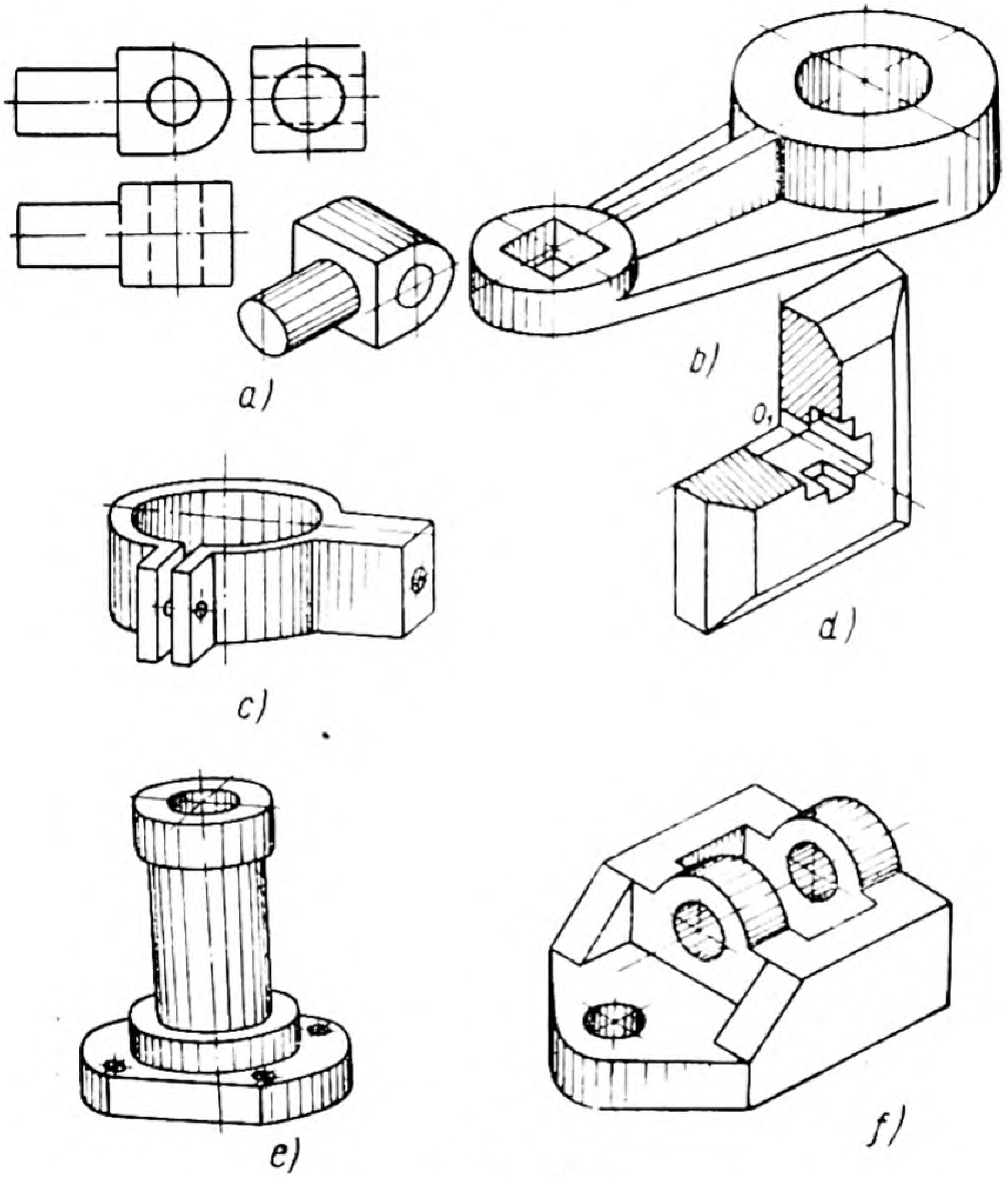


Fig. 408

without foreshortening along axes $O'x'$ and $O'z'$, but with foreshortening along axis $O'y'$, the factor being 0.5. On the dimetric projection of a sphere three ellipses are shown. These are the projections of the three major circles of the sphere and they lie in the coordinate planes. One eighth of the sphere is removed.

18. If an object with a rounded end is to be represented in axonometric projection, as in Fig. 406, the outlines of the object are first constructed without any rounding off as is shown by broken lines. Only after that is the half of the ellipse, representing the rounded end, constructed.

The construction of the bracket (Fig. 407, *a*) both in isometric projection (Fig. 407, *b*) and dimetric projection (Fig. 407, *c*), should be begun by drawing its base which appears as a parallelepiped. The position of the axis of the cylindrical bush is determined next. Then the bush itself is drawn in. After this the edges and lugs with cylindrical bores are constructed. Finally, $^1/_4$ of the bush is removed. In Figs. 407, *b* and 407, *c* all the ellipses were constructed by means of chords.

Some details drawn in rectangular axonometric projection are shown in Fig. 408. These details are a lug (isometry, Fig. 408, *a*), a connecting rod (isometry, Fig. 408, *b*), a collar (dimetry, Fig. 408, *c*), a base plate with a section removed (isometry, Fig. 408, *d*), a king-post (dimetry, Fig. 408, *e*) and the body of a jig (isometry, Fig. 408, *f*).

§ 46. Oblique Axonometric Projections

1. Oblique axonometric projections may be classified as isometric, dimetric and trimetric, depending on the position in space of the planes of axonometric projection and the direction of the projectors. The most widely used oblique projections are those in which the plane of axonometric projection Π' is parallel to the coordinate plane xOz or parallel to the horizontal coordinate plane xOy.

When plane Π' is in such a position the distortion factor along two of the axonometric axes is unity. Along the third axis, depending on the direction of projection, we may obtain the same distortion factor, or any other. In the first case the oblique axonometric projection is isometric, in the second—dimetric. The axonometric representation obtained when the factor of distortion is greater than unity is not employed since it is distorted.

Axonometric projections, in which the plane of projection Π' is parallel to one of the coordinate planes xOz or xOy, are advantageous because all lines, angles and figures lying in planes, parallel to Π', are projected onto the latter without distortion.

In such a case a condition is that the direction of projection should not be perpendicular to plane Π'. This follows from the definition of the oblique axonometric projection itself. Mereover, the direction of projection should not be parallel to the third coordinate plane yOz, otherwise the axis $O'y'$ would become perpendicular to axis $O'x'$, and the representation would not be sufficiently pictorial (Fig. 409).

2. The value of the distortion factor along axis $O'y'$ and the angle between this axis and the axis $O'x'$ may be chosen arbitrarily. In Fig. 410 the following elements are shown: the plane of projection Π', the projection of the point of origin of coordinates — point O', and the projection $O'y'$ of axis Oy.

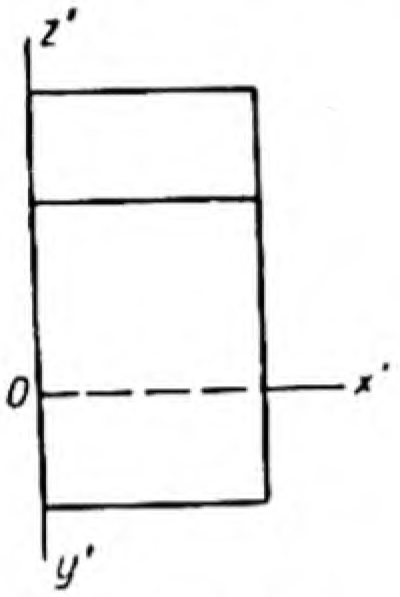


Fig. 409

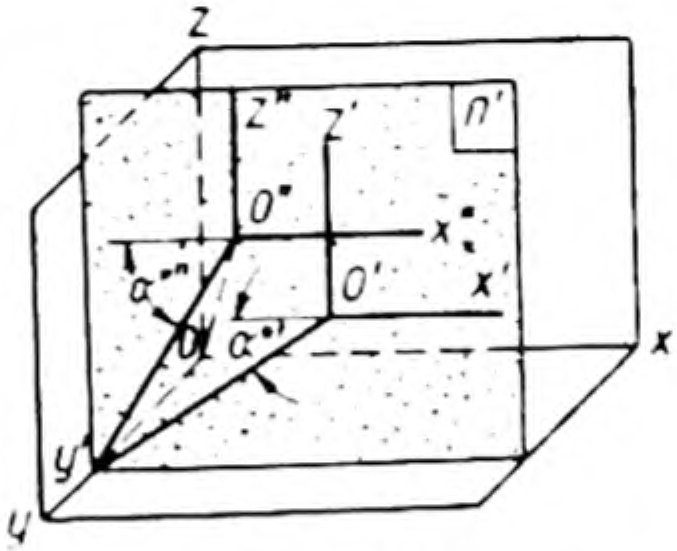


Fig. 410

If triangle $OO'y'$ is revolved about axis Oy, the side Oy' remains stationary, and the vertex O' will move in plane Π' along an arc of a circle with its centre in point y' and whose radius is $O'y'$. In point O'' we have a new position of the vertex O' of the triangle. Since $O'y' = O''y'$, the distortion factor along axis y' remains the same ($O'y : Oy' = O''y' : Oy'$). The direction of the axonometric projection, when point O' moves into position O'', changes. The angle between axis $O'y'$ and the extension of axis $O'x'$ also changes, see angles $\alpha^{\circ\prime}$ and $\alpha^{\circ\prime\prime}$ in Fig. 410.

Leaving $O'y'$ stationary, we can transfer the origin of the axonometric axes O' to a new position O_2'' on the axis $O'y'$ (Fig. 411). In this case the direction of the projection changes, but the angle $\alpha^{\circ\prime}$, between axis $Q_2''y'$ and the extension of the axonometric axis $O''x''$ remains equal to the angle $\alpha^{\circ\prime}$ formed by the same axis $O'y'$ and extension of axis $O'x'$. The distortion factors $O'y' : Oy'$ and $O''y' : Oy'$ also change.

From the analyses given above it follows that in oblique axonometric projections the angle between the axis $O'y'$ and the extension of axis $O'x'$ may be changed, as may the factor of distortion along axis $O'y'$. In other words, we may select both of these values at random. In each case the combination will correspond to a definite direction of the oblique axonometric projection.

This basic concept of the theory of axonometric projection was first put forward by Prof. Polke in 1853 in his theorem: Three lines of any length, lying in one plane and emerging from one point at any angles to each other, represent a parallel projection of three lines of equal length laid off on rectangular axes of coordinates from point of origin of the coordinates O.

This theorem was later proved for the more general case of central projection by Prof. Chetveroukhin, Prof. Glagolev and other Soviet mathematicians.

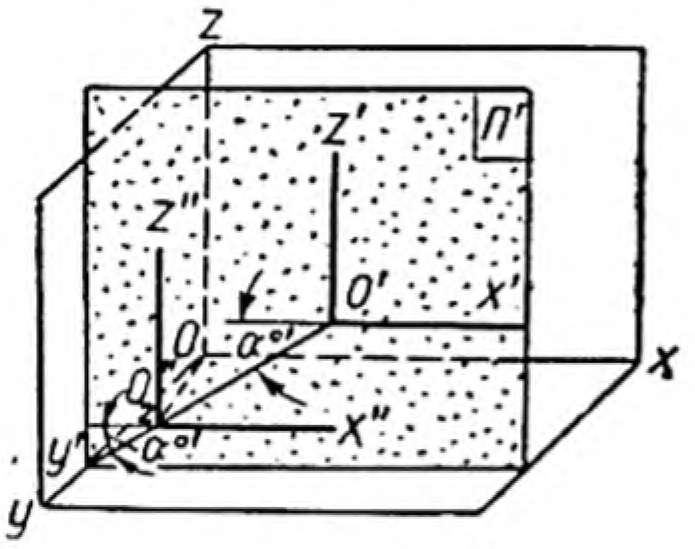


Fig. 411

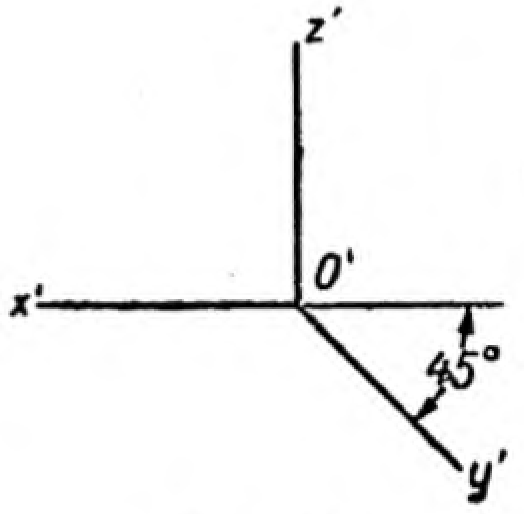


Fig. 412

Axonometric oblique projections obtained in plane Π' parallel to plane zOx are called *frontal projections*. The angle $\alpha^{\circ\prime}$ between axis $O'y'$ and the extension of axis $O'x'$ in these projections is assumed as 30°, 45°, or 60°. When angle $\alpha^{\circ\prime} = 45°$, and the distortion factor along axis $O'y'$ is $^1/_2$, these projections are called *cabinet oblique projections* (Fig. 412).

When plane Π' is horizontal, and the object is viewed from above, the oblique axonometric projections obtained are called *military projections*.

Three frontal projections of a cube with the angle α equal to 45°, 60° and 30°, and the distortion factor equal to $^1/_2$ along axis $O'y'$ are shown in Fig. 413, *a*.

Three frontal isometric projections of the same cube with a distortion factor equal to unity along all three axes are shown in Fig. 413, *b*. In Fig. 414 a cube is shown in oblique axonometric projection. The angles of inclination of axes $O'x'$ and $O'y'$ to the horizontal are both 45°.

A cube, with circles inscribed in its faces, is shown in frontal projection in Fig. 415. To construct the ellipses — the projections

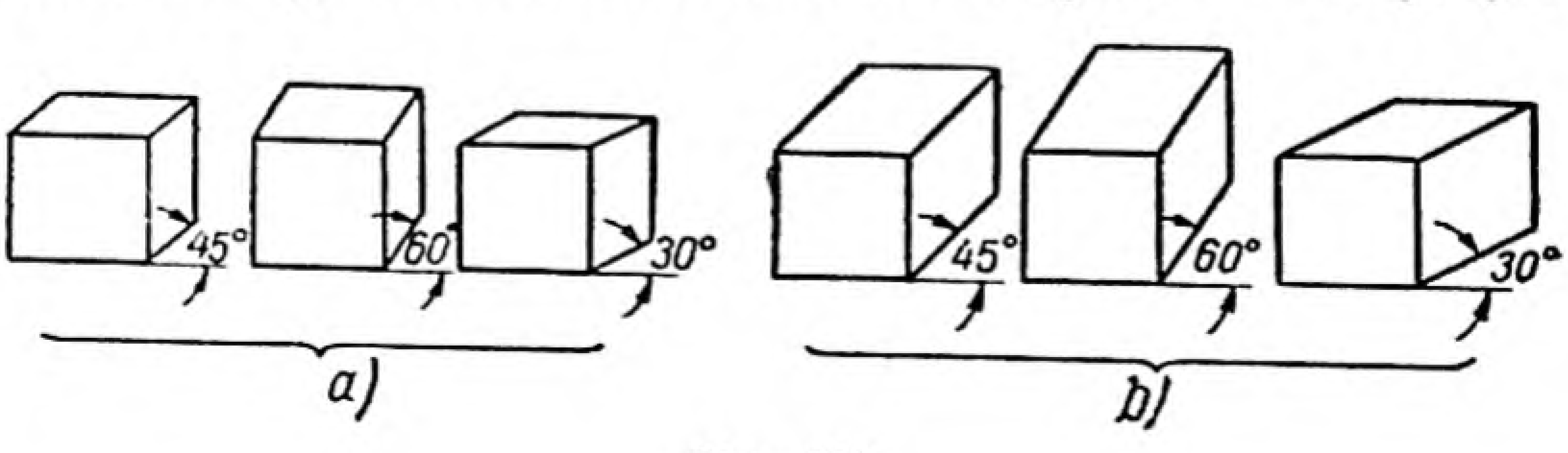


Fig. 413

of the circles — chords are drawn. For example, to construct the points *1'* and *2'* of the upper ellipse, through any point *K'*, taken on line *C'K'*, which is parallel to axis *O'y'*, we draw a horizontal line. Point *K* should then be found on the horizontal projection of the cube, Fig. 415, *b*. The distortion factor along axis *O'y'* is 0.5, so $CK=2C'K'$. Through K_1 we draw a chord and transfer the distances $K_1 1_1$ and $K_1 2_1$ to the axonometric representation.

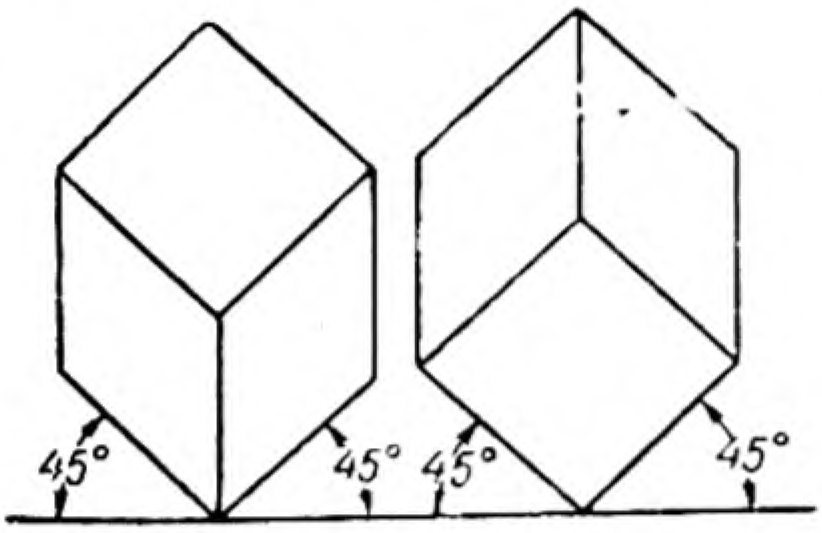


Fig. 414

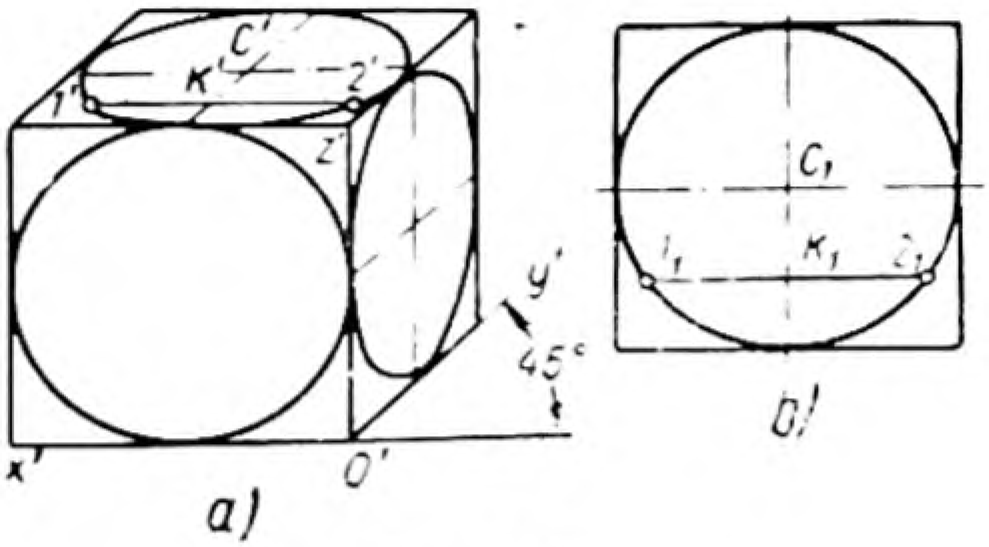


Fig. 415

It should be kept in mind that the major axis of the top ellipse in the frontal axonometric projections is not horizontal. Similarly in the lateral face the major axis of the ellipse will not be perpendicular to axis *O'x'*.

On the front face the inscribed circle in the frontal projection remains the same as in the orthographic projection. This is why frontal projections are the most convenient forms of pictorial representations when the axis of a body of revolution being depicted is perpendicular to the frontal plane of projection Π_2.

3. Oblique axonometric projections are constructed by the same methods as used for rectangular axonometric projections.

For example, let us examine the construction of an oblique axonometric projection of the line of intersection of two cylinders (Fig. 416). This construction may be carried out with the help of auxiliary cutting planes. In this case the cutting planes chosen are parallel to plane Π_3. These planes cut both cylinders along their generatrices. The points contained in the line of intersection are given by the intersection of the respective generatrices in one and the same cutting plane. Point *K* of this line, for example, is found at the intersection of generatrices lying in the plane passing through the axes of the cylinders. The points of intersection, given

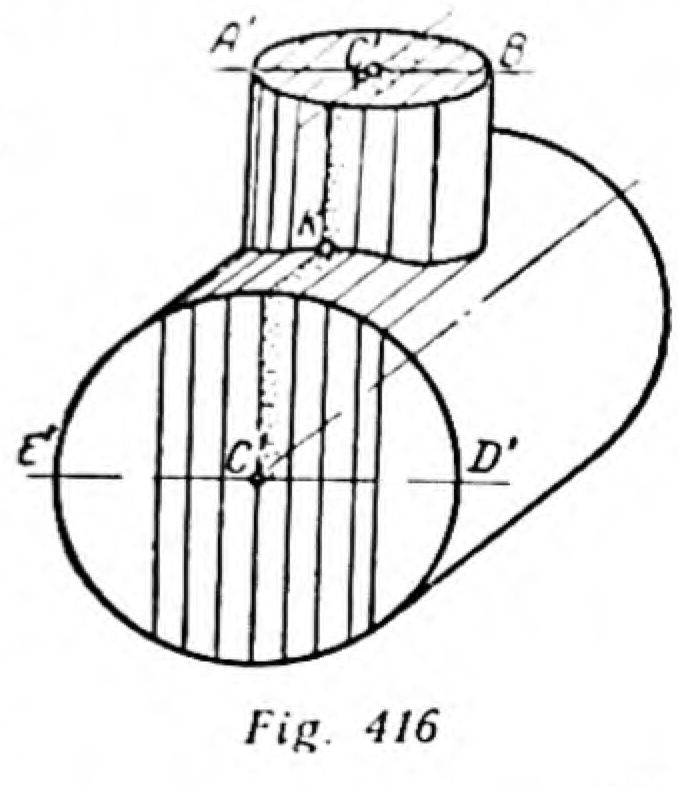


Fig. 416

by other auxiliary planes parallel to the axes of the cylinder, are also points contained in the line of intersection of the given cylinders.

Problems and Exercises

1. How are axonometric projections classified?
2. What is a trace triangle?
3. What are the values of the factors of distortion in isometric rectangular projections?

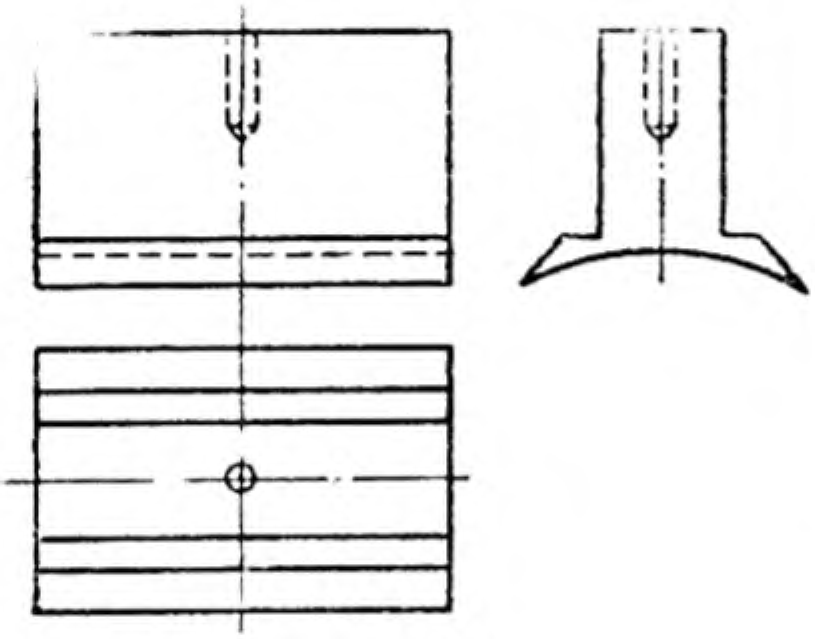


Fig. 417

4. How is an angular scale constructed for a foreshortening of 0.47 in dimetric projection?

5. Explain the method of constructing the ellipses which are the isometric projections of circles in the coordinate planes.

6. What are the dimensions of the major and minor axes of an ellipse, which is the projection of a circle contained in plane xOy, if the isometric projection is constructed a) with a foreshortening of 0.82 along axes Ox, Oy and Oz, b) without any foreshortening along these axes?

7. What are the dimensions of the major and minor axes of the ellipse, which is the projection of a circle contained in plane zOx, if the dimetric projection is constructed a) without any distortion along the axes Ox and Oz and with a distortion factor of 0.5 along axis Oy, b) with a distortion factor of 0.94 along axes Ox and Oz and 0.47 along axis Oy?

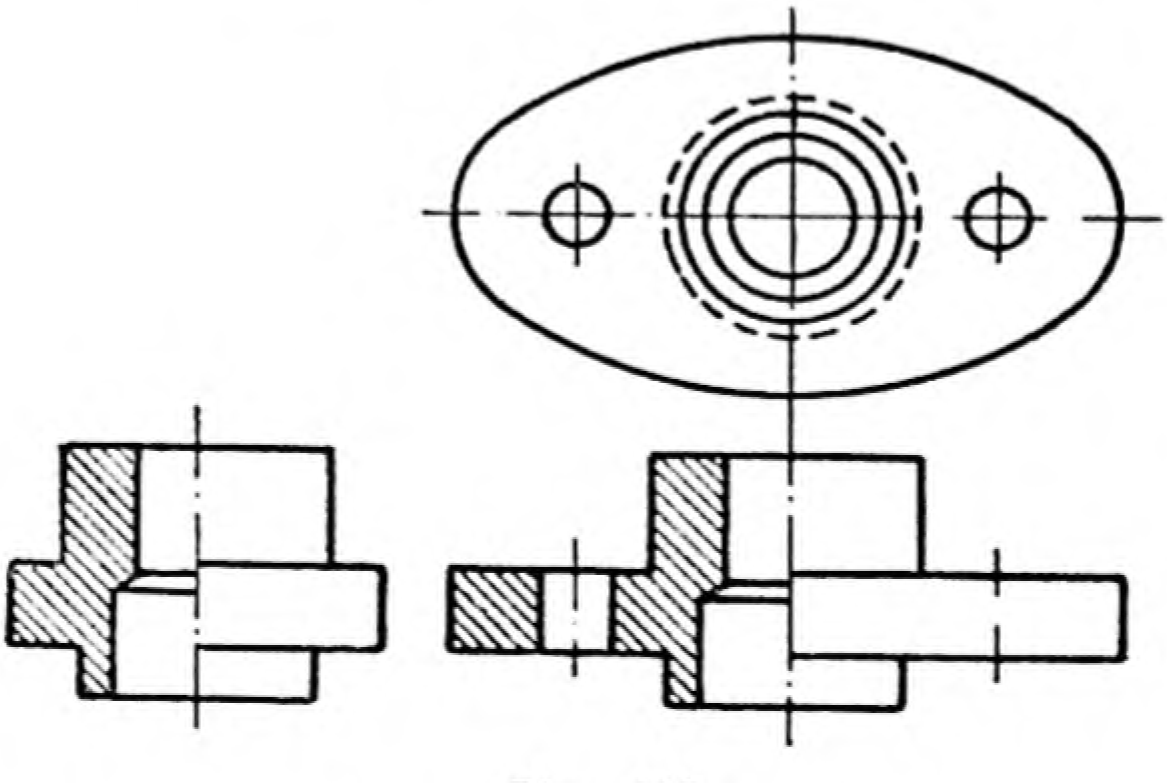


Fig. 418

8. Which axonometric projections are called a) frontal, b) cabinet and c) military?

9. Construct the rectangular isometric projection of the magnet core, shown in Fig. 417. There should be no foreshortening along the axes.

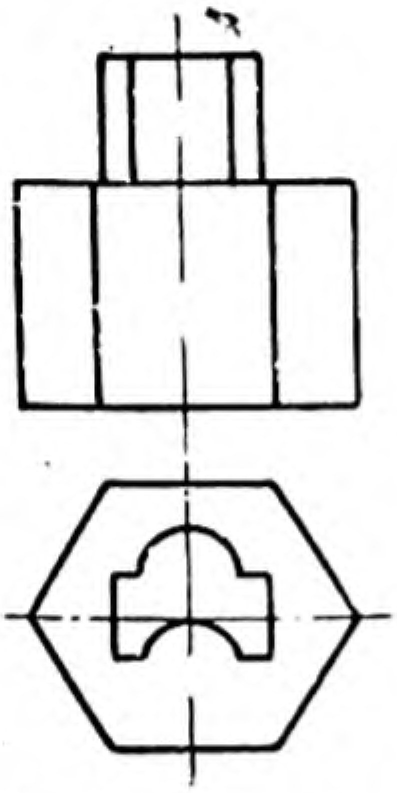


Fig. 419

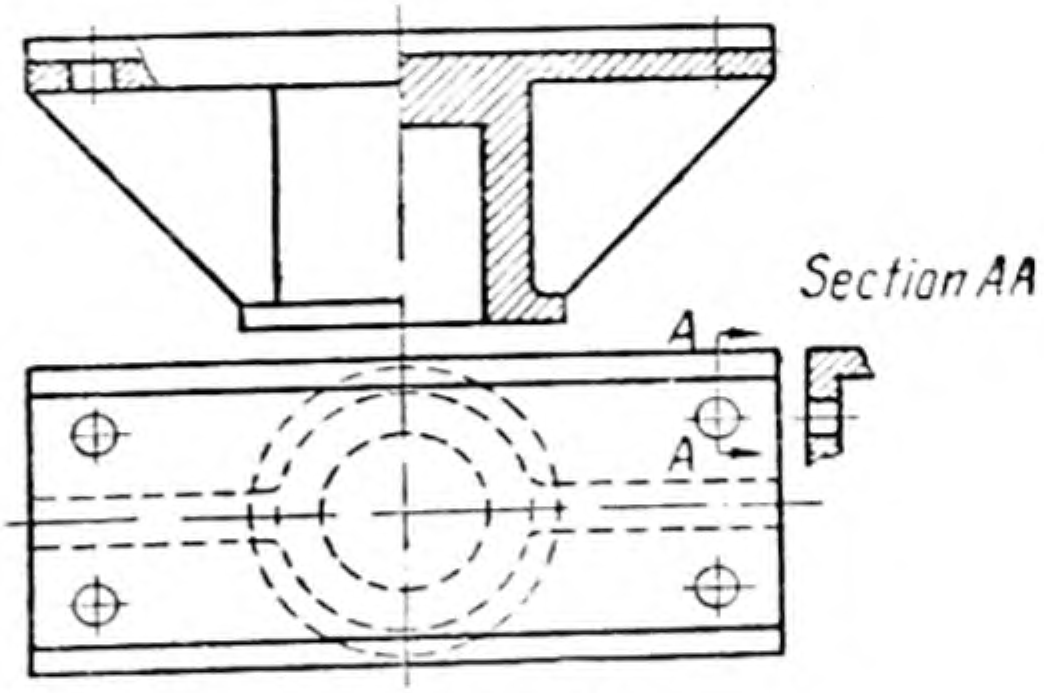


Fig. 420

10. Construct the rectangular isometric projection of the gland cover, shown in Fig. 418, with no foreshortening along the axes.

11. Construct the isometric (or dimetric) projection of a sphere, diameter $D=60$ mm. One quarter of the sphere should be removed.

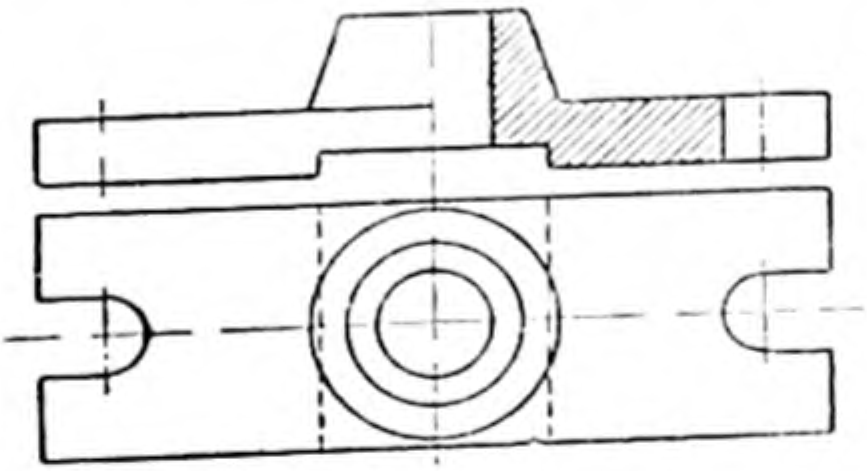


Fig. 421

12. Construct the frontal axonometric projection ($\alpha=45°$) of the fixture, shown in Fig. 419.

13. Construct the rectangular dimetric projection of the bracket, shown in Fig. 420.

14. Construct the rectangular isometric projection of the cross socket, shown in Fig. 421.

Chapter XII

CURVED LINES AND CURVED SURFACES

§ 47. Curved Lines

1. There are two general classes of curved lines, plane curves and space curves, often referred to as single-curved and double-curved lines.

The construction of the second projection of a plane curve, having one of its given projections, is possible only if two projections of a plane in which this curve is contained are given. The construction of the horizontal projection of a curve *ABCDE* lying in the plane of triangle *TMN*, by means of its frontal projection, is shown in Fig. 422.

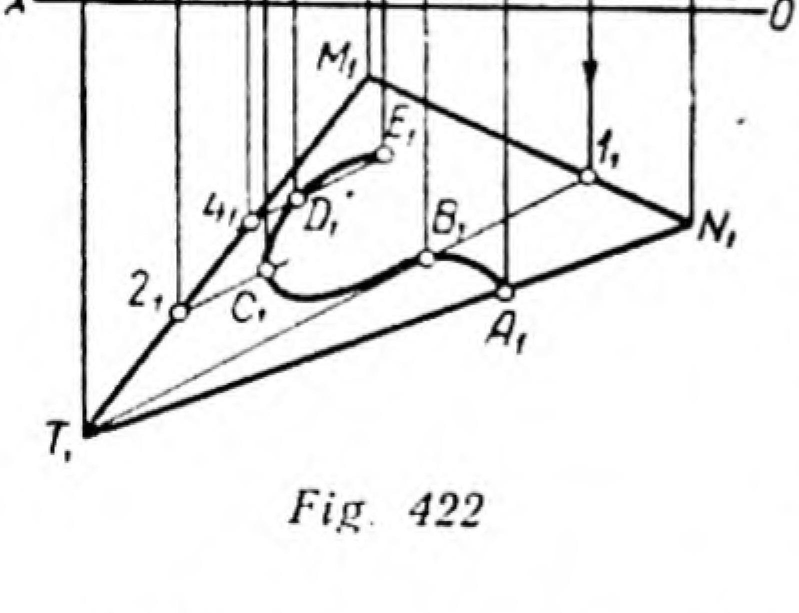


Fig. 422

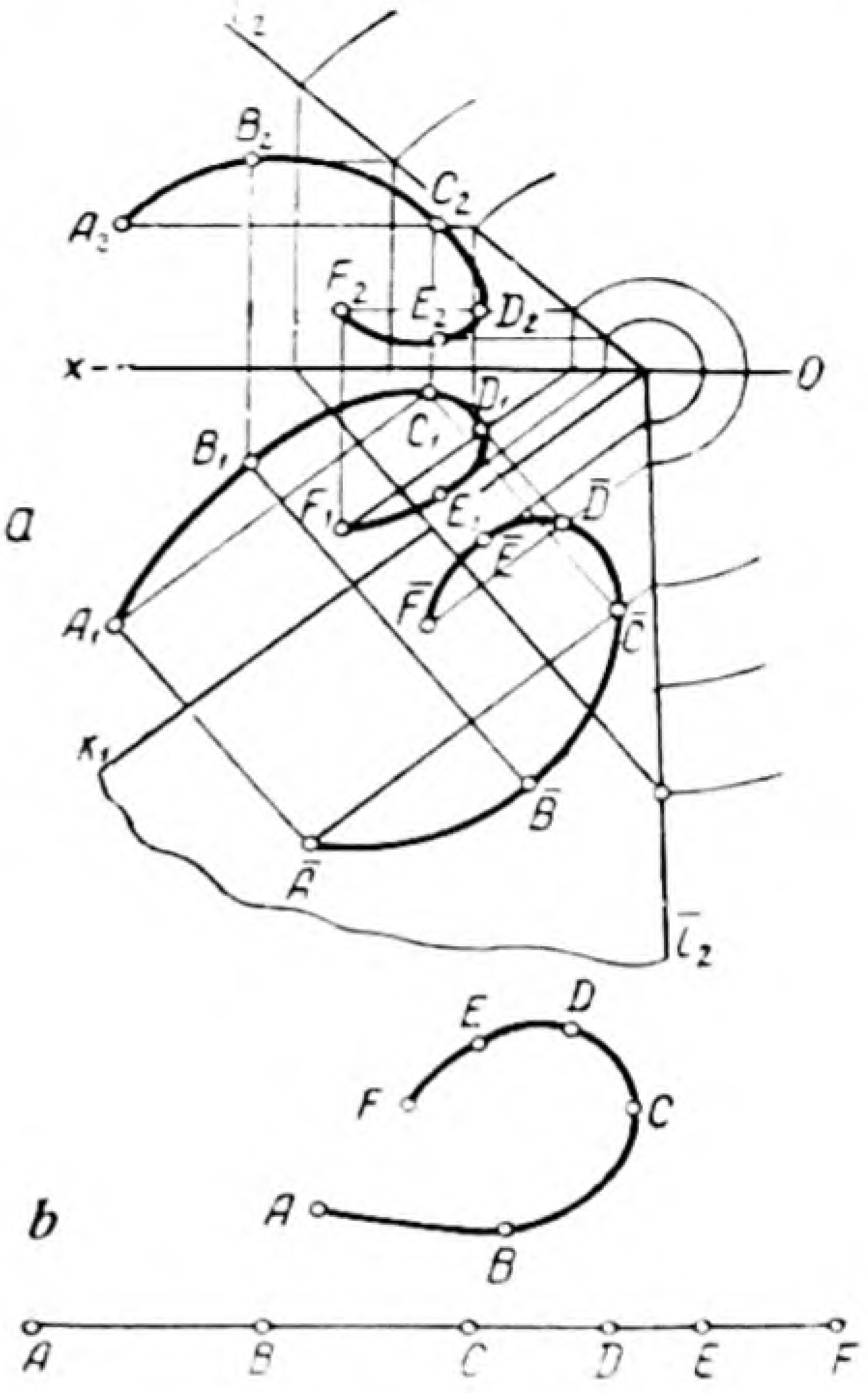
The construction consists in drawing horizontals of the plane of the triangle through points *B*, *C*, *D* and *E* of the curve. The horizontal projections B_1, C_1, D_1 and E_1 of these points are found on the horizontal projections of the respective horizontals. The horizontal projection A_1 of point *A* is determined directly, since point *A* belongs to the side *TN* of the triangle.

A continuous curve, the horizontal projection of curve *ABCDE*, is drawn through points A_1, B_1, C_1, D_1 and E_1.

2. The construction of the curve *ABCDEF* in the oblique plane α is shown in Fig. 423, *a*. First, plane α is made to coincide with one of the planes of projection, say with plane Π_1. The curve $\overline{ABCDEF}$ is then constructed in the new position of plane α. Horizontals of plane α are drawn through points $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$, $\overline{E}$ and $\overline{F}$. With the help of these horizontals we find the horizontal projections

A_1, B_1, C_1, D_1, E_1, F_1 and the frontal projections A_2, B_2, C_2, D_2, E_2, F_2 of the points of the given curve. Through the corresponding projections of points A, B, C, D, E and F we draw the projections of the curve lying in the oblique plane α.

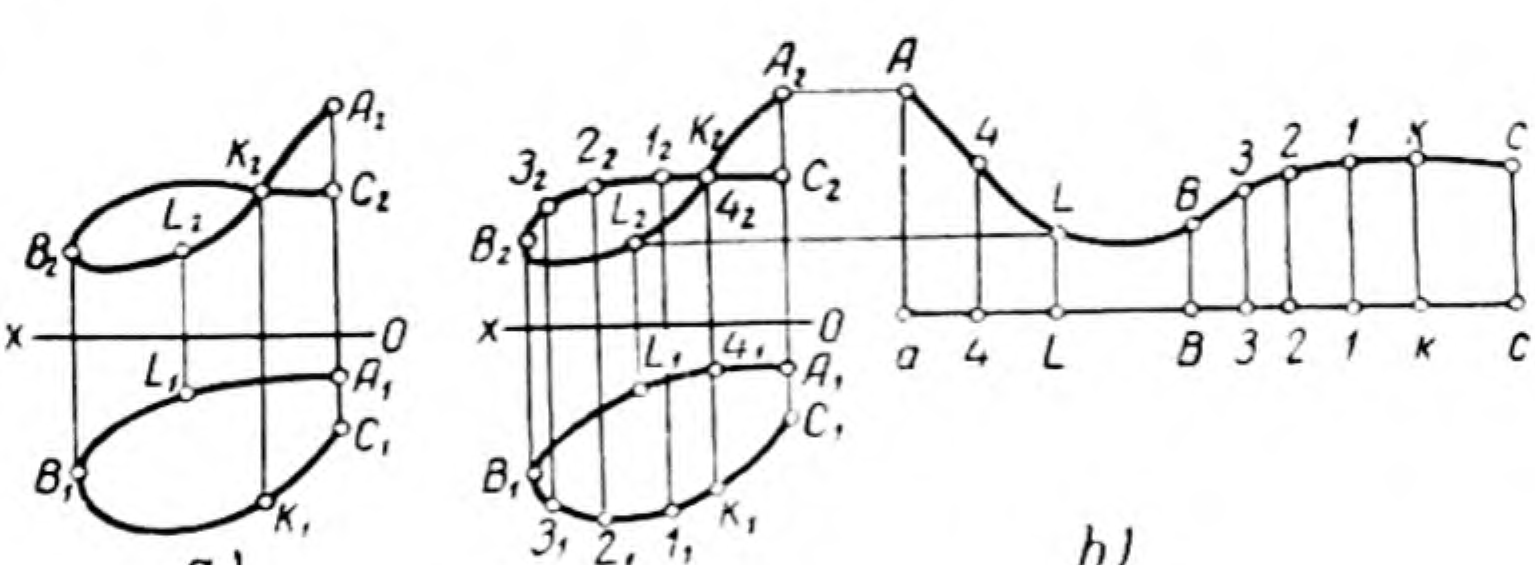
3. To determine the length of a plane curve the plane in which the curve lies is brought into coincidence with one of the planes of projection. The curve in this position is then divided into small parts. Each part is assumed to be a segment of a straight line. These segments are then laid off on a straight line in the same order as on the curve. The sum of all the segments is approximately equal to the length of the curve. This method is used to find the length of the curve $ABCDEF$, as shown in Fig. 423, *b*, and is often referred to as the development of the curve.



a
B_2
C_2
A_2
F_2
E_2
D_2
x
0
D_1
C_1
B_1
F_1
E_1
$\bar{D}$
$\bar{E}$
$\bar{F}$
$\bar{C}$
A_1
K_1
$\bar{B}$
$\bar{A}$
$\bar{l}_2$
b
E
D
F
C
A
B
A
B
C
D
E
F

Fig. 423

4. When making the multiview drawing of a space curve it is often necessary to letter certain of its points in order to avoid confusion when reading the drawing if different parts of the projections of the curve lie on the one and same projectors. For example, if points K and L were not denoted on the projections of curve ABC (Fig. 424, *a*) it would not be clear which of the two parts AB or BC is nearer to plane Π_2.



A_2
K_2
C_2
L_2
B_2
x
0
L_1
A_1
C_1
B_1
K_1
A_2
A
3_2
2_2
1_2
K_2
C_2
L_2
4_2
B_2
x
0
L_1
4_1
A_1
C_1
B_1
3_1
2_1
1_1
K_1
4
L
B
3
2
1
K
C
a
4
L
B
3
2
1
K
C

a) b)

Fig. 424

5. In order to determine the approximate length of a space curve one of its projections, say the horizontal projection, is divided into parts, as seen in Fig. 424, *b*. These segments are marked off successively on a straight line. Through the points of division perpendiculars to the line are erected. On these perpendiculars the distances, at which the frontal projections of the corresponding points lie from the axis *Ox*, are marked. A smooth curve is then drawn through these points. The space curve is said to have been "unrolled" or developed into a plane curve.

§ 48. Cylindrical and Conical Helical Lines

1. Helical lines are the most widely used space curves in engineering practice.

A cylindrical helical line or *helix* is a spiral in space, contained in the surface of a circular cylinder and intersecting all its generatrices at the same angle. The helix may be considered as being generated by a point contained in the surface of a right circular cylinder and simultaneously rotating about the axis of the cylinder as it moves parallel to the axis. The longitudinal movement is assumed to be proportional to the angular movement.

The axis and diameter of the helix are the axis and diameter of the cylinder in which the helix lies. On the surface of one and the same cylinder different helices may be obtained by varying the ratio of the two velocities of the generating point. In this case the pitch of each of the helices generated on the cylinder will differ from that of the others. The pitch or lead *h* of a helix is the distance between the two nearest points in which the helix intersects one of the generatrices of the cylinder. In other words, the pitch is the distance the point moves parallel to the axis as it completes one revolution.

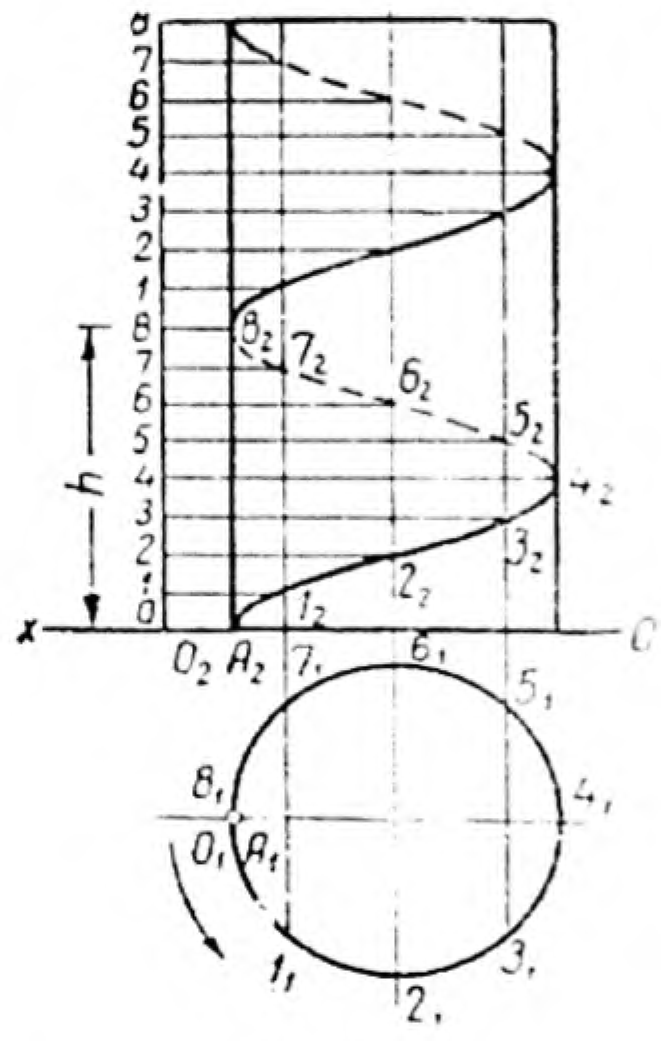


Fig. 425

The part of a helix between the nearest points on the generator, namely the part generated in one revolution, is generally called a *turn* of the helix.

A circular cylinder on which a helix has been plotted is shown in Fig. 425.

Since all the points of the cylindrical helix are located on the lateral surface of the cylinder, which is projected onto plane Π_1 as a circle, it is obvious that the horizontal projections of all points of the helical line also lie on this circle.

In order to construct the frontal projection of a cylindrical helix we divide the circular base of the cylinder into a number of equal parts, in this case into eight parts. A definite value must be taken as the pitch h of the helix. The pitch is then divided into the same number of equal parts as the base, since the linear movement of point A is proportional to its angular movement.

After point A has moved from its initial position O to position 1, i. e., after it has moved through one eighth of the pitch h, its horizontal projection will occupy position 1_1, and its frontal projection will be 1_2.

A horizontal line is drawn through the first division of the line of pitch. Then, through the first division 1_1 of the circle a vertical is passed. The intersection of these two lines determines the position of the frontal projection 1_2 of point A in its new position. The other points, $2_2, 3_2, \ldots, 8_2$, of the frontal projection of the helical line are constructed in a similar manner.

The frontal projection of the first turn of the cylindrical helix is obtained by drawing a curve through the points $O_2, 1_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2$ and 8_2. The frontal projection of the second turn is constructed in a similar manner.

Helices may be right-hand or left-hand, depending on which direction the point generating the helix moves. Right-hand helices, in other words helical lines with a right-hand thread, are those generated by the clockwise rotation of a point about an axis when the point moves along the axis away from the observer. Left-hand helices are generated by counter-clockwise rotation.

A helix with a right-hand thread is shown in Fig. 425. One of the characteristics of such a helix is that the visible parts of the curve rise to the right and the hidden parts rise to the left. The characteristic feature of a helix with a left-hand thread is that the visible parts of the curve rise to the left and the hidden parts rise to the right.

A helix is also obtained on the cylinder if a right triangle $A_0B_0C_0$ is wrapped around a cylinder (Fig. 426). The horizontal side A_0B_0 of the triangle is equal to the length of the circle of the base of the cylinder. The vertical side B_0C_0 is equal to the pitch h of the helix. It follows that the hypotenuse A_0C_0 of the right triangle $A_0B_0C_0$ is the development of one turn of the cylindrical helix. The angle α is called the *lead angle*, or the *helix angle*. The size of this angle depends on the pitch h of the helix and length of the circumference of the base of the cylinder. We have: $\tan \alpha^\circ = \frac{h}{2\pi R}$ where R is the radius of the circumference of the cylinder. The length L of one turn of the helix may be determined from the formula $L = \sqrt{h^2 + (2\pi R)^2}$.

The frontal projection of the cylindrical helix is a sine wave. This may easily be proved. Let us assume the axis of projection as the axis x and the frontal projection J_1J_2' of the axis of the helix as the axis z (Fig. 427). We take any point K on the helix. The angle of revolution of point K from its initial position O is denoted by γ° ($\measuredangle\gamma^\circ = \measuredangle O_1J_1K_1$). The radius of the helix is denoted by r and the pitch by h. The coordinates x and z of point K are shown in Fig. 427.

From the right triangle $J_1N_1K_1$, we have

$$x = r \sin \gamma^\circ. \tag{1}$$

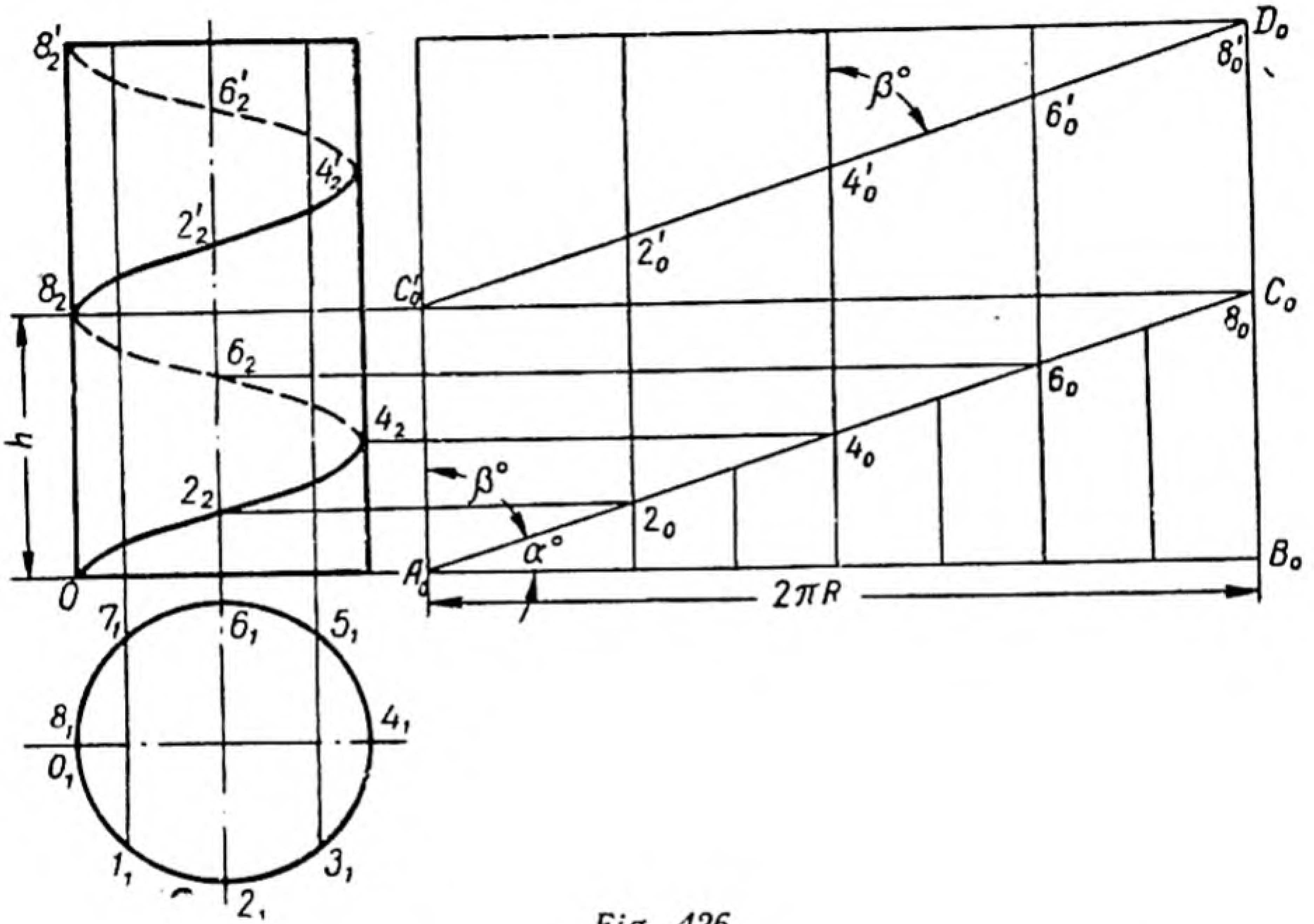


Fig. 426

Since the linear and angular movements of the point forming the helix are proportional, then $\frac{z}{h} = \frac{\gamma^\circ}{2\pi}$,

$$\text{hence } \gamma^\circ = \frac{z \cdot 2\pi}{h} = 2\pi \frac{z}{h}. \tag{2}$$

Substituting the value found for γ° in equation (1), we obtain $x = r \cdot \sin 2\pi \frac{z}{h}$, which is the equation of a sine wave.

The angle β on the development of the helix (Fig. 426) is equal to the angle between the generatrix of the cylinder and the helix. This angle β in space is equal to the angle between one of the generatrices of the cylinder and a tangent to the helix.

Since all the generatrices of the cylinder form the same angle β with the helix it follows that all tangents to the helix have the same slope relative to the generatrices of the cylinder passing through the corresponding tangent points.

The generatrices of the cylinder are perpendicular to plane Π_1, so the slope of all the tangents to the helix, relative to plane Π_1, is the same and equal to the angle α formed by the hypotenuse and horizontal side of the right triangle ABC. On the basis of this the tangent to the helix at any point on it can be constructed.

Let us consider the right triangle KLM (Fig. 427) whose frontal projection $K_2L_2M_2$ is shown by hatching. This triangle consists of line KL, a segment of the generatrix of the cylinder passing through the tangent point K, the tangent KM to the helix and line ML. Since the triangle KLM is located in a horizontal-projecting plane, the line ML, called the *subtangent,* may be considered as the horizontal projection of the tangent KM.

It is obvious that angle $KML=\alpha°$, i. e., the angle between the tangent to the helix and plane Π_1.

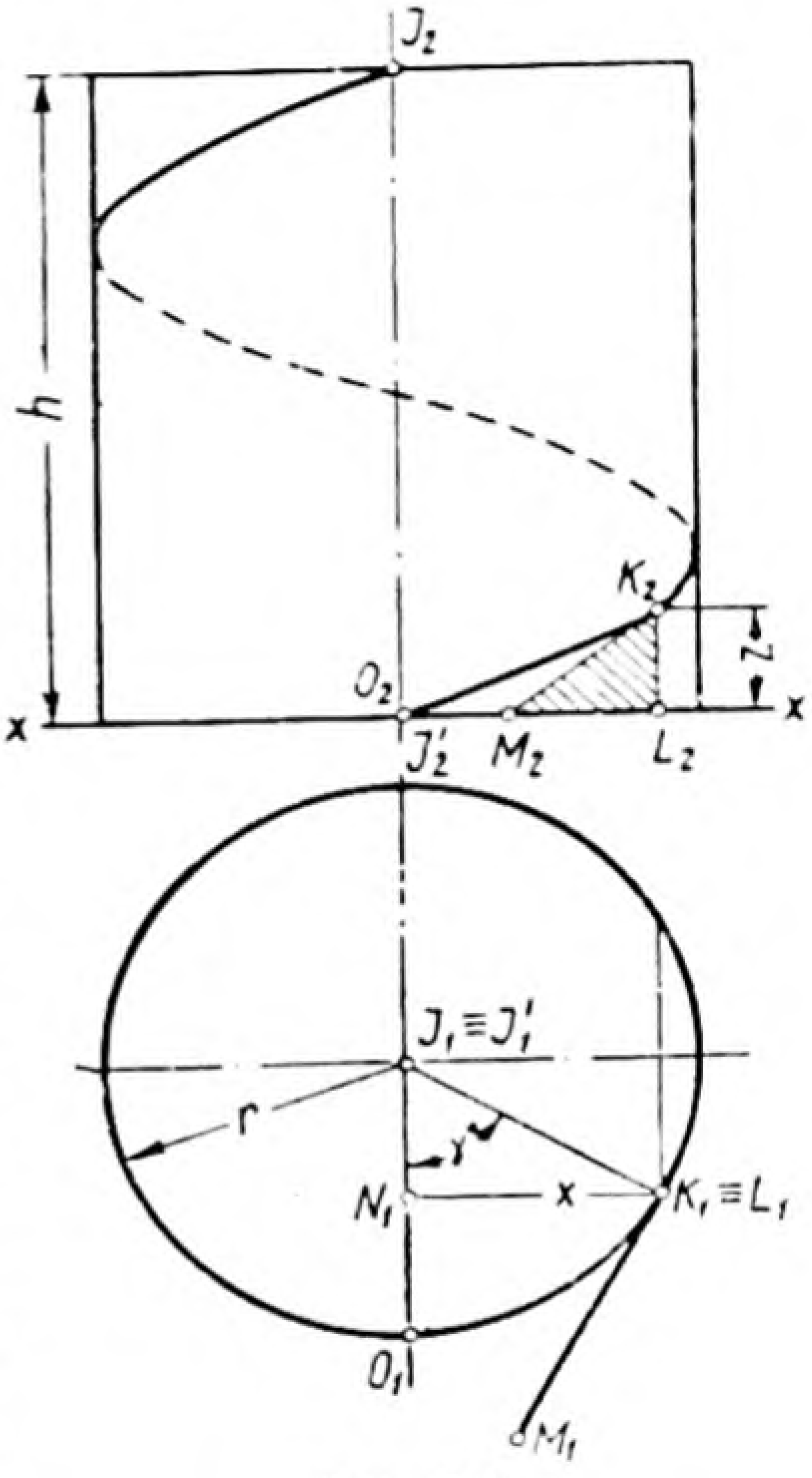


Fig. 427

From the right triangle KLM we have $ML=KL \cdot \cot \alpha°$, where $\alpha°$ is the angle between the hypotenuse KM and side ML. But $OL=KL \cdot \cot \alpha°$. Therefore $ML=OL$, i. e., the length of the subtangent is equal to the length of the straightened arc of the circle — the horizontal projection of the helix — from the initial point O_1 up to the horizontal projection K_1 of the tangent point.

The subtangent ML is the horizontal projection of the tangent. The tangent is the hypotenuse and the subtangent is the base of the right triangle KLM. Therefore, the length of the tangent is equal to the length of the straightened arc of the helix itself from the initial point to the point of tangency.

Using these properties of the tangent and subtangent of a helix these lines may be constructed. In order to construct a tangent to the helix through point K (Fig. 427) we draw a tangent to the circle which is the horizontal projection of the helix. The tangent point is K, the horizontal projection of point K_1. Then from point K_1 we lay off K_1M_1, equal to the length of the straightened arc of the circle from point O_1 to point L_1. Having obtained point M_1 we find point M_2 by drawing a projector. Point M, whose projections

M_1 and M_2 have been found, is the horizontal trace of the required tangent. Joining points M_2 and K_2 by a straight line we obtain the frontal projection K_2M_2 of the tangent KM to the helix at point K.

If we revolve triangle KML about axis KL into the frontal position, the true value of angle $KML = \alpha^\circ$ will be found in the multiview drawing. The lead angle α of the helix is determined by the development of the helix (Fig. 426). The angle α is formed by the hypotenuse and horizontal side of triangle ABC.

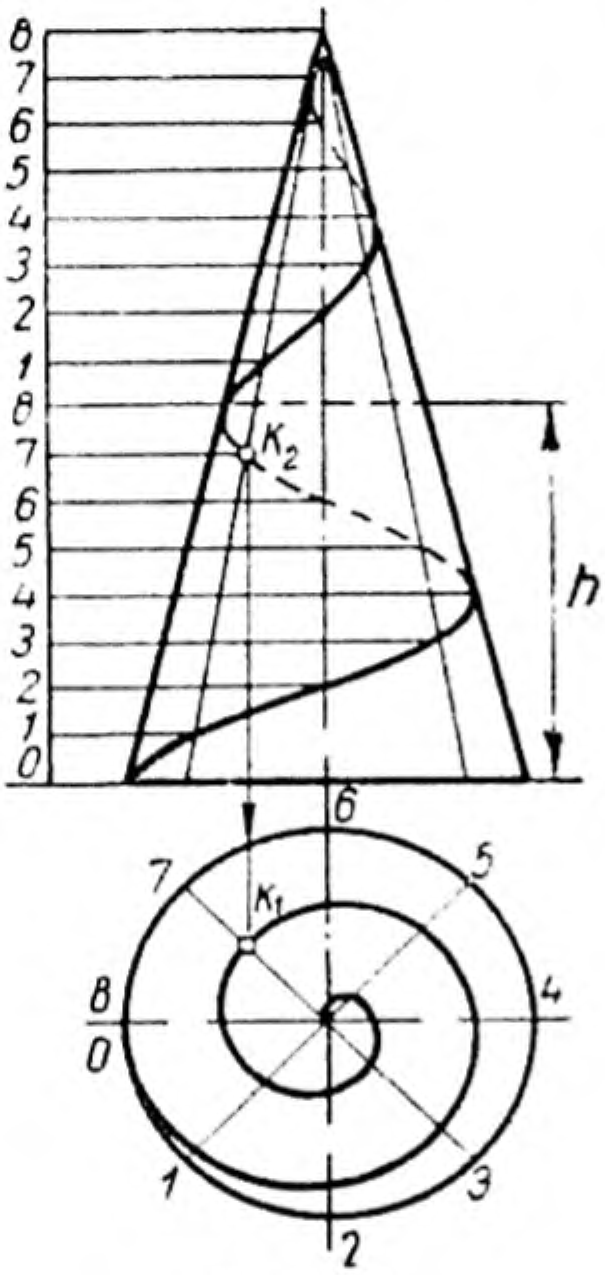


Fig. 428

2. The **c o n i c a l h e l i x** (Fig. 428) is generated by a point moving along the surface of a right circular cone so that its movement along the generatrices of the cone is proportional to the angular movement about the axis of the cone.

The movement of the point along the generatrices of the cone may be considered as composed of a movement along the axis of the cone and a movement in a direction perpendicular to this axis.

Since the two linear movements and the angular movement of the generating point are proportional, the construction of a conical helix may be carried out as follows. The circle of the base of the cone is divided into several equal parts, in this case into eight parts. Then the eight corresponding generatrices of the cone are drawn in and their horizontal and frontal projections are constructed. The pitch h is then divided into the same number of equal parts. Through each division a horizontal is passed to intersect the frontal projection of the corresponding generatrix. For example, point K_2 lies at the intersection of the horizontal line drawn through division *7* of the pitch and the frontal projection of the generatrix drawn through division *7* of the circle of the base.

The points found in this way are the frontal projections of the points of the helix. By using the frontal projections we can obtain the horizontal projections of these points. To do this projectors are dropped from the frontal projections of the points to intersect the horizontal projections of the respective generatrices of the cone. The construction of point K is shown in Fig. 428. By passing curves through the corresponding projections of these points we obtain two projections of the conical helix. The horizontal projection of the conical helix is an Archimedean spiral. The frontal projection is a sine curve, the amplitude of which decreases. The construction of two turns of the R. H. conical helix is shown in Fig. 428. The pitch h is equal to one half of the altitude of the cone.

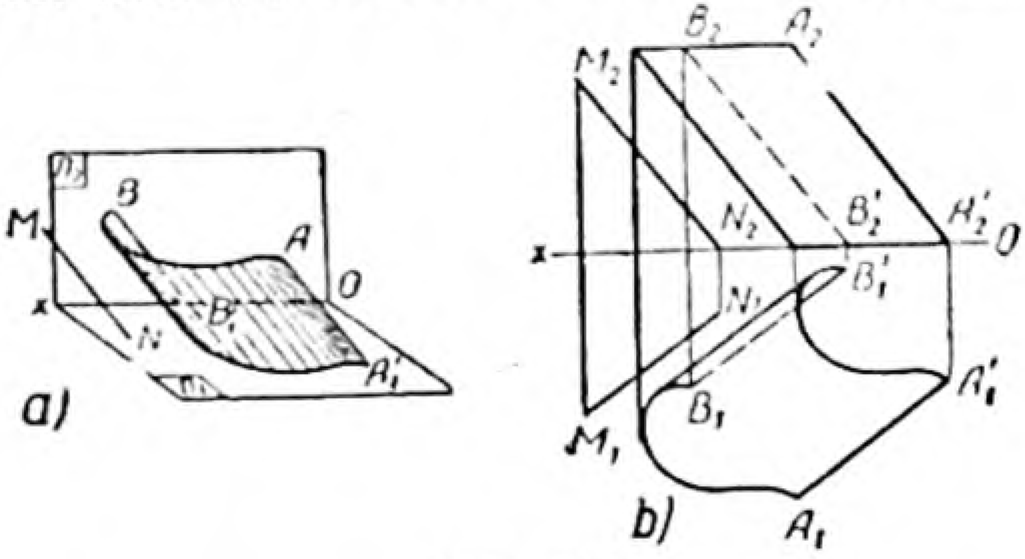
Problems and Exercises

1. How is the projection of a plane curved line constructed on a given plane?
2. How is the length of a space curve determined?
3. Define a) a cylindrical helix, b) a conical helix.
4. How are the projections constructed a) for a cylindrical helix, b) for a conical helix?
5. Construct a L. H. cylindrical helix, the diameter of which is 40 mm and the pitch $h=30$ mm.
6. Construct a L. H. conical helix, given the diameter of the base circle of the cone as 50 mm, the altitude of the cone 80 mm, and the pitch $h=40$ mm.

§ 49. Curved Surfaces

1. Curved surfaces are generated by lines moving in accordance with definite laws. Depending on the form of these lines, or generatrices, curved surfaces are divided into two major classes, single-curved surfaces, or ruled surfaces, generated by moving straight lines, and double-curved surfaces, the generatrices of which are curves. Cylinders and cones are examples of single-curved surfaces.

A sphere is an example of a double-curved surface. The stationary line along which the generatrix of the curved surface moves is called the *directrix*. When a plane, tangent to the surface, passes through all the points contained in each of the generatrices of a single-curved surface consecutively, the single-curved surface is said to be *developable*, i. e., it can be "unwrapped". When this condition is not fulfilled the single-curved surface is called a *warped surface*. It cannot be unwrapped, so it is not developable. Summing up, a developable surface may be "unwrapped", so that it coincides with a plane and then has no folds or breaks. Warped surfaces and double-curved surfaces cannot be developed.



a)

b)

Fig. 429

2. Cylindrical surfaces. A cylindrical surface is generated by a straight line moving along the directrix AB, so that it always remains parallel to a given line MN* (Fig. 429). In this case the directrix AB must not lie in the same plane as the generatrix.

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* Each position that the generatrix assumes is called an element of the surface.

The line in which a curved surface intersects a plane of projection is called a *trace* of the surface. In Fig. 429 the curve $A_1'B_1'$ in plane Π_1 is the horizontal trace of the cylindrical surface $A'B'BA$.

If a trace of a cylindrical surface* and the direction of the generatrices are known, the cylindrical surface may be constructed and on it any generatrix may be plotted and any of its points can be determined (Fig. 430).

Of all the generatrices of the cylindrical surface the outline generatrices, AA' and CC', BB' and DD', are of particular significance.

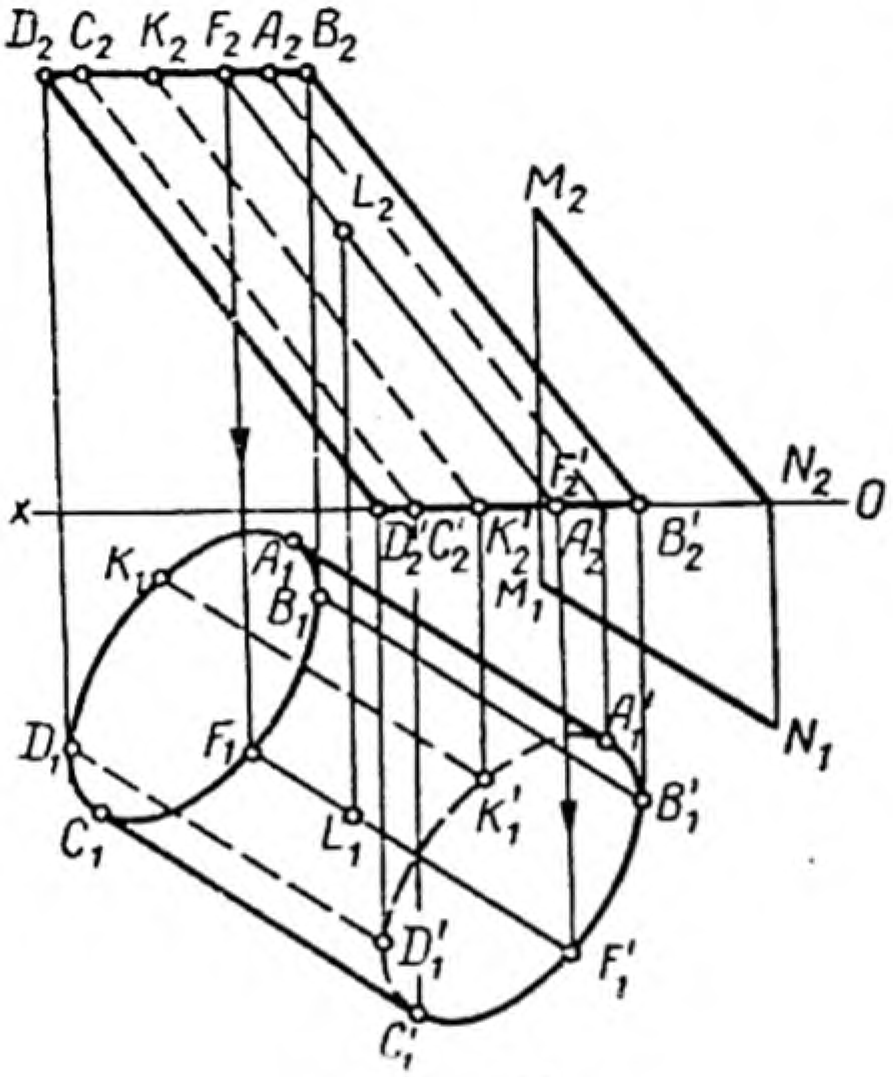

Fig. 430

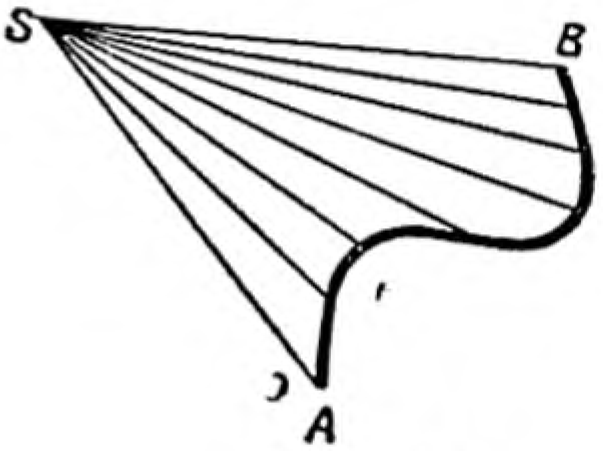

Fig. 431

In the multiview drawing the projections of two outline generatrices are the boundaries of the visible and hidden parts of the surface. In Fig. 430 that part of the cylinder passing through the segment $A'B'C'$ of the horizontal trace, when projected onto plane Π_1, is visible. The rest of the surface is hidden. When projected onto plane Π_2 that part of the cylinder surface passing through the segment $B'C'D'$ of the horizontal trace is visible. The rest of the surface is hidden.

3. Conical surface. The surface generated by the movement of a straight line SA along curve AB, when the generating line in all its positions passes through a point S, is called a *conical surface* (Fig. 431).

Line SA is the *generatrix*. Point S is called the *vertex*. Curve AB

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* The trace of the surface is at the same time the directrix of this surface.

is the *directrix*. The conical surface consists of two parts called the *nappes* of the surface (Fig. 432).

If the vertex of the cone and its directrix, or the trace of the cone surface, are known, then the conical surface may be constructed. On the surface any generatrix may be drawn and any point determined. For example, to construct the frontal projection K_2 of point K given on the surface of the cone (Fig. 433), the horizontal projection K_1 of the point being known, it is sufficient to draw through point K_1 the horizontal projection S_1L_1 of the generatrix SL of the cone and then find the frontal projection of the generatrix S_2L_2, after which, from point K_1, a projector is drawn to intersect S_2L_2 in point K_2.

In Fig. 433 the lines S_2A_2 and S_2B_2 and also S_1C_1 and S_1D_1 are the projections of the outline generatrices. These generatrices separate the visible and hidden parts of the projections of the conical surface.

4. The developable helicoid. The surface formed by the continuous movement of a straight line, which in all its positions remains tangent to a cylindrical helix, is called a *developable helicoid*, or *helical convolute* (Fig. 434). The cylindrical helix, which in this case is the directrix of the developable helicoid, is the *helical directrix of tangency*. So the developable helicoid belongs to that group of curved surfaces which have a directrix of tangency.

Any section of a developable helicoid given by a plane perpendicular to the axis of a cylindrical helix is an *involute of a circle*.

Part of a helicoid cut by a cylindrical surface whose axis coincides with the axis of a helicoid is shown in Fig. 435.

The line of section in this case will be a helix with the same axis as the helicoid. This surface is called a *developable annular **helicoid***.

5. Cylindroid. The surface generated by a straight line, which constantly remains parallel to a given plane as it moves along two directrices not located in one plane, is called a *cylindroid*. The plane to which the generatrix of the cylindroid remains parallel as it moves is called the *plane of parallelism*, or *plane **director***.

A representation of a cylindroid, generated by the movement of the generatrix AB along two space curves, the directrices, is given in Fig. 436, *a*. The multiview drawing of this same cylindroid is shown in Fig. 436, *b*. The straight line AB in each of its position remains parallel to the given plane α, which is the plane of parallelism.

6. The hyperbolic paraboloid. This surface, sometimes called a warped plane, is a particular case of a cylindroid (Fig. 437).

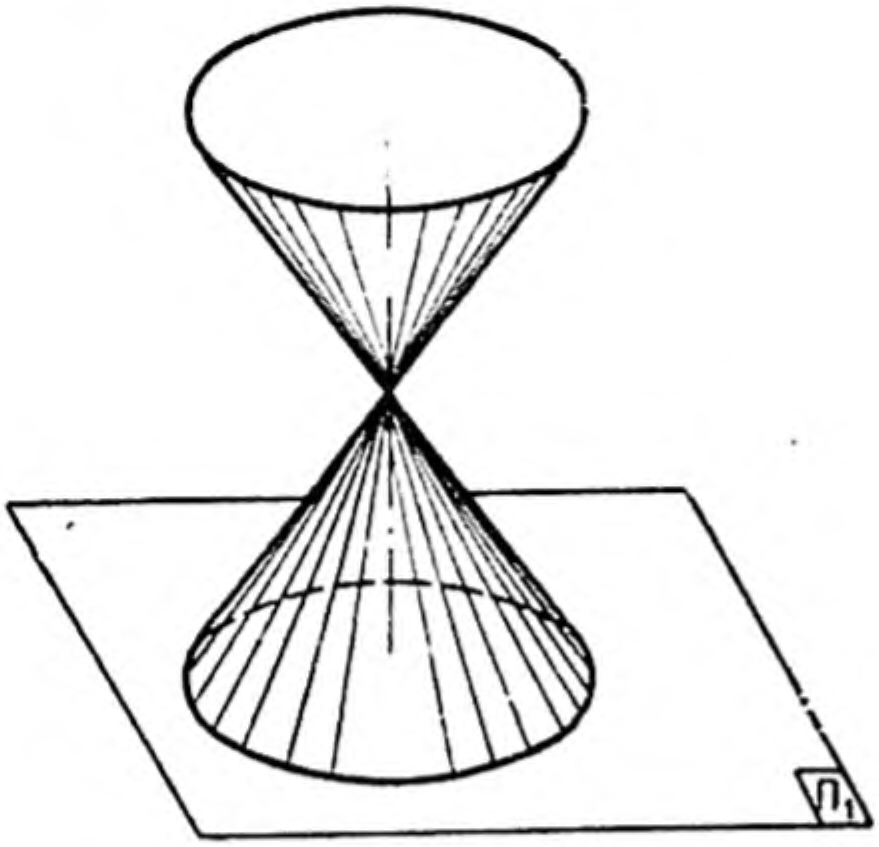


Fig. 432

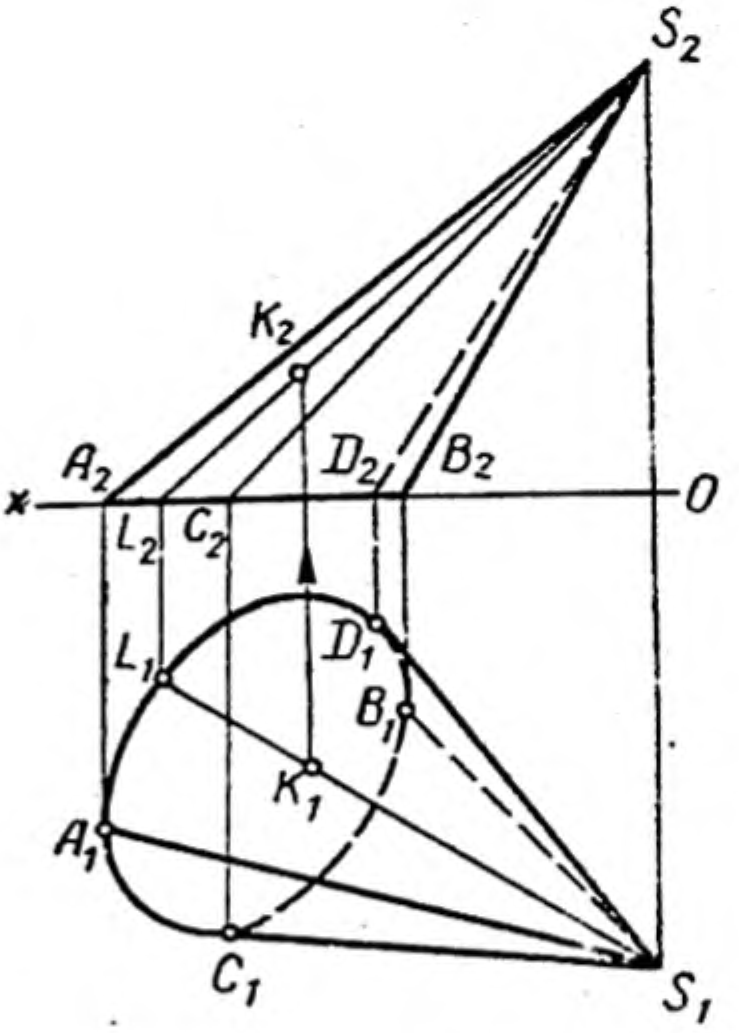

S2
K2
A2
D2
B2
x
0
L2
C2
D1
L1
B1
K1
A1
C1
S1

Fig. 433

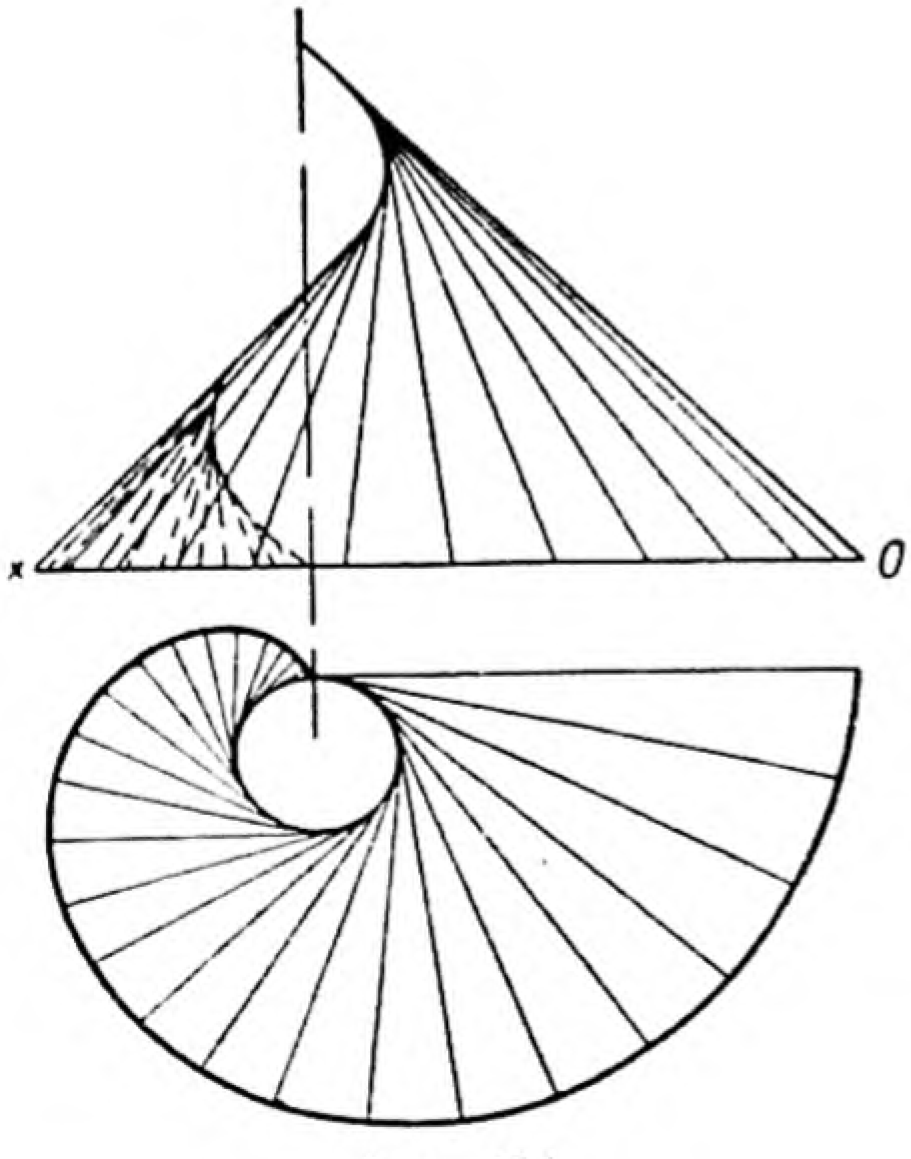


Fig. 434

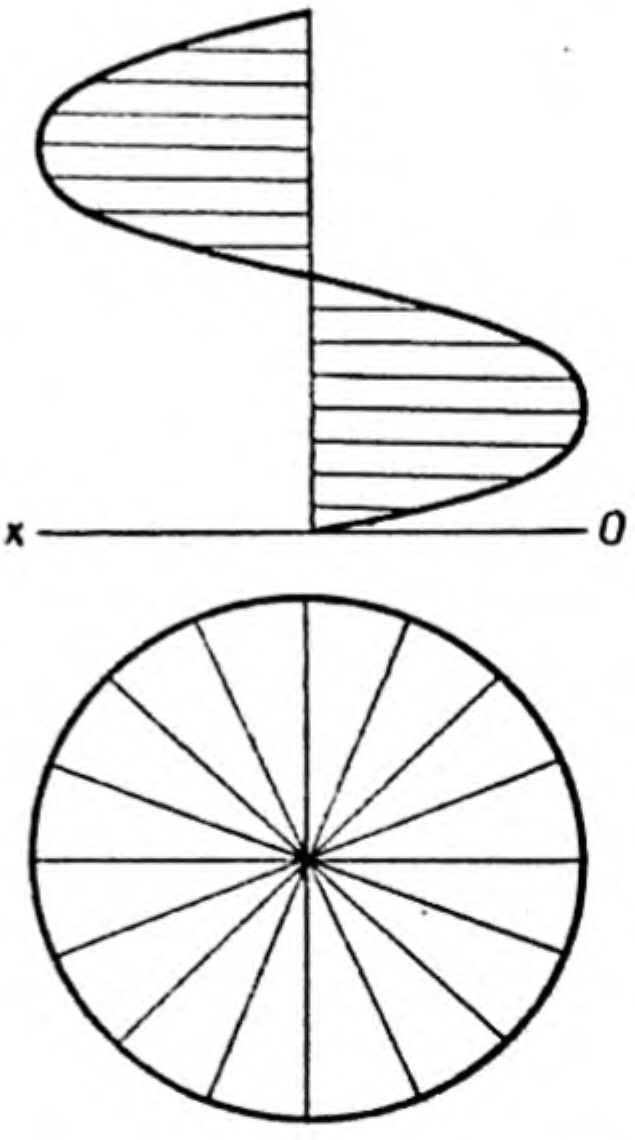
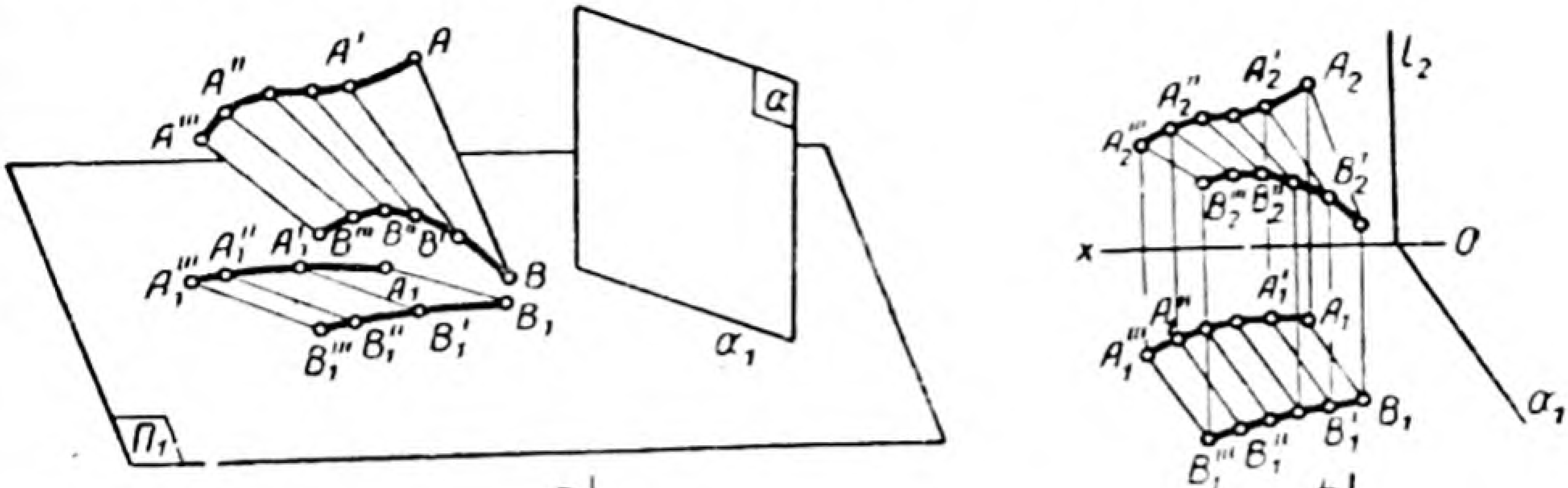


Fig. 435

The directrices of such a surface are two non-intersecting, or skew lines. The horizontal plane of projection Π_1 in this case is assumed as the plane of parallelism for the hyperbolic paraboloid (Fig. 437).

Three projections of a hyperbolic paraboloid, having two planes of parallelism α and β, perpendicular to plane Π_1 which form angles of $\alpha°$ with plane Π_2, are shown in Fig. 438.



a) b)

Fig. 436

If we cut such a hyperbolic paraboloid by horizontal planes γ, γ', γ'' the sections obtained will be hyperbolas. If such a hyperbolic paraboloid is cut by planes parallel to Π_2 or Π_3 the sections will be parabolas.

7. The conoid is a particular case of the cylindroid. One of the directrices of the conoid is a straight line, the other is a curve. The generatrix is a straight line which constantly remains parallel to a plane of parallelism.

A conoid is shown in Fig. 439. One directrix of this surface is line AA', in this case perpendicular to the plane of parallelism Π_1, the second directrix is curve BB'.

If the curved directrix of the conoid is a cylindrical helix, the conoid is called a *helical conoid* or *non-developable helicoid* (Fig. 440).

If the helical conoid is cut by the surface of a right circular cylinder, whose axis MM' coincides with the straight directrix of the conoid, the line of intersection obtained will be a cylindrical helix having the same pitch as the helical directrix $BB'B''B'''B''''$ of the conoid (Fig. 441). The surface of the helical conoid contained between helical lines is called an *annular helical conoid.*

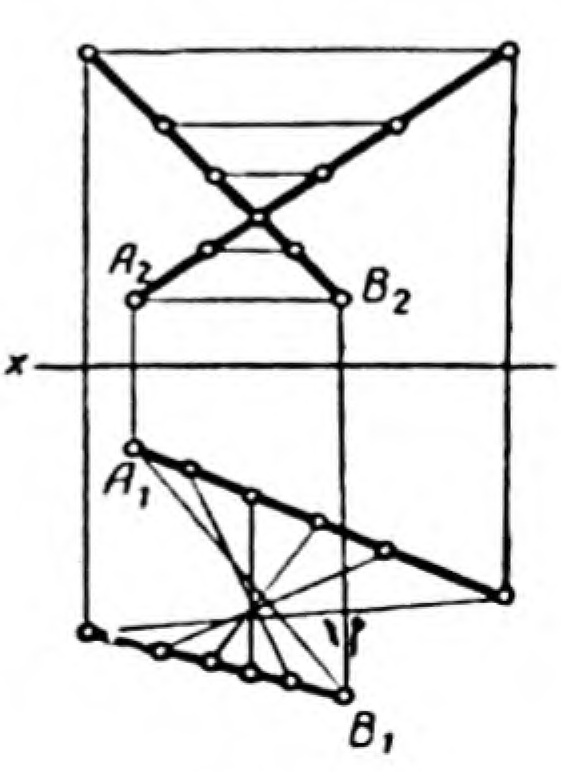


Fig. 437

Annular helical conoids find wide application in engineering practice. Nuts and bolts with rectangular or square threads, cylindrical springs of rectangular section, helical staircases and the Archimedean screw all contain surfaces of helical annular conoids.

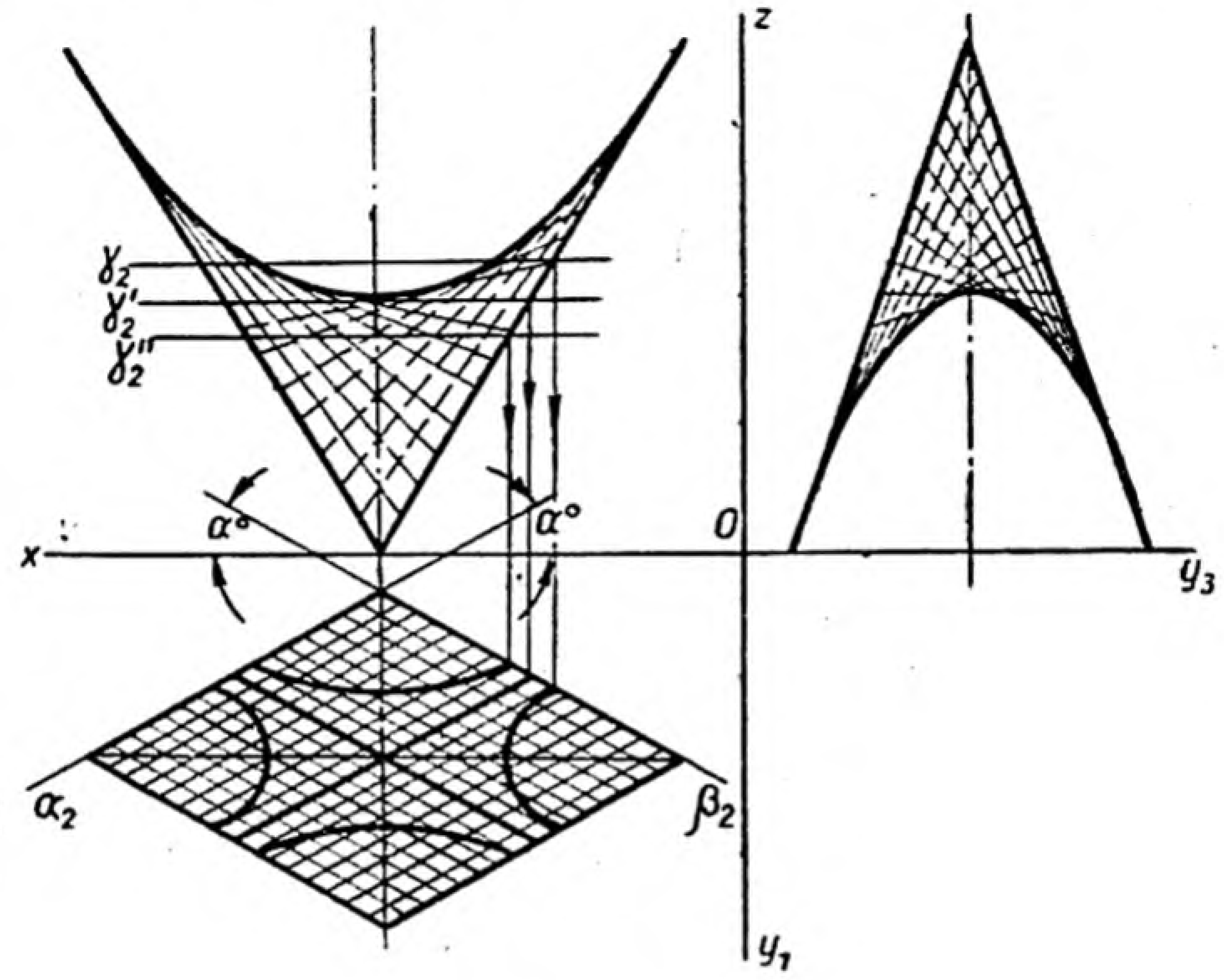


Fig. 438

The points and lines on non-developable, or warped ruled surfaces are constructed the same way as on developable ruled surfaces.

It is required to construct the frontal projection K_2 of a point K on the surface of a conoid, the horizontal projection K_1 of this point being known (Fig. 439). Through point K_1 the horizontal projection of generatrix AK is drawn to intersect the horizontal projection of the curved directrix in point L_1. From point L_1 we draw a projector to intersect the frontal projection of the curved directrix in point L_2.

The frontal projection of the generatrix, on which point K lies, will pass through point L_2 parallel to axis Ox. A projector is then drawn through K_1 to the frontal projection of the generatrix LK. In this way frontal projection K_2 of point K is obtained.

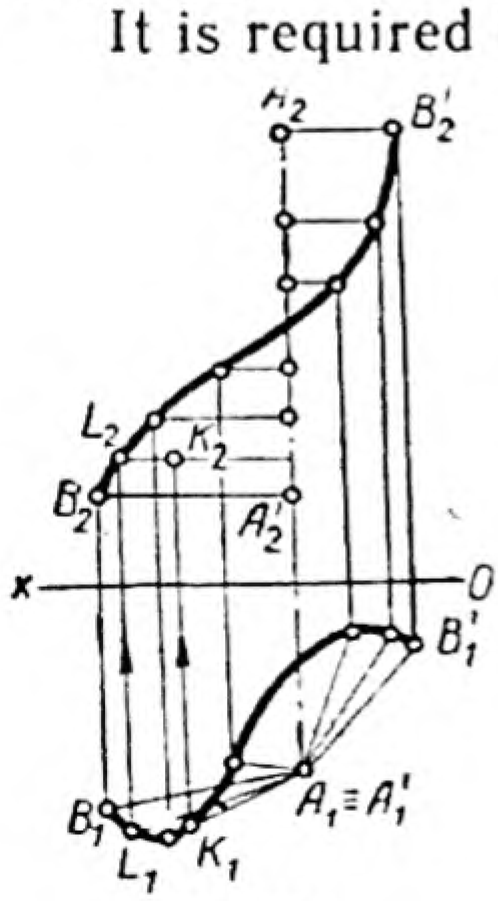


Fig. 439

8. The **oblique helicoid** is one whose generatrix is a straight line that intersects the axis at a constant angle but not at right angles. One point of the generatrix describes a cylindrical helix, while a second point slides along the axis.

If an oblique helicoid is cut by a circular cylinder having the same axis, an *oblique annular helicoid* is obtained (Fig. 442).

The surface of an oblique annular helicoid, which is used for screws with triangular and trapezoid threads, is limited by two cylindrical helical lines of the same pitch. If the generatrices *1-1′*, *2-2′*, etc., of this surface are extended, they intersect the axis *AB*, making with it the same angle α in space.

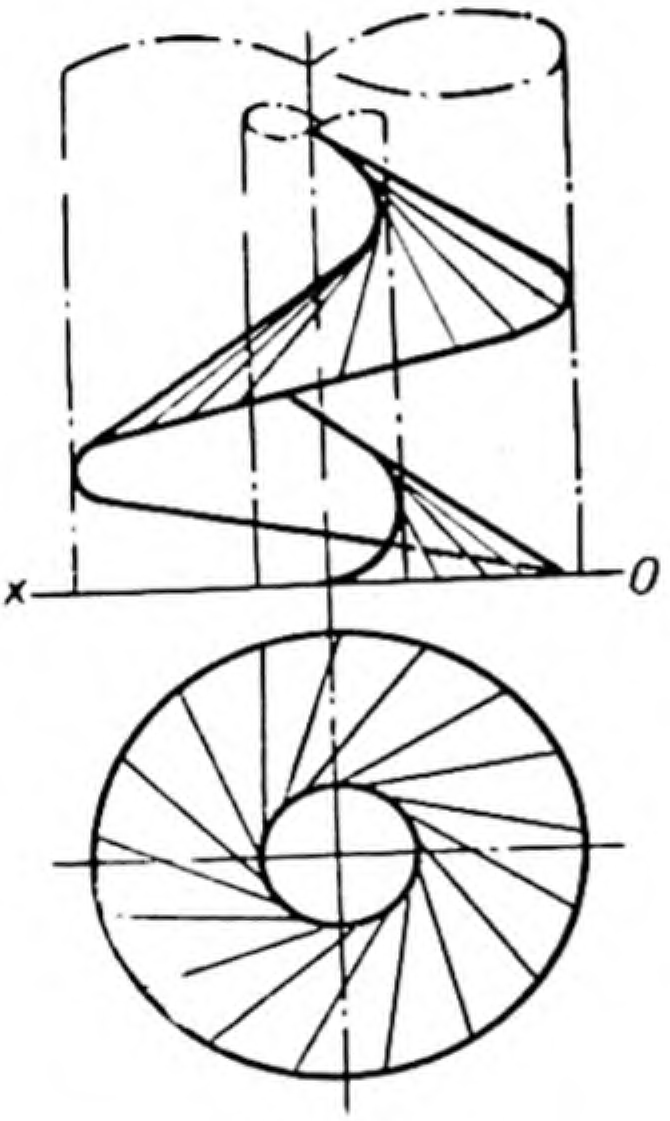


Fig. 440

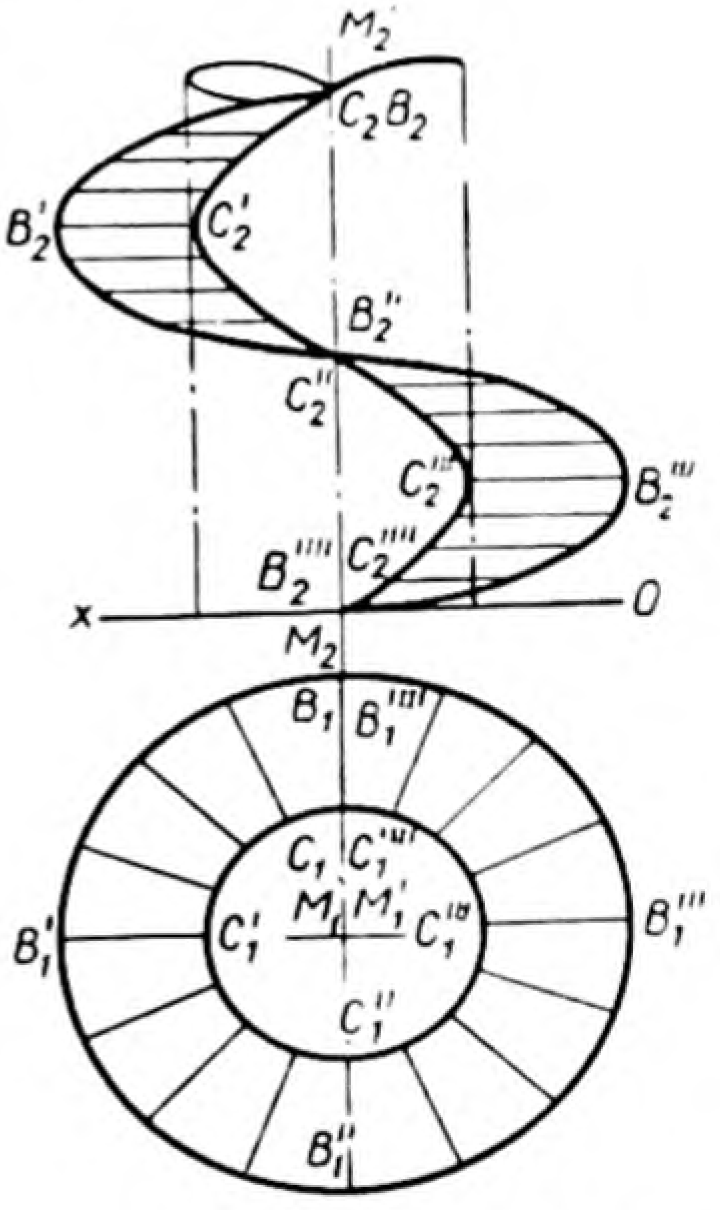


Fig. 441

To construct an oblique annular helicoid we first construct two auxiliary cylinders with a common axis *AB*. The circles, which are the horizontal projections of these cylinders, are divided into a number of equal parts, in this case into 12 parts. The division points are joined by straight lines 1_1-$1'_1$, 2_1-$2'_1$, etc. The lines 1_1-$1'_1$, 2_1-$2'_1$, etc., are the horizontal projections of the generatrix of the oblique annular helicoid in its various positions, i. e., of its elements. To find the frontal projections of these lines, we divide the pitch of the helix into the same number of equal parts. These twelve segments are laid off upwards from points O_2 and O'_2. The latter are the frontal projections of the extreme points of the generatrices which describe helices of the same pitch *h* on the surface of the cylinders.

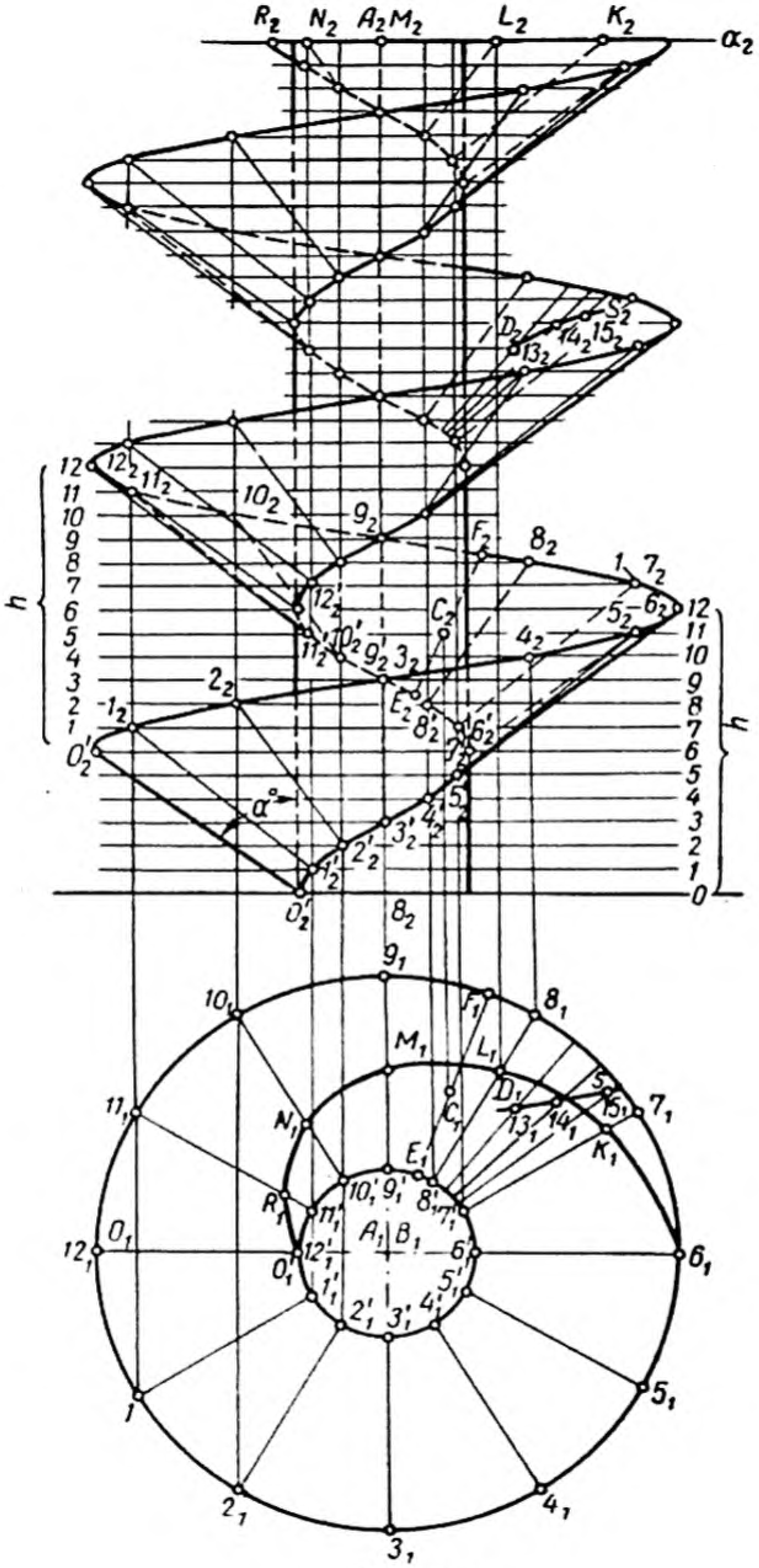

R_2
N_2
A_2M_2
L_2
K_2
α_2
D_2
S_2
13_2
14_2
15_2
12_2
11_2
10_2
9_2
F_2
8_2
7_2
6_2
5_2
4_2
3_2
2_2
1_2
C_2
E_2
$12'_2$
$11'_2$
$10'_2$
$9'_2$
$8'_2$
$7'_2$
$6'_2$
$5'_2$
$4'_2$
$3'_2$
$2'_2$
$1'_2$
$0'_2$
$\alpha°$
h
h
0
1
2
3
4
5
6
7
8
9
10
11
12
B_2
9_1
10_1
F_1
8_1
M_1
L_1
11_1
N_1
C_1
D_1
S_1
13_1
14_1
15_1
7_1
K_1
E_1
R_1
$10'_1$
$9'_1$
$8'_1$
$7'_1$
$11'_1$
O_1
12_1
$12'_1$
A_1
B_1
$6'_1$
6_1
O'_1
$1'_1$
$5'_1$
$2'_1$
$3'_1$
$4'_1$
1
5_1
2_1
3_1
4_1

Fig. 442

The line $O_2O'_2$ is drawn beforehand at the given angle α to the axis of the helicoid.

Verticals are erected from the division points of the major and minor circles. Horizontals are drawn through the dividing points of the pitch. The intersection of corresponding lines gives us points 1_2-$1'_2$, 2_2-$2'_2$, etc. By joining up these pairs of points by straight lines we obtain the frontal projections of the generatrices. Curves are passed through points 1_2, 2_2, . . . , 12_2, contained in the lateral surface of the major cylinder, and through points $1'_2$, $2'_2$, . . . , $12'_2$, contained in the lateral surface of the minor cylinder. These curves are two cylindrical helices and represent the outlines of an oblique annular helicoid.

The construction of the curve $RMNLK$, in which the oblique annular helicoid is cut by the horizontal plane α, consists in determining the points of intersection of a number of elements, or generatrices, with plane α. The frontal projections of these points are the points of intersection of the frontal projections of the elements with the trace-projection α_2 of plane α.

The horizontal projections R_1, N_1, M_1, L_1, K_1 of points R, N, M, L and K are obtained by dropping projectors from points R_2, N_2, M_2, L_2 and K_2 to intersect the horizontal projections of the respective elements.

The curve $RNMLK$, the projection of which on plane Π_1 is a true view, is a Spiral of Archimedes.

To construct the frontal projection of any point, such as C, contained in the surface of the helicoid, given the horizontal projection C_1 of this point, we first draw through point C_1 a radial line E_1F_1. We then construct the frontal projection E_2F_2 of this element and pass a projector through point C_1 to intersect E_2F_2 in point C_2.

Let the frontal projection of a curve, such as DS, contained in the surface of the helicoid, be given. It is required to find the horizontal projection of this curve. On the helicoid, in the region where the line DS must lie, we draw a series of elements. Then from points 13_2, 14_2, 15_2, in which the frontal projections of the generatrices intersect the given projection D_2S_2 of the curve, we drop projectors to the horizontal projections of the respective elements. This gives us points 13_1, 14_1 and 15_1 through which a curve is drawn. This curve is the horizontal projection of curve DS contained in the oblique annular helicoid.

§ 50. Surfaces of Revolution

1. *Surfaces of revolution* are those surfaces that may be generated by revolving a straight or curved line about a stationary axis. Two examples of surfaces of revolution are right circular cylinders and spheres.

The section of a surface of revolution by a plane perpendicular to the axis of revolution is a circle. The most commonly used method of constructing points on surfaces of revolution is based on this fact.

A body of revolution is shown in Fig. 443. Its surface is formed by the revolution of a plane curve *ABC* about the axis *MN* which

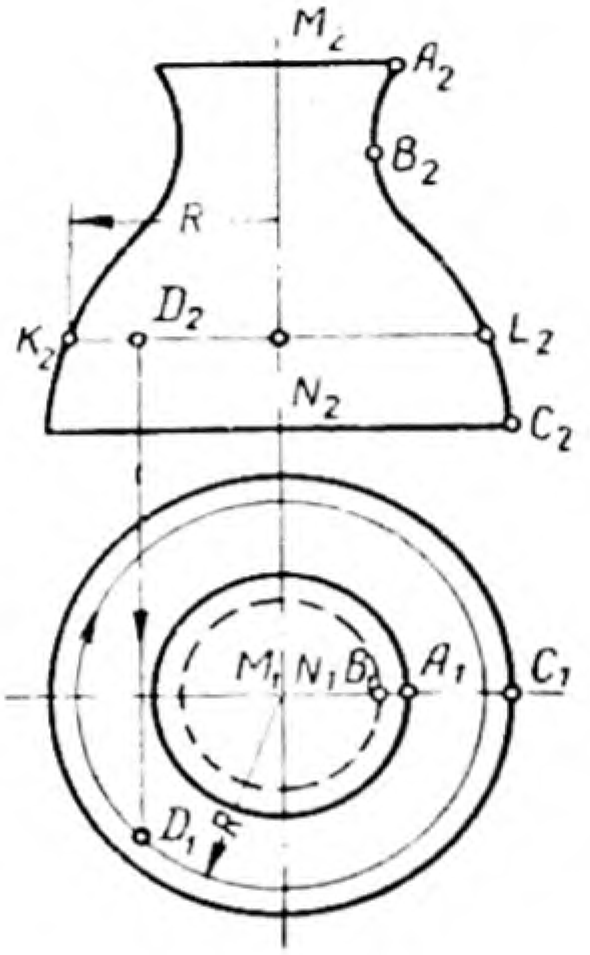


Fig. 443

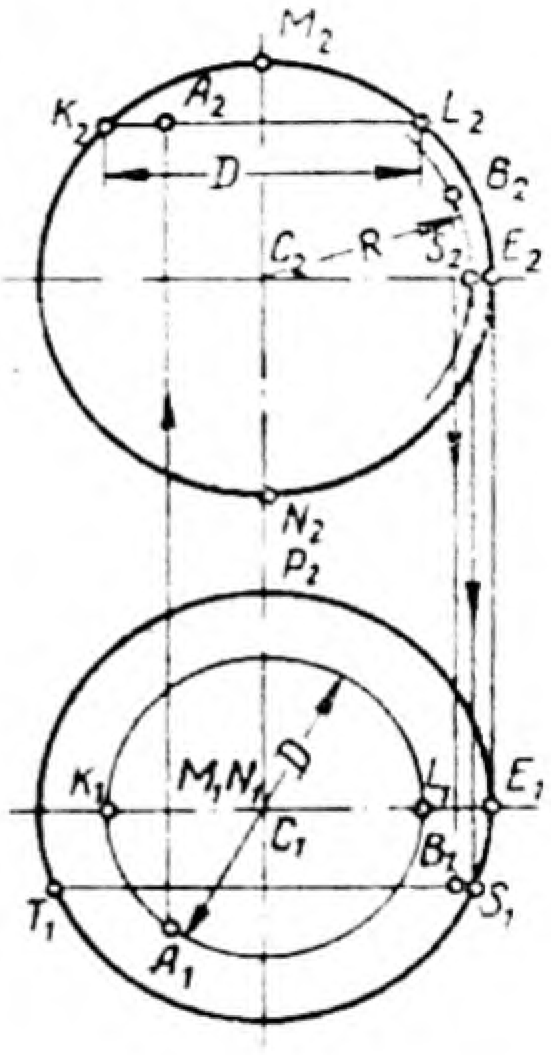


Fig. 444

is perpendicular to plane Π_1. Let us find the horizontal projection of point *D* belonging to this surface, if the frontal projection D_2 of point *D* is given.

Point *D* must lie on a circle, the plane of which is perpendicular to the axis of revolution *MN* and, so also perpendicular to plane Π_2. Therefore through point D_2 we draw a horizontal K_2L_2 which is the frontal projection of the required circle. Next, we describe a circle with point M_1N_1 as its centre and radius $R=\frac{1}{2}K_2L_2$. This is the horizontal projection of the circle on which point D_1 is found by dropping a projector from point D_2.

2. Given the horizontal projection A_1 of point *A* on the surface of a sphere, Fig. 444, construct its frontal projection. First draw a circle with its centre at C_1 through point A_1. Having obtained points K_1 and L_1 we then find the frontal projection K_2L_2 of this circle. This will be a line parallel to ground line Ox. A projector is then drawn through point A_1 to line K_2L_2. This gives us the frontal projection A_2 of point *A* on the surface of the sphere.

In order to construct a point *B* on the surface of this same sphere, given its frontal projection B_2, first we draw a circle through point B_2 with centre C_2. Having obtained point S_2 from it we draw a projector to this circle, which is the horizontal projection of the sphere. This gives us point S_1. Next a line S_1T_1 is passed through point S_1 parallel to line Ox. This is the horizontal projection of the circle with a radius CS. On it we find point B_1 with the

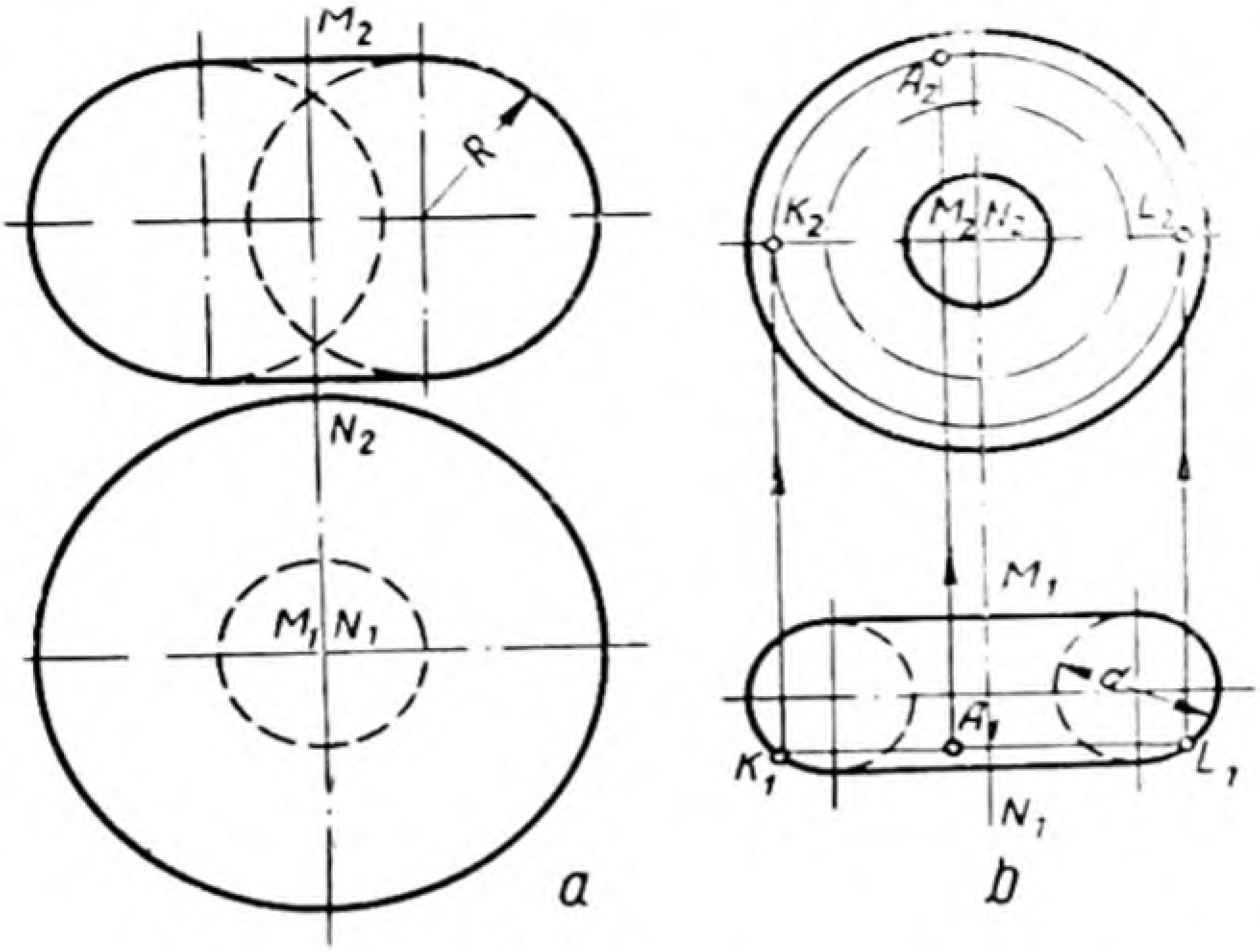


Fig. 445

help of a projector passed through point B_2. Point B_1 obtained in this way is the horizontal projection of point B on the surface of the sphere.

It should be noted that the circles KAL and BST may be considered as sections of a sphere cut by projecting planes drawn respectively through points A and B. Sections of a sphere cut by planes perpendicular to axis MN (Fig. 444) are called *parallels*. Sections cut by planes passing through the axis MN of the sphere are called *meridians*. The parallel cut by the plane passing through the centre of the sphere is the *equator*.

The meridian whose plane is parallel to the frontal plane of projection is called the *principal meridian*.

3. The surface represented in Fig. 445, *a* is obtained by the revolution of a circle of radius R about axis MN which intersects the circle, but does not pass through its centre. If we were to revolve a circle of diameter d (Fig. 445, *b*) about axis MN, also located in the plane of the circle, but not cutting it, we would obtain the surface called an *annular torus* or *tore*.

To construct a point on the surface of a torus it is sufficient to draw an auxiliary circle KL on the surface (Fig. 445, *b*). On the projections K_1L_1 and K_2L_2 of the circle points A_1 and A_2 are taken. These are the projections of point A belonging to the surface of the torus. Points on the surface of revolution, shown in Fig. 445, ***a***, are constructed in a similar manner.

4. The surface formed by the revolution of an ellipse about one of its axes is called an *ellipsoid of revolution,* or a *spheroid* (Fig. 446).

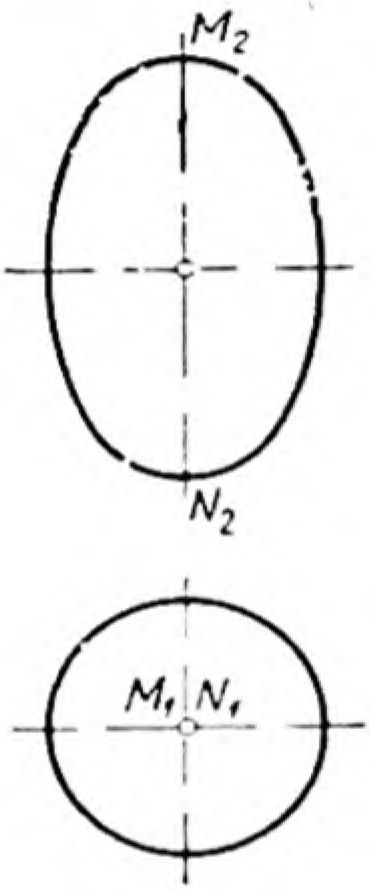


Fig. 446

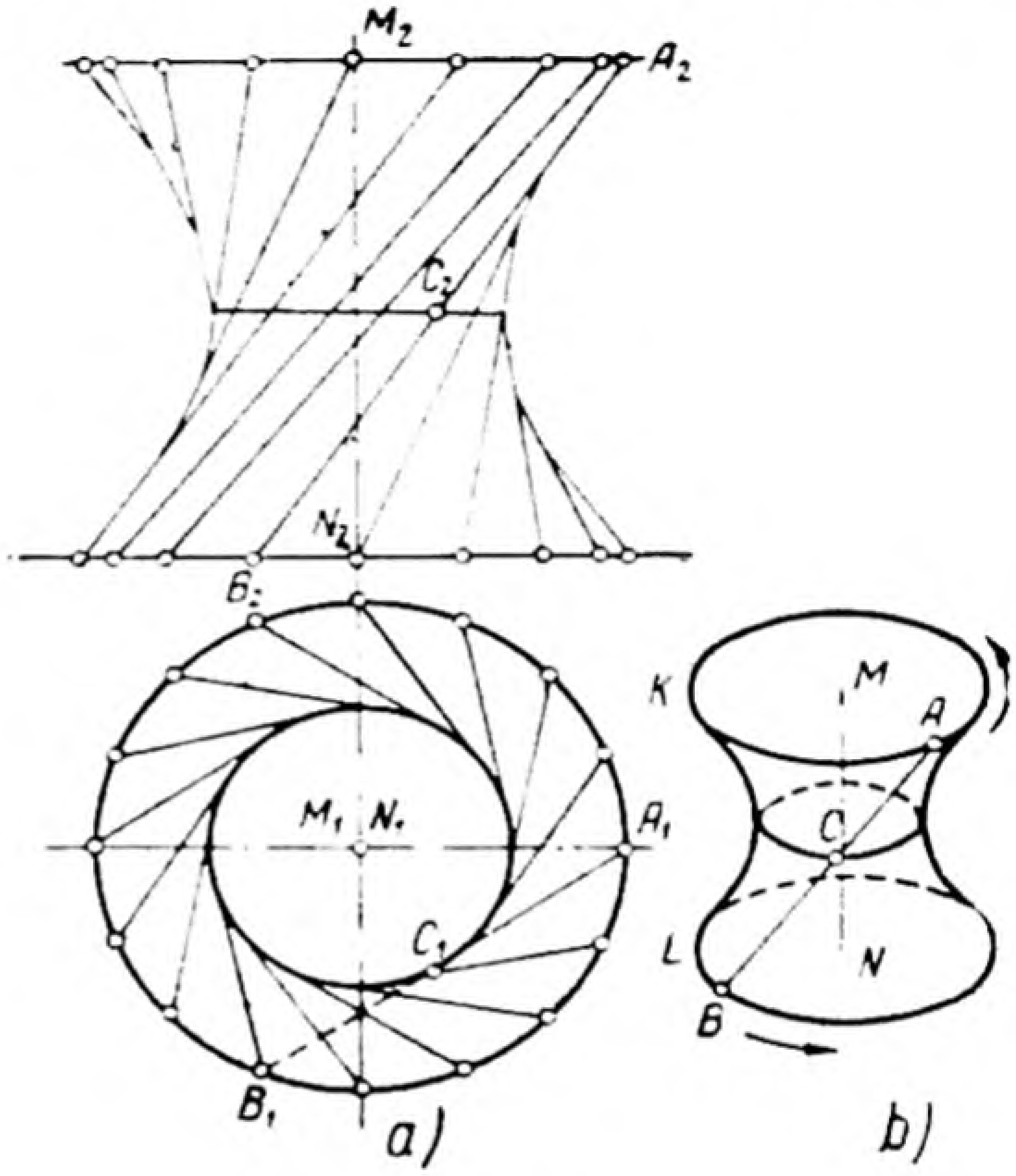


Fig. 447

5. *A hyperboloid of revolution of one nappe* is the surface generated by a line that rotates about the axis, but does not intersect it (Fig. 447).

All points of the generatrix AB of the hyperboloid of revolution of one nappe describe circles with their centres on axis MN. The circle described by the point C, nearest to the axis MN, is called the *neck*.

A hyperboloid of one nappe may also be obtained by revolving a hyperbola about an imaginary axis. In Fig. 447, *b* the imaginary axis is MN and the generatrix is the hyperbola KL.

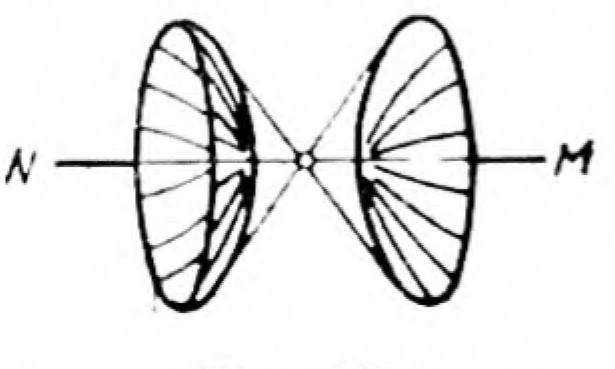


Fig. 448

The section of a hyperboloid of revolution cut by a plane perpendicular to its axis is a circle, and the section cut by a plane passing through the axis of the hyperboloid is a hyperbola.

The surface of revolution we have described is termed a hyperboloid of one nappe in order to distinguish it from the hyperboloid of two nappes, shown in Fig. 448. The latter is obtained by revolving a hyperbola about its real axis.

The single-nappe hyperboloid of revolution, which is a warped surface, finds many applications in engineering practice, for example, in the design of radio towers and supports for tanks, etc.

6. If a parabola is rotated about its axis the surface obtained is called a *paraboloid of revolution*.

Problems and Exercises

1. What is meant by single-curved surfaces and double-curved surfaces? Give examples of such surfaces.
2. Name and define developable single-curved surfaces.
3. What are undevelopable surfaces?
4. Give the definition of the following surfaces: a cylindroid, a hyperbolic paraboloid, a conoid, a helical conoid, an annular helical conoid, an oblique helicoid and an oblique annular helicoid.
5. What are surfaces of revolution? Give examples of these surfaces.
6. What is the usual procedure for constructing points and lines on surfaces?
7. Construct a right L. H. helicoid.
8. Construct an oblique L. H. helicoid.

§ 51. Cylindrical Screws

1. The surface generated by the movement of a plane figure over the surface of a cylinder along a helix contained in it, so that the plane of the figure always passes through the axis of the cylinder, is called a *helical surface*. The body limited by this surface is called a *screw*. The threads of screws are of different shapes, depending on the form of the generating profile, i. e., on the shape of the plane figure being moved. Among the threads most commonly used are those of square, rectangular, triangular, trapezoidal and semicircular cross sections.

In Fig. 449, *a* the construction of one turn of a triangular thread is shown.

Threads may be right-hand (R. H.) and left-hand (L. H.). When a R. H. thread is generated the generating profile moves counter-clockwise, when the screw is viewed from above. In the drawing this means that the front, visible side of the thread rises from left to right. When a L. H. thread is generated the reverse is true.

When inserting a screw into a nut with a R. H. thread the screw is rotated clockwise. When inserting a screw into a nut with a L. H. thread the screw is turned counter-clockwise.

If the screw is generated by the movement of only one profile, as in Fig. 449, *a*, it is called a *single-thread,* or a *single-cut screw.*

If two or more similar generating profiles are moved, assuming them joined together, *double-thread screws, triple-thread screws* and so on are obtained.

When drawing a double-thread screw it is necessary to construct helical lines equal in number to the number of vertices of the two generating profiles. For example, when constructing a double-thread screw (Fig. 449, *b*) it is necessary to construct eight helical lines, corresponding to the number of vertices of the two generating squares which are shown by hatching.

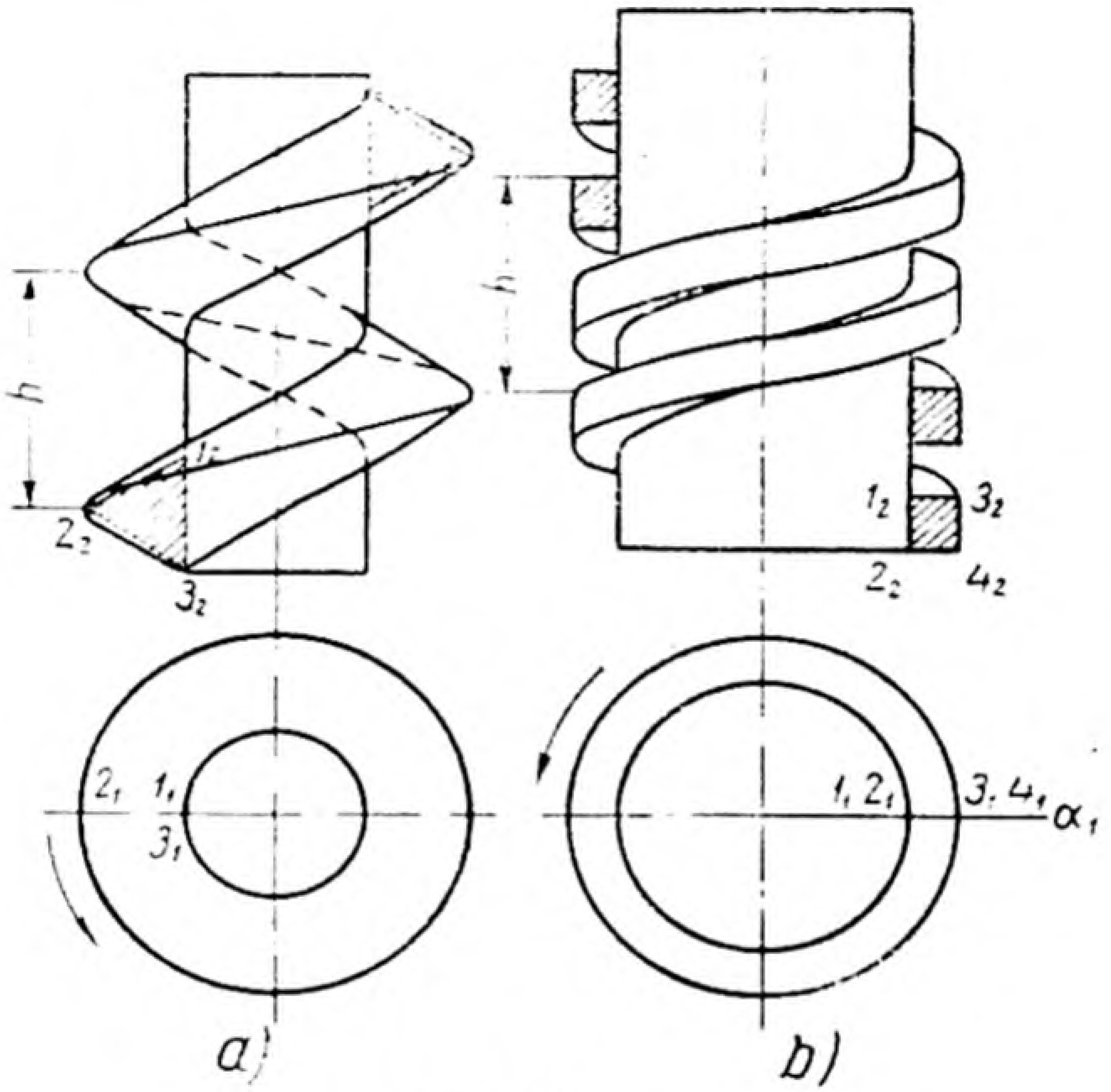


Fig. 449

2. A screw and the corresponding nut, having a R. H. single-cut rectangular thread, are shown in Fig. 450. Since the cross section *1-2-4-3* of the protruding thread of the screw is the same as the cross section of the recess *1-2-4-3*, the sum of the dimension 3_2-1_2 of the protrusion and the dimension 1_2-3_2 of the recess is equal to the pitch h of the screw.

The protrusions and recesses consist of cylindrical surfaces, exterior and interior, generated by the movement of the sides *1-3* and *2-4* of the generating rectangles, and of two right helical surfaces, or right helicoids, obtained by the movement of the sides *1-2* and *3-4* of the rectangles.

The construction of the screw and nut is carried out by the drawing helices with a pitch h through points *1, 2, 3* and *4*. For this purpose two coaxial cylinders are constructed, the radii of the bases of which differ by the depth *1-2* of the thread recess. Having divided the circle of the base of the cylinders into a number of equal parts, say six, we divide the pitch h into the same number of equal parts and construct four helical lines, i. e., two on the inside cylinder and two on the outside. Having determined the visible outline of the protrusions and recesses of the thread we draw them in with continuous lines. The hidden outlines are usually shown only within the limits of one or two screw pitches h.

The cross-section of the screw cut by the horizontal plane α is also shown in Fig. 450. This plane cuts the exterior and interior cylinders and two right helical surfaces. The section of the cylindrical surfaces are arcs of circles, the sections of the right helical surfaces are the diametrically opposite segments KL and MN. The horizontal projections K_1, L_1, M_1 and N_1 of points K, L, M and N are determined by drawing projectors from points K_2, L_2, M_2 and N_2, in which the frontal trace of the cutting plane cuts the frontal projections of the helices. These projectors intersect the horizontal projections of the lateral cylinder surfaces in the points required.

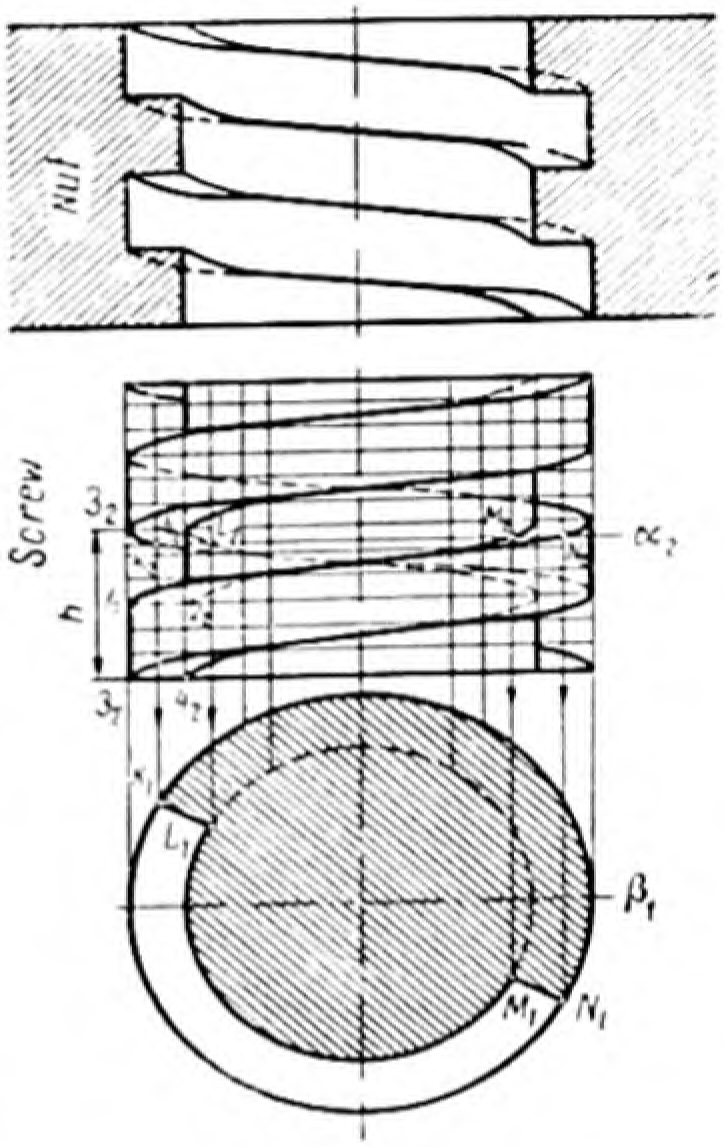


Fig. 450

The nut is represented in cross-section cut by a frontal plane β passing through the axis of the screw and nut. On this cross-section the visible part of the thread is seen as rising from right to left. This corresponds to the way in which the rear, hidden part of the R. H. thread of the screw rises, i. e., from right to left. The hidden lines shown in the cross-section of the nut are those parts of the rear helices which are covered by other parts of the thread.

3. A right double-thread, screw with a thread of triangular section is shown in Fig. 451. The screw is cut by a horizontal plane perpendicular to its axis.

The circles of the bases of the cylinders and the screw pitch h are divided into 12 equal parts. Then, the six helices, corresponding to the number of vertices of two generating triangles, are constructed.

The surface of this screw consists of two oblique helicoids. The frontal projections of the visible outlines on these surfaces are obtained by drawing tangents to the frontal projections of the exterior and interior helical lines. In Fig. 452 these outlines are shown separately on an enlarged scale.

To construct the section of the screw cut by plane α auxiliary planes β and γ are drawn through the axis of the screw. These planes intersect each protrusion in triangles, the projections of which are shown in Fig. 451. The points K and L, in which plane α intersects sides MN and $M'N'$ of these triangles, are points contained in the lines of intersection of the screw and plane α. Having constructed the horizontal projections K_1 and L_1 of these points and joining the points 6_1, K_1, L_1 and 6_1 by a continuous line, we obtain the horizontal projection of one of the branches of the line of intersection of the screw and plane α. The other three branches of this curve, which are Archimedean spirals, are constructed in a similar manner. The line of intersection has two axes of symmetry. The L. H. and R. H. parts of intersection are symmetrical because plane α cuts a double-thread screw.

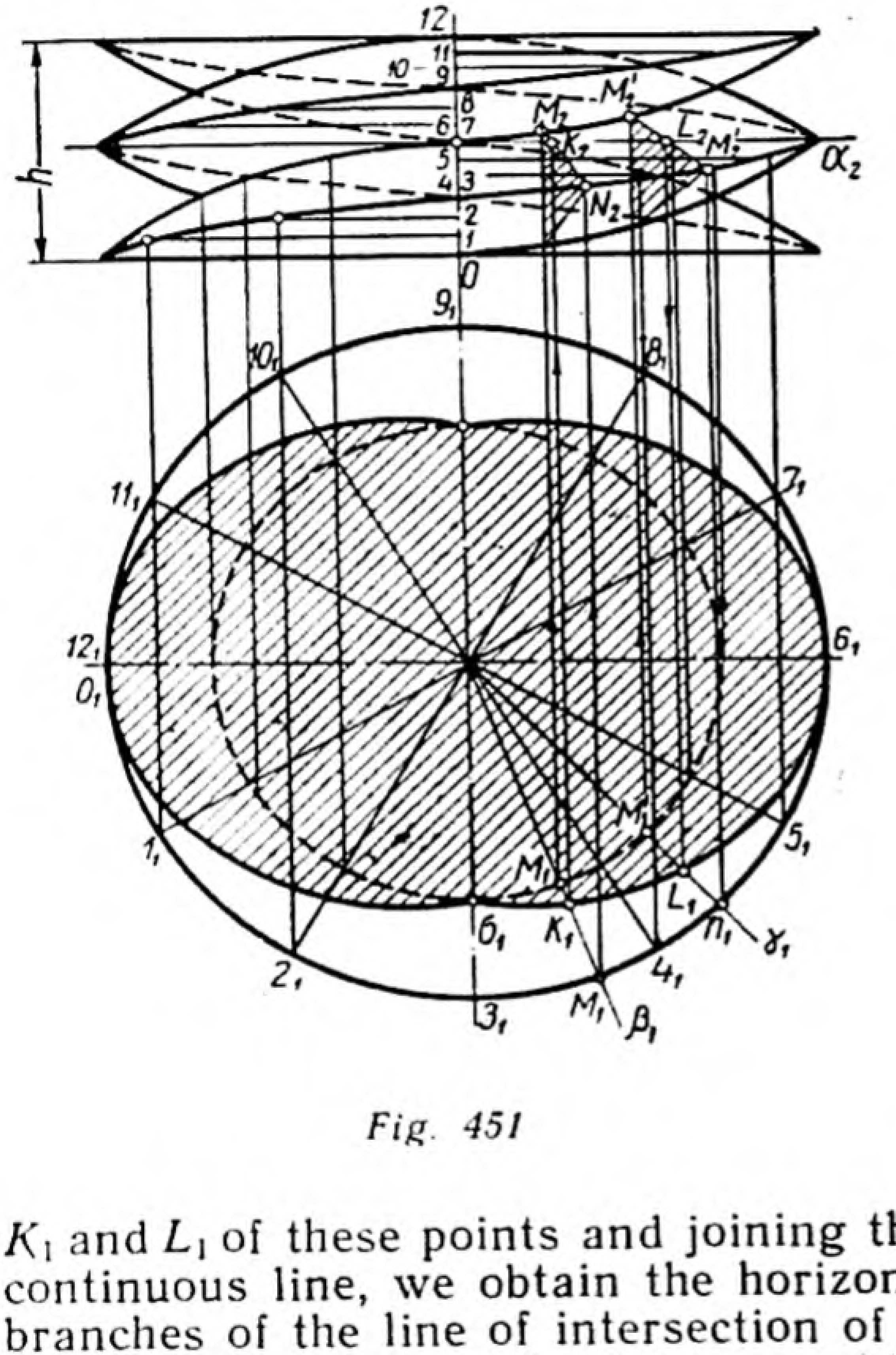


Fig. 451

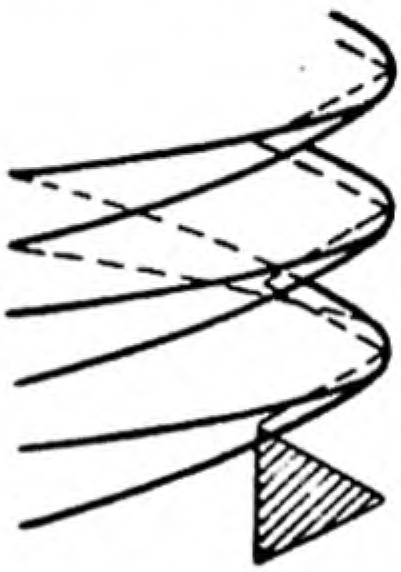


Fig. 452

4. A single-cut R. H. screw, the cross section of the thread of which is an equilateral trapezoid, is represented in Fig. 453. The surface of such a screw is limited by two cylinders and two oblique helicoids. The cross-section of the screw given by cutting plane α is also constructed with the help of auxiliary planes passing through the axis of the screw. The points K, L, Q and S belonging

to the outline of the section are given by the intersection of plane α with the sides MN, $M'N'$, $M''N''$, $M'''N'''$ of the trapezoid along which the screw protrusions are intersected by the auxiliary axial planes. Since the screw in this case is single-cut, the intersection obtained is limited by arcs of two circles and two branches of the Archimedean spiral.

5. A screw and nut, with a thread of trapezoid section, are shown in Fig. 454. There are two generating trapezoids. The direction of the turns is R.H. The surface of the screw consists of two cylinders, right helicoids and oblique helicoids. The cross-section of the screw and nut given by the cutting plane α is constructed with the help of auxiliary planes β, β', β'' and β''' which are passed through the axis of the screw. The circular parts of the outline of the section correspond to the parts of the right helicoid cut by plane α. The curves, which are Archimedean spirals and lie between the outer and inner circles, are the sections of the oblique helicoids cut by plane α.

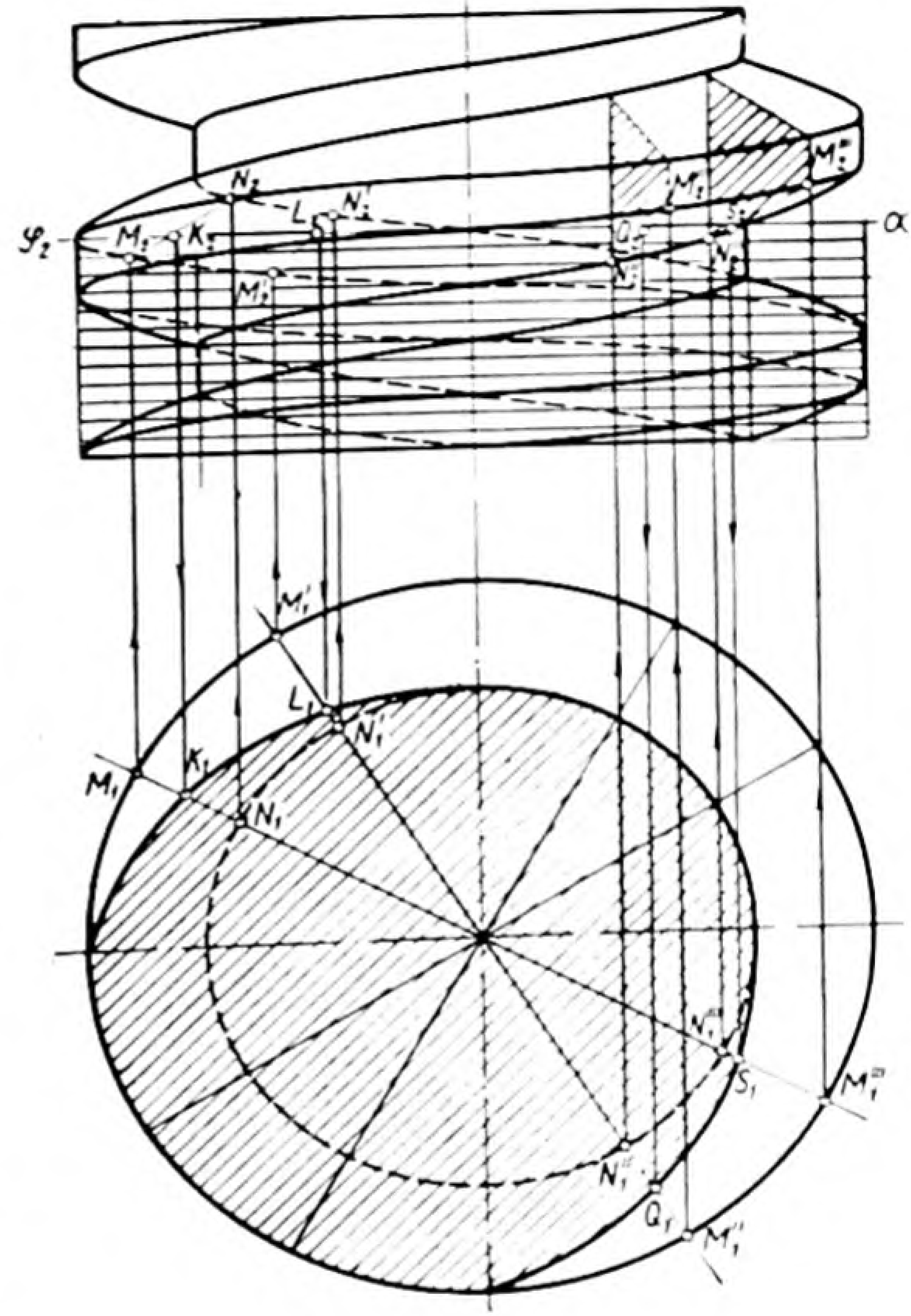
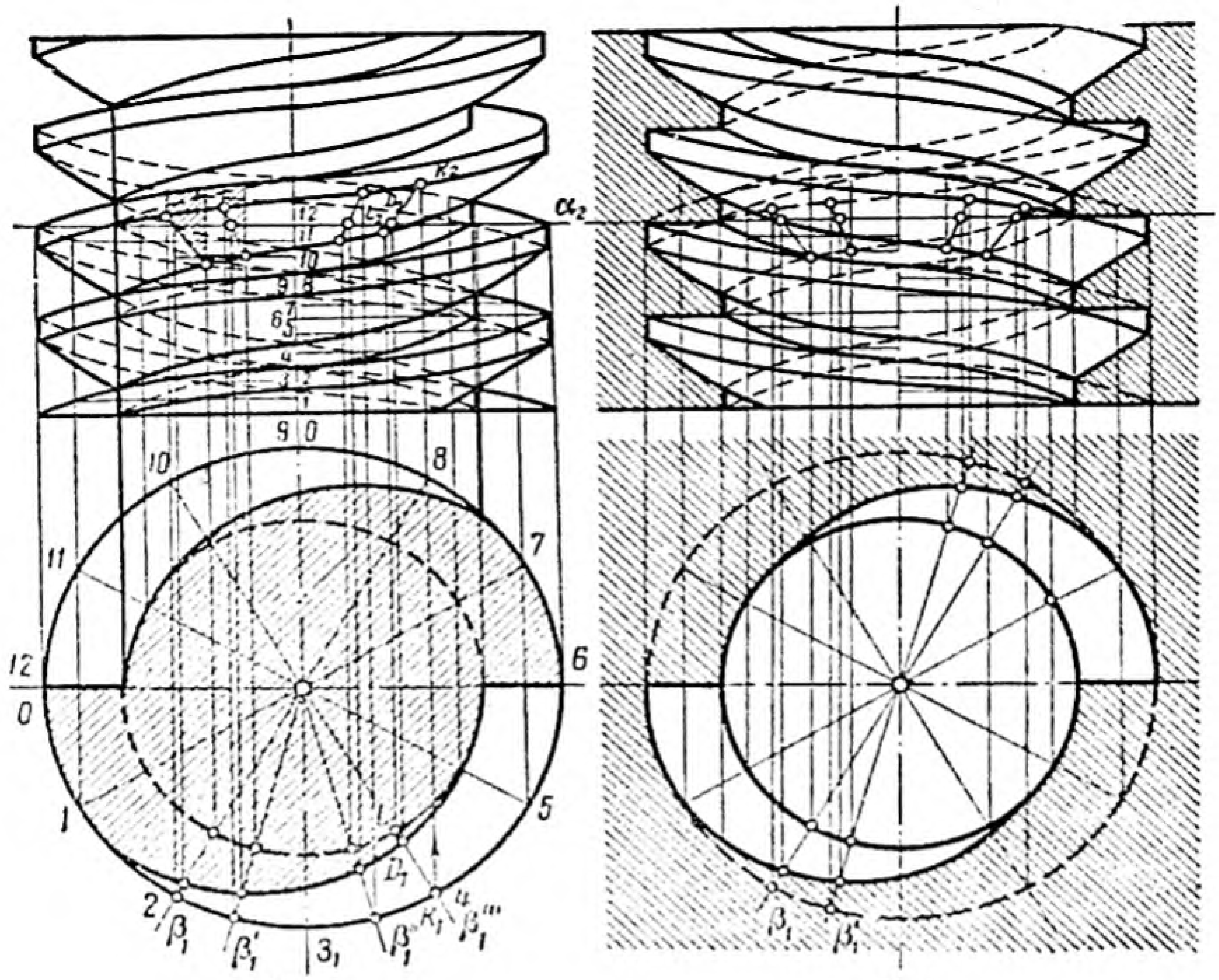


Fig. 453

6. A screw and nut with a R. H. triple-turn trapezoid thread is shown in Fig. 455. The cross-sections of the screw and nut given by plane α consist of three equal parts symmetrically located.

In the section of the nut, seen on the left of Fig. 455, the exterior helix, which is covered by the helical surface in front of it, is shown as a broken line.

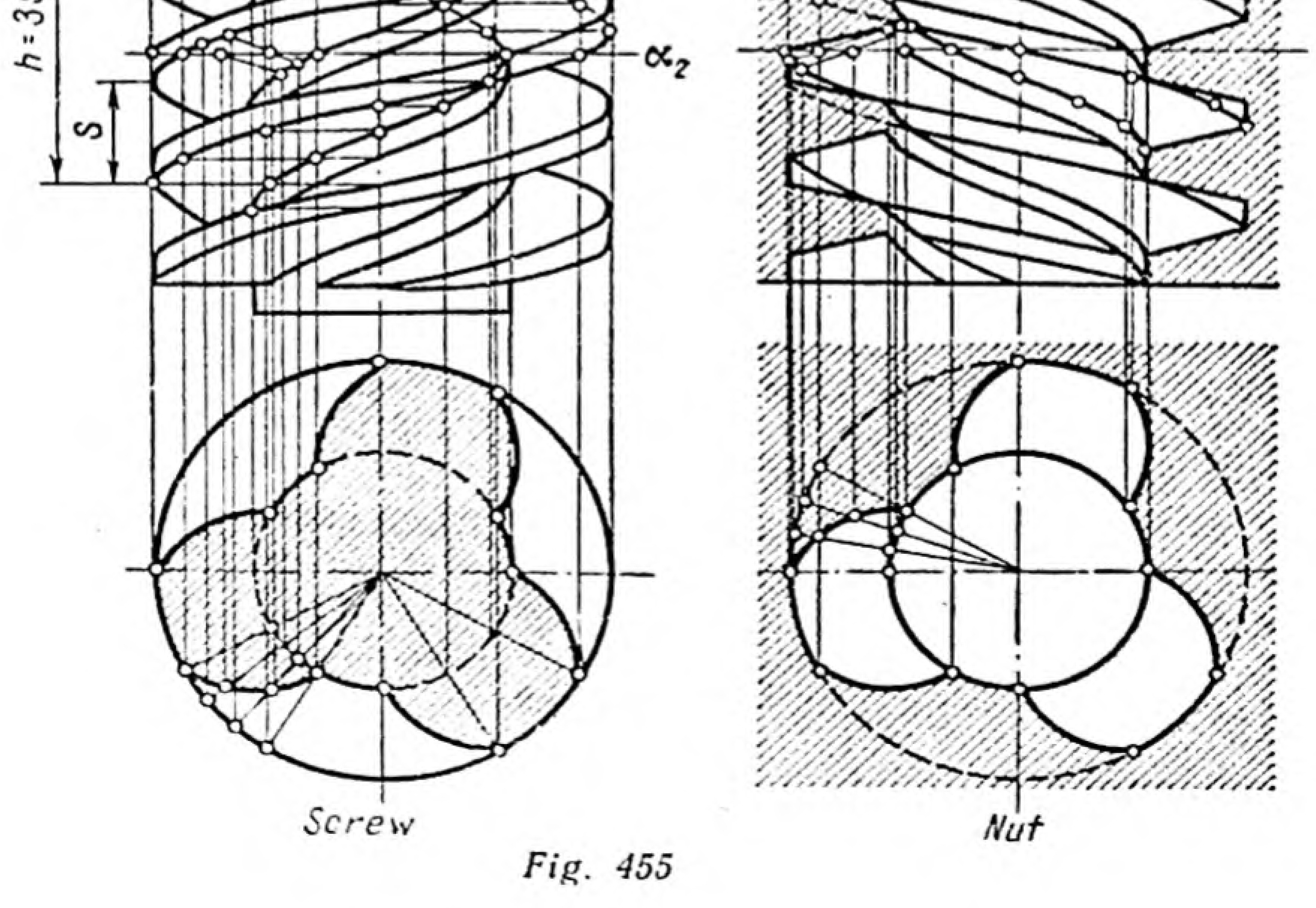


K_2
α_2
12
11
10
9
8
7
6
5
4
3
2
1
0
β_1
β_1'
3_1
β_1''
K_1
β_1'''

Screw

Nut

Fig. 454



h = 3s
S
α_2

Screw

Nut

Fig. 455

§ 52. Planes Tangent to Curved Surfaces

1. A plane may be tangent to a curved surface in one or more points, depending on the form of the surface. In the first case the tangent plane may be determined by drawing through the point of tangency, taken on the curved surface, two straight lines respectively tangent to two curved lines passing through this point and lying on the curved surface, for example, to two circles on the surfaces of a sphere (Fi. 456, *a*). An example of the second case is a plane which is tangent to a single-curved surface, such as a cylindrical surface, on which all the points of contact lie on one line, i. e., on a generatrix of the cylinder. This is the *line of tangency* (Fig. 456, *b*). The line of tangency is one of the lines determining the tangent plane. A second line of this plane is the tangent to any curved line on the given surface, such as the circular base of the cylinder.

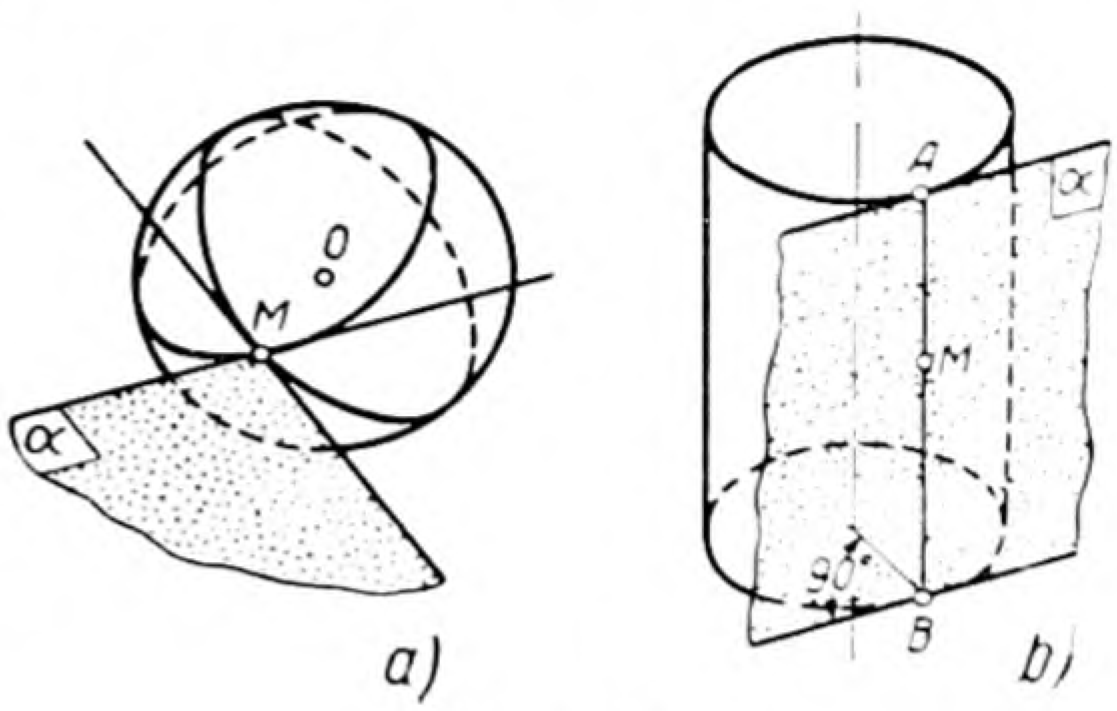


Fig. 456

2. Let it be required to draw a plane tangent to a cylindrical surface, passing through a given point *M* on the surface (Fig. 457).

The generatrix *AB* passes through point *M*. Through point *B*, i. e., the horizontal trace of this generatrix, draw line *CD* tangent to the circle of the base of the cylinder. The plane determined by the intersecting lines *AB* and *CD* is the required tangent plane.

The line *CD* is the horizontal trace of the tangent plane. To construct the frontal trace, which is not shown in the drawing, it is necessary to find the frontal traces or piercing points of two lines

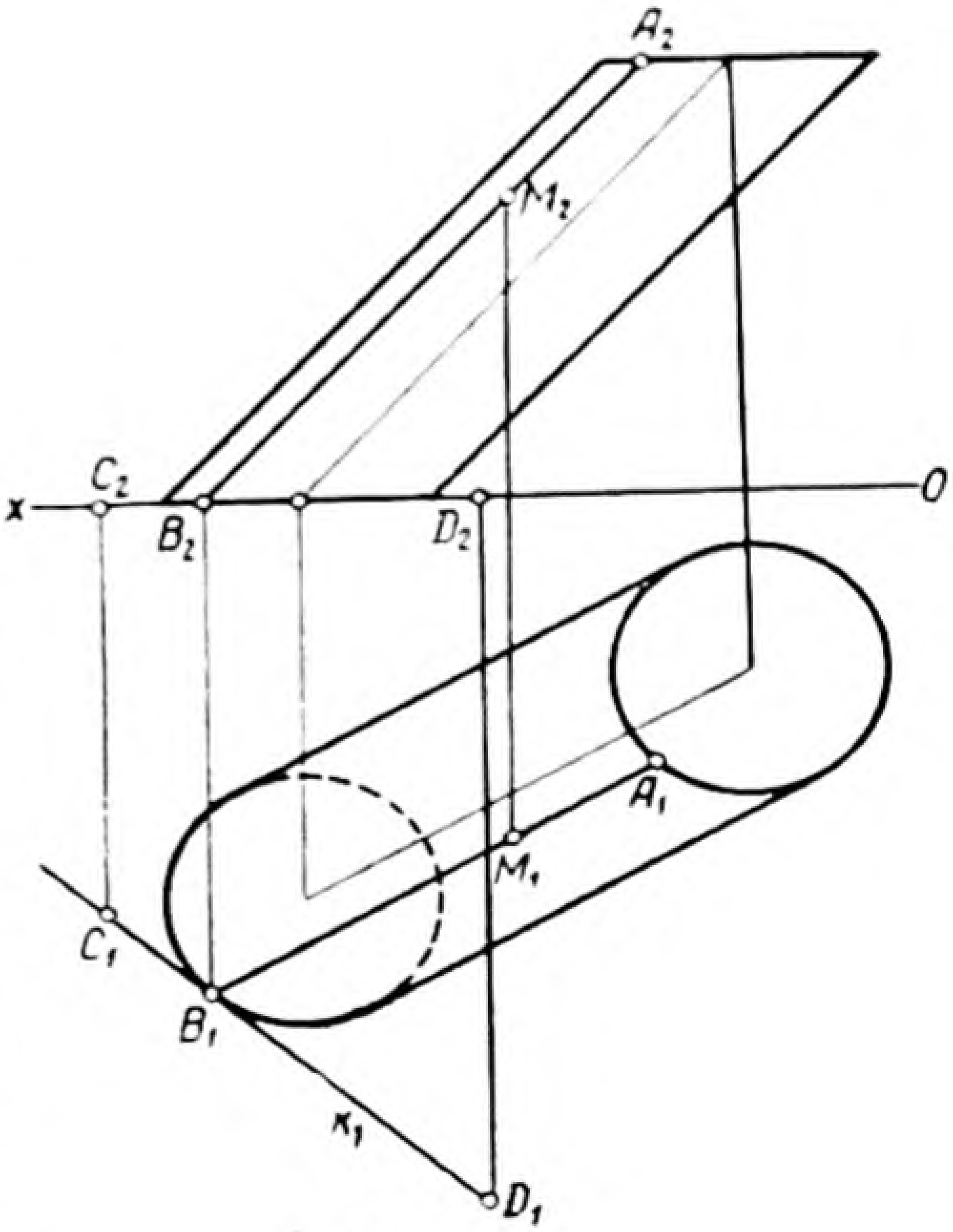


Fig. 457

contained in the plane, for example, the frontal trace of the generatrix AB and of a horizontal passing through point M. The frontal trace of the plane will pass through these frontal piercing points.

The plane tangent to the surface of a cone and passing through point M on the surface is constructed in a similar way (Fig. 458).

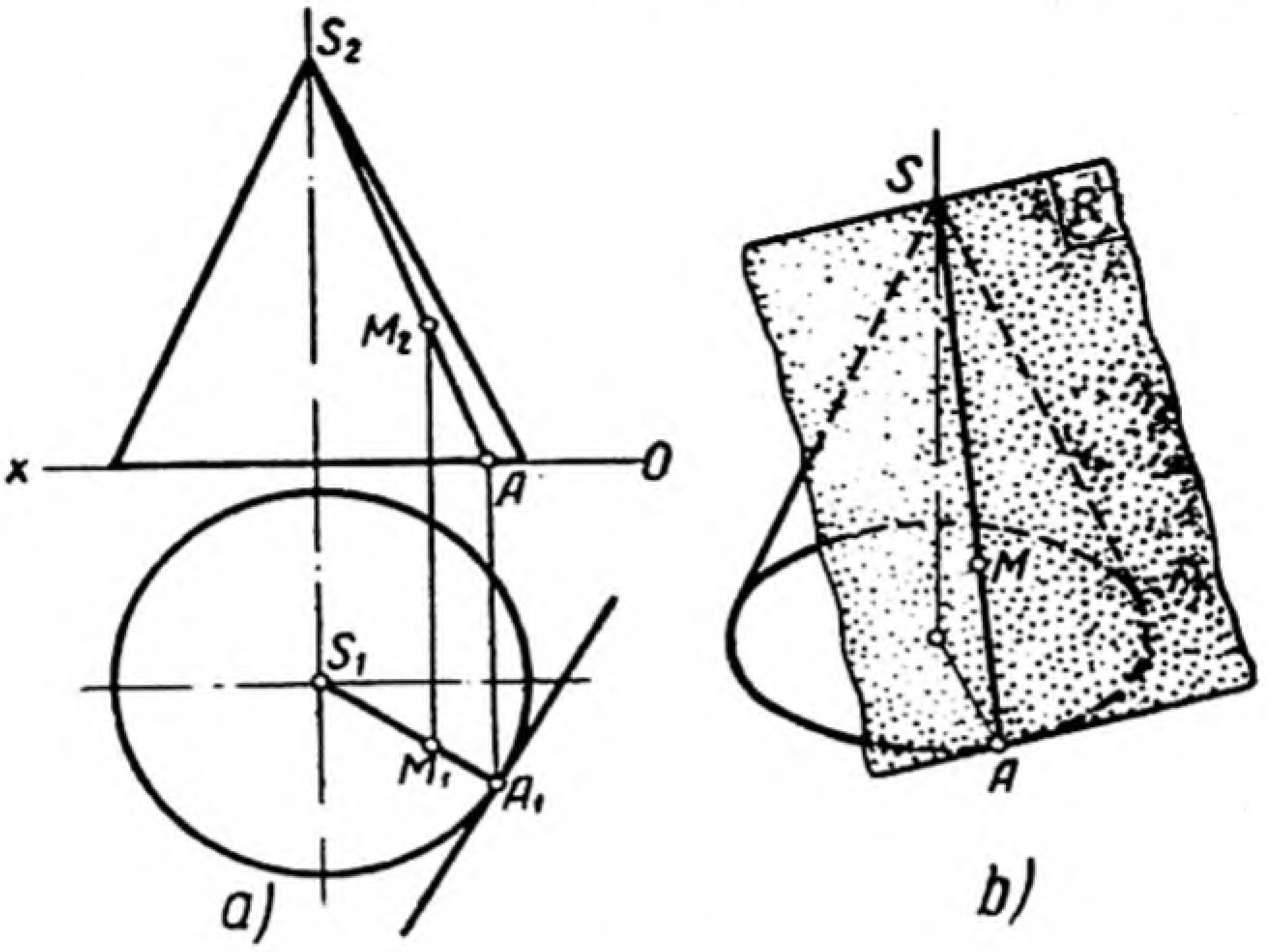


Fig. 458

3. A plane tangent to a spherical surface is perpendicular to the radius of the sphere at the point of tangency. Therefore, to construct a plane tangent to the sphere at a given point M on its surface (Fig. 459), it is sufficient to draw through point M a horizontal AB and frontal ED perpendicular to radius CM of the sphere. The plane defined by these intersecting lines will be the required tangent plane.

4. Let it be required to draw a tangent plane to a surface of revolution and passing through a given point M on the surface (Fig. 460). The method consists in drawing two curves on the surface of revolution through point M, the tangents to these curves at point M determining the required plane.

Through point M we draw a parallel, i. e., a horizontal section, and the tangent AB to this section at point M. Then a meridian is drawn. The frontal projection of this meridian is not shown in the drawing (Fig. 460). At point M a tangent to the curve should be drawn. In order to draw this tangent we can consider that the meridian, together with point M, is turned about the axis of the surface of revolution until it takes up a position parallel to plane Π_2. Point M will then occupy the position M'. The meridian will coincide with the principal meridian, i. e., with the generatrix to the right of the surface. Next the tangent CM' to the curve at

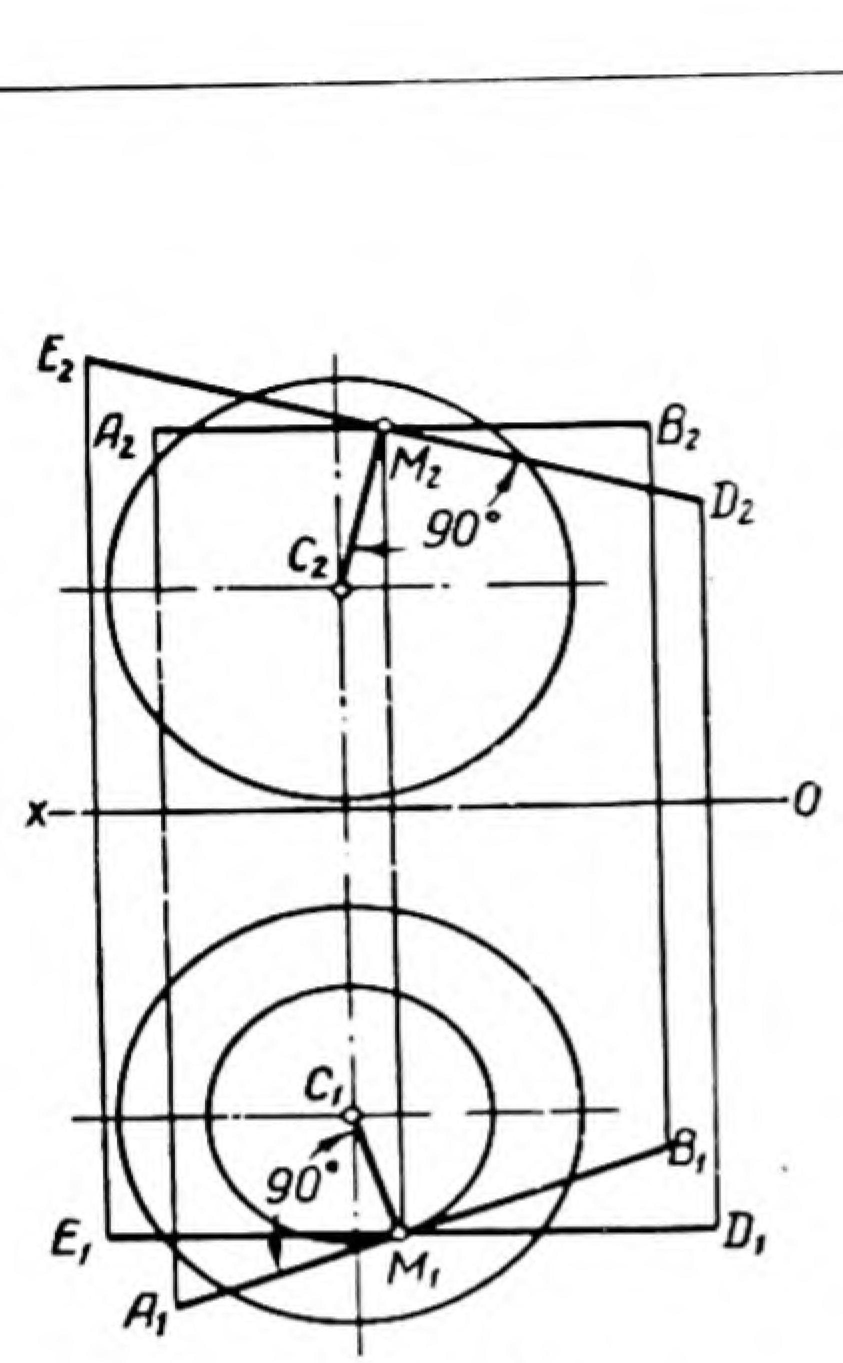

E_2
A_2
M_2
B_2
D_2
90°
C_2
x
0
C_1
90°
B_1
E_1
D_1
A_1
M_1

Fig. 459

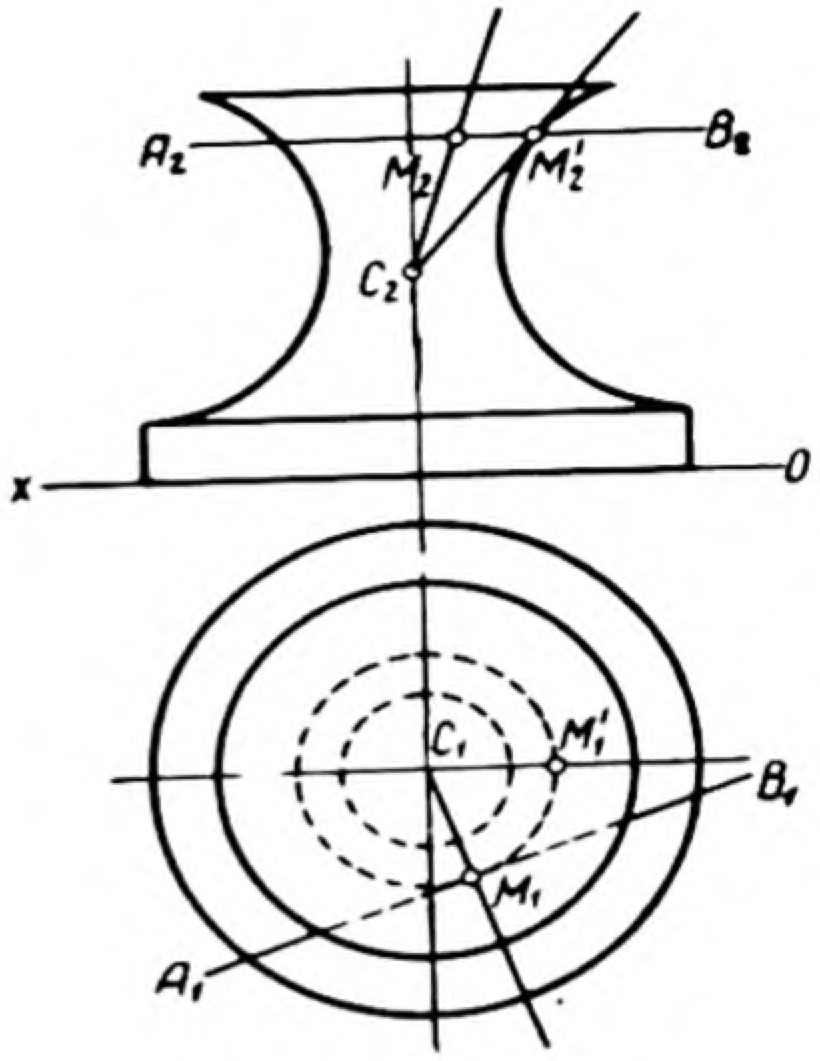

A_2
M_2
M_2'
B_2
C_2
x
0
C_1
M_1'
B_1
M_1
A_1

Fig. 460

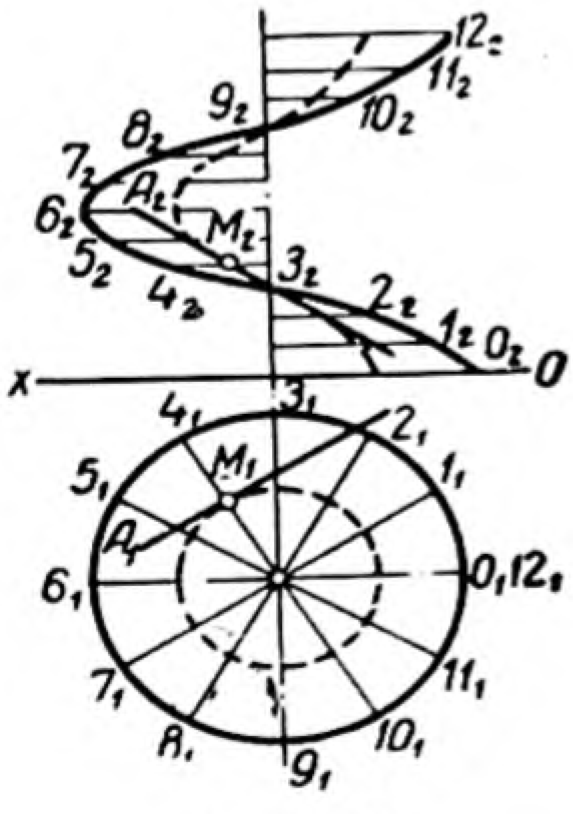

12_2
11_2
9_2
10_2
8_2
7_2
A_2
6_2
M_2
3_2
5_2
4_2
2_2
1_2
0_2
x
0
4_1
3_1
2_1
M_1
1_1
5_1
A_1
6_1
$0_1 12_1$
7_1
11_1
8_1
10_1
9_1

Fig. 461

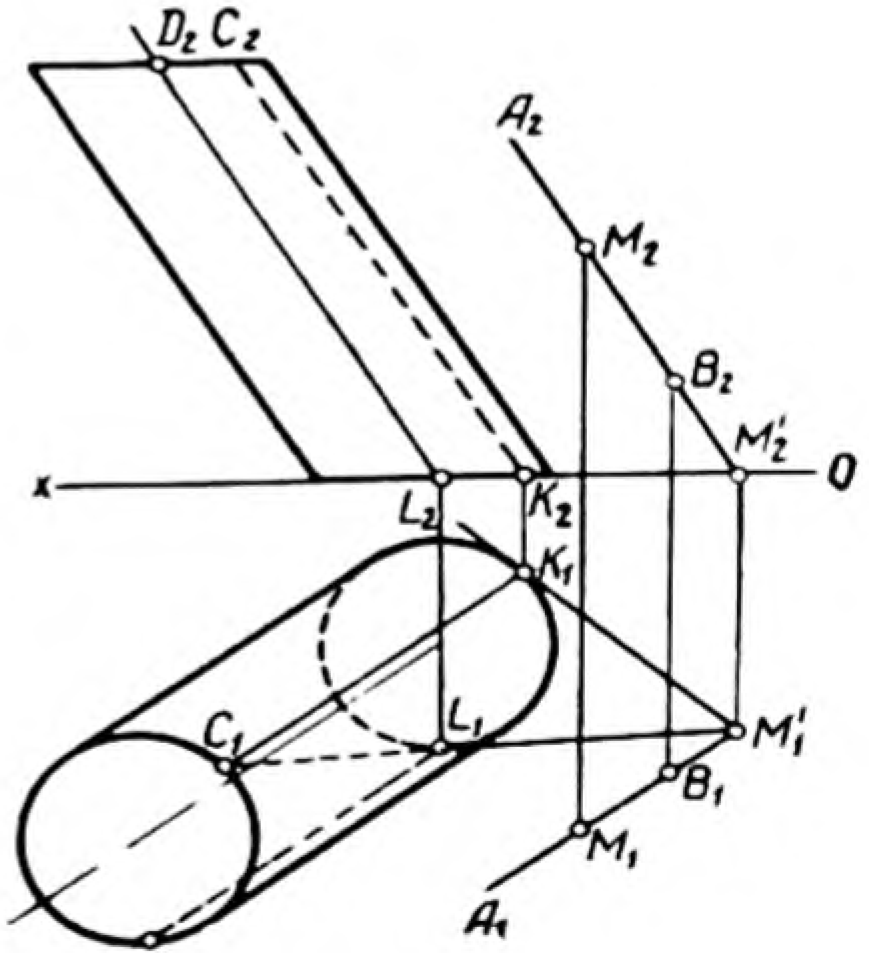

D_2 C_2
A_2
M_2
B_2
M_2'
x
0
L_2
K_2
K_1
C_1
L_1
M_1'
B_1
M_1
A_1
D_1

Fig. 462

point M' is drawn. With the reverse revolution of line CM' point C remains stationary, since it lies on the axis of revolution. The intersecting lines AB and CM define the required plane.

5. The construction of a plane tangent to a helical conoid in a given point M is seen in Fig. 461.

The generatrix $M4$ is one of the two straight lines determining the required plane. As the second line AM we take the tangent to the coaxial helix, passing through point M.

6. Let it be required to construct a plane tangent to the surface of a cylinder, passing through a point M lying outside the surface (Fig. 462). The required plane is determined by the line AB drawn through point M parallel to the generatrices of the cylinder and by the tangent KM to the horizontal trace of the surface of the cylinder drawn through the horizontal piercing point M' of line AB. The line of tangency on the surface of the cylinder will be generatrix KC. Since a second tangent LM' may be drawn to the circle of the base from point M' this problem has two solutions. The tangents KM' and LM' are the horizontal traces of planes tangent to the surface of the given cylinder.

7. The construction of two planes tangent to the surface of a cone and passing through a point Q not contained in this surface is shown in Fig. 463. By joining point Q with the vertex S of the cone we find the horizontal piercing point M of line SQ. Through this point M we draw two tangents to the circle of the base of the cone. Lines M_1K_1 and M_1F_1 are the horizontal traces of the planes which pass through point Q and are tangent to the surface of the cone. Each of these traces together with line SM determines one of the tangent planes.

8. Let it be required to pass through a point Q a plane tangent to the surface of a sphere (Fig. 464).

This problem is solved by the construction of a conical surface with its vertex in point Q and described about the given sphere. A straight line is passed through the centre of the sphere C and point Q. Next we find the true length C_4Q_4 of this line CQ on a new plane of projection Π_4. Having constructed the projection of the sphere on plane Π_4, through point Q_4 we draw tangents to this projection. These tangents are the projections on plane Π_4 of the limiting elements of a cone described about the sphere. By substituting plane Π_1 by plane Π_5 perpendicular to the axis of the cone, we obtain a circle with centre Q_5. This is the projection of the cone on the new plane Π_5.

Any plane tangent to the surface of this cone will be tangent to the given sphere and will pass through point Q. One of these planes is shown, i. e., that determined by triangle QNK, of which the side QN is a generatrix of the cone and the side NK a tangent to the base of the cone. The point of tangency L of this plane on the surface of the sphere lies on the circle passing through points A and B. All the generatrices of the auxiliary cone touch the surface of the given sphere on this circle.

The plane of the triangle QNK can be constructed first in the system of planes of projection Π_4-Π_1, and then in the system of planes of projection Π_1-Π_2.

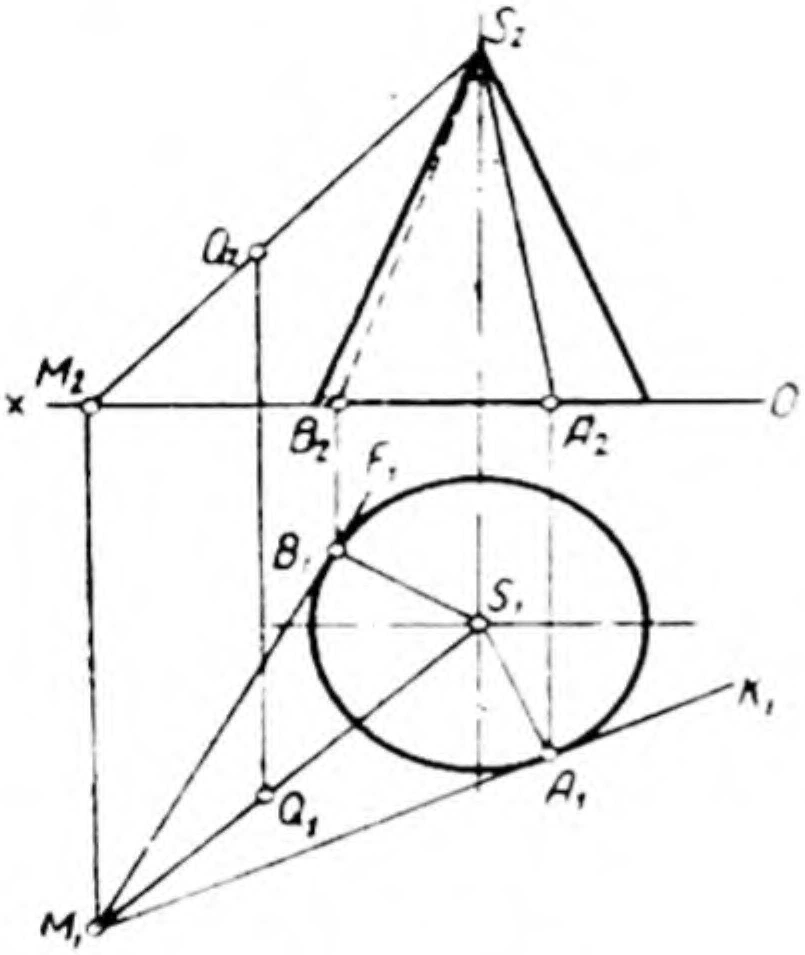


Fig. 463

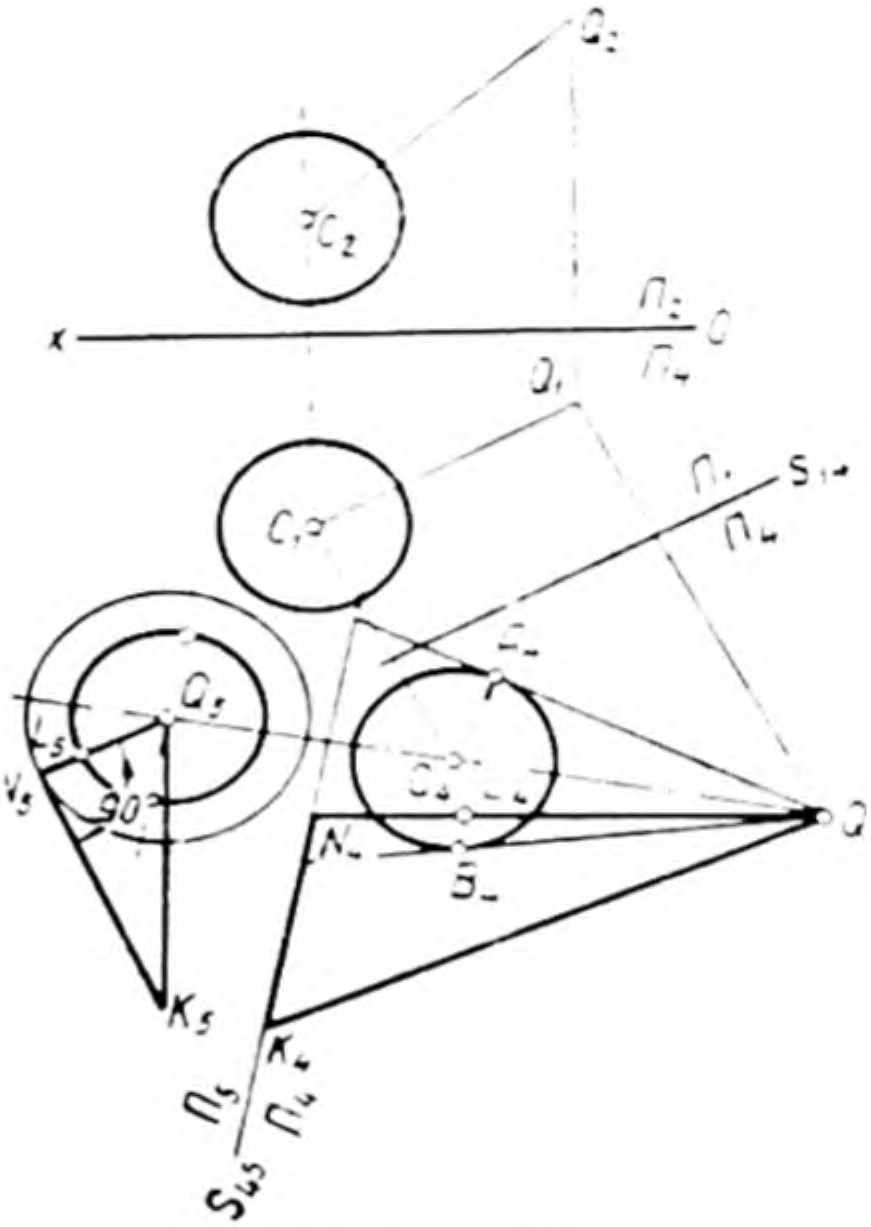


Fig. 464

9. The construction of a tangent plane to the surface of a cylinder and parallel to a given line AB is shown in Fig. 465.

Let us pass a plane γ parallel to the given line AB through the axis of the cylinder. To do this through any point C on the axis of cylinder we draw a line CM parallel to line AB. The line drawn through points M_1 and O_1 is the horizontal trace of plane γ. Since plane γ passes through the axis of the cylinder, it cuts the surface along generatrices. The planes α and α' are drawn parallel to the plane γ and their horizontal traces α_1 and α'_1 are tangents to the circle of the base of the cylinder. These are the required tangent planes. The lines of tangency are the generatrices of the cylinder passing respectively through points K and L. The trace α_1 together with the generatrix passing through point K and the trace α'_1, together with the generatrix passing through point L, determine the two planes tangent to the surface of the cylinder and parallel to the given line AB.

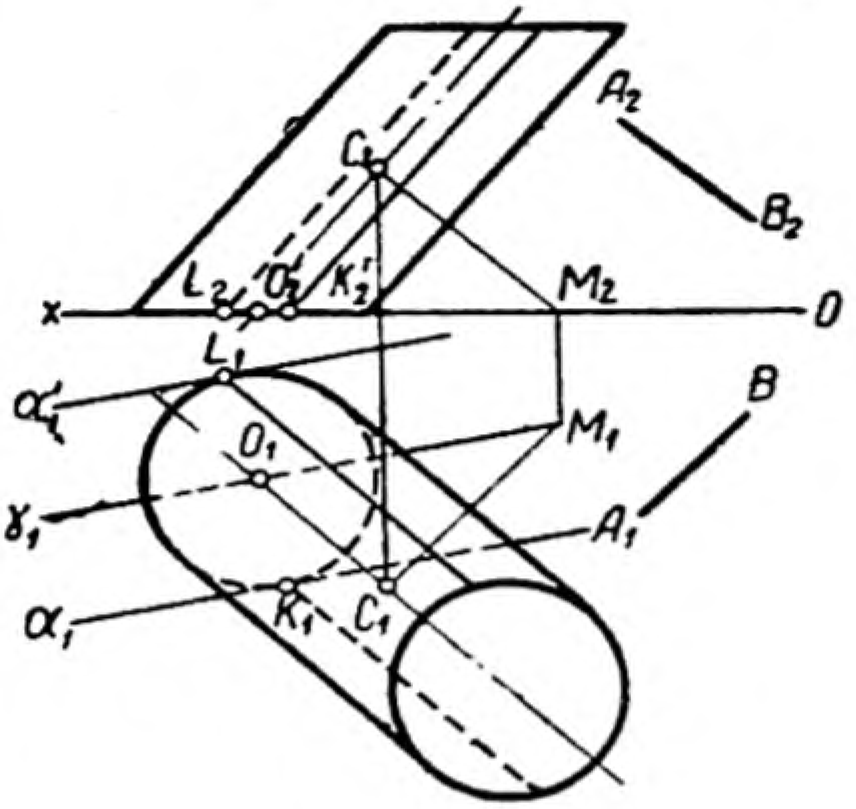

A₂
C₂
B₂
x
L₂
O₂
K₂'
M₂
0
L₁
α₁'
B
M₁
O₁
γ₁
A₁
α₁
K₁
C₁

Fig. 465

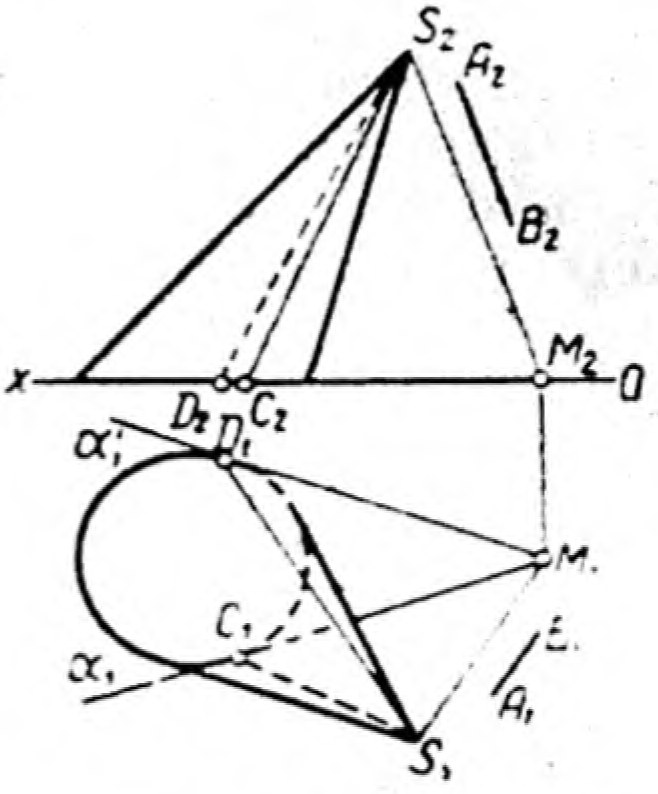

S₂
A₂
B₂
x
M₂
0
D₂
C₂
α₁'
D₁
M₁
C₁
E₁
α₁
A₁
S₁

Fig. 466

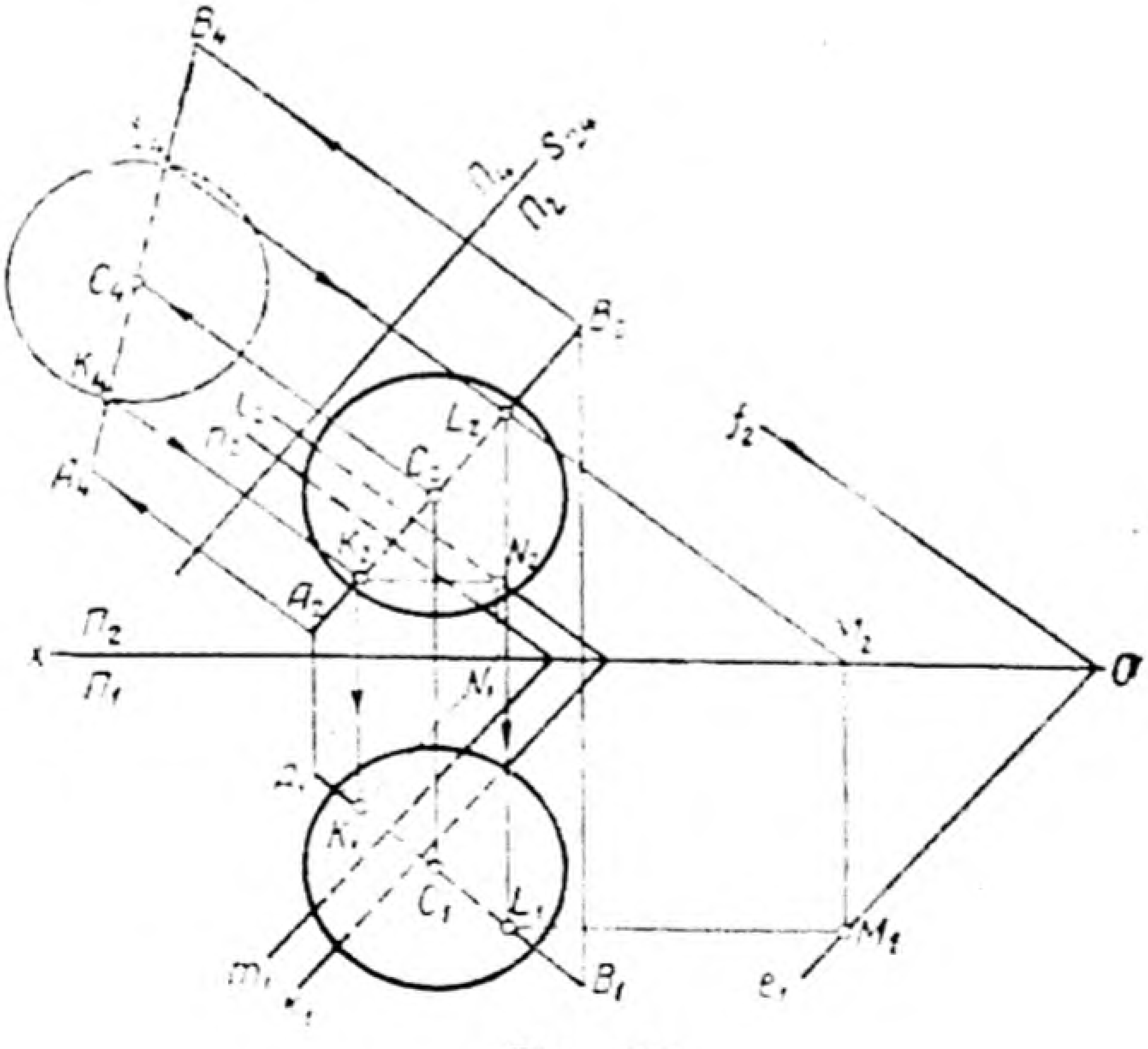

B₄
L₄
Π₄
Π₂
C₄
B₂
K₄
L₂
A₄
C₂
f₂
K₂
N₂
A₂
x
Π₂
Π₁
N₁
M₂
0
A₁
K₁
C₁
L₁
M₁
m₁
B₁
e₁

Fig. 467

10. Let it be required to draw a plane tangent to the surface of a cone and parallel to line AB (Fig. 466).

Through the vertex S of the cone we draw a line parallel to the given line AB and find the horizontal piercing point M of line SM. Then through point M_1 we draw tangents to the circle of the base of the cone. These tangents will be the horizontal traces of planes tangent to the surface of the cone. These planes are parallel to line AB and the generatrices SC and CD are their lines of tangency on the surface of the cone. This problem has two solutions.

11. The construction of a plane tangent to the surface of a sphere and parallel to a given plane γ is shown in Fig. 467. Since a plane tangent to the surface of the sphere and parallel to the given plane γ must pass through the limiting point of a radius of the sphere drawn perpendicular to plane γ, this problem has two solutions.

We draw line AB through the centre C of the sphere perpendicular to plane γ. We then determine points K and L in which this line AB pierces the surface of the sphere. When line AB is projected onto a new plane of projection Π_4, to which it is parallel, it is seen in its true size. This new projection A_4B_4 will intersect the projection of the surface of the sphere on plane Π_4 in points K_4 and L_4. These are the projections of the extreme points of diameter KL, which is perpendicular to plane γ. Having constructed the projections of points K and L on planes Π_2 and Π_1, as shown by arrows, we draw the required planes α and α' through points K and L parallel to the given plane γ. This is done with the help of the horizontal KN and frontal LM, also parallel to plane γ.

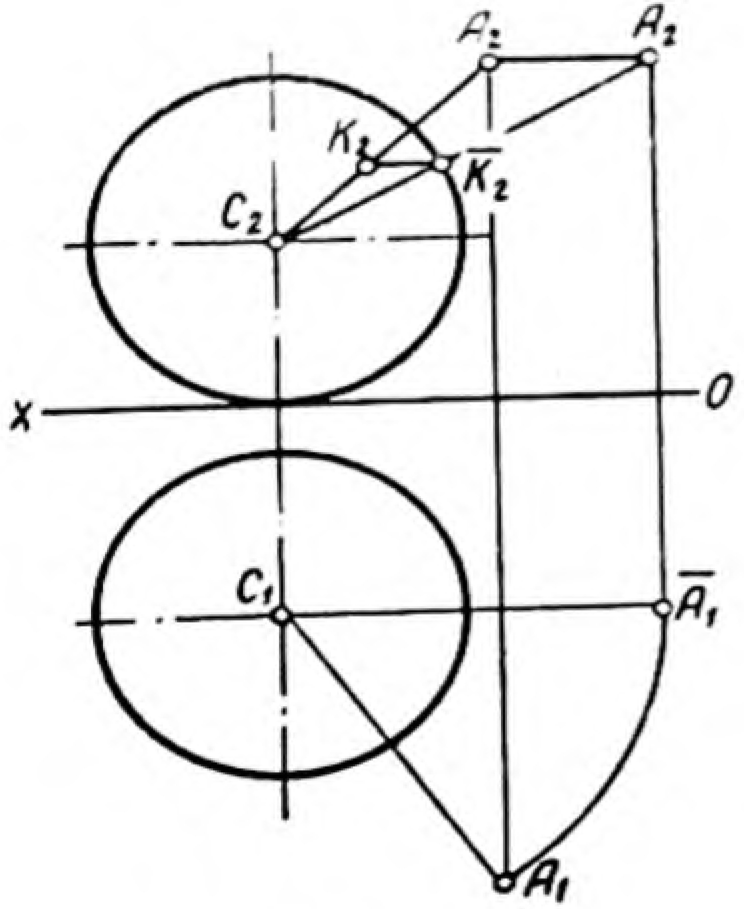


Fig. 468

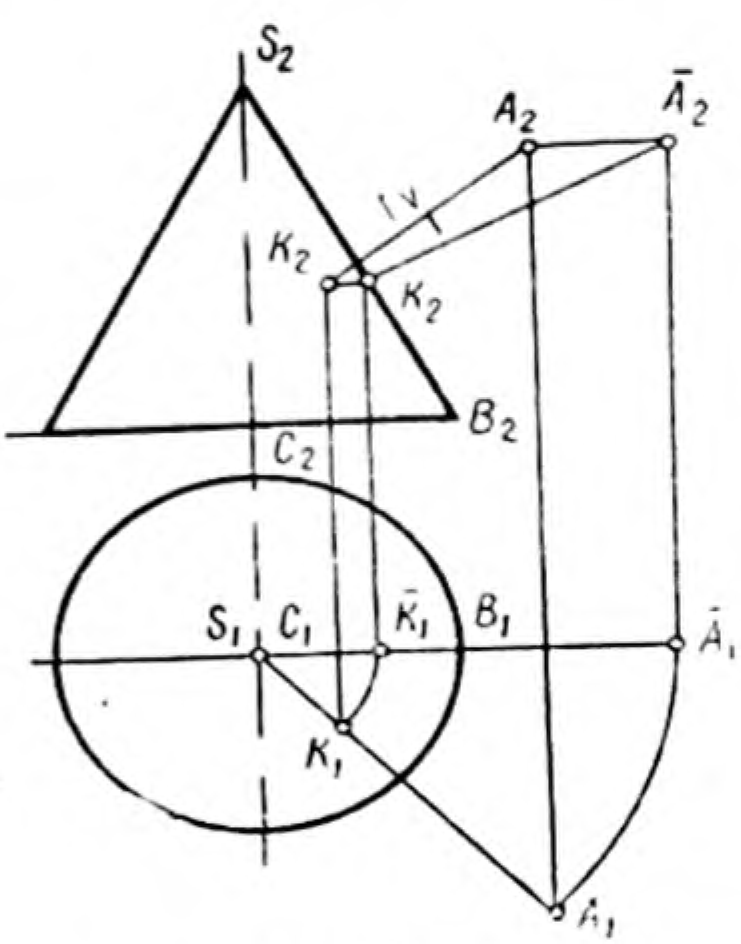


Fig. 469

12. Normals to curved surfaces. In general, when constructing a normal to a curved surface at a point on the surface, using the procedure described above, it is first necessary to draw a plane tangent to the surface through the given point. After that a perpendicular to this plane is drawn through the point.

In certain cases normals to curved surfaces are drawn without auxiliary lines, for example, the normal to the surface of a sphere in a given point passes through this point and the centre of the sphere.

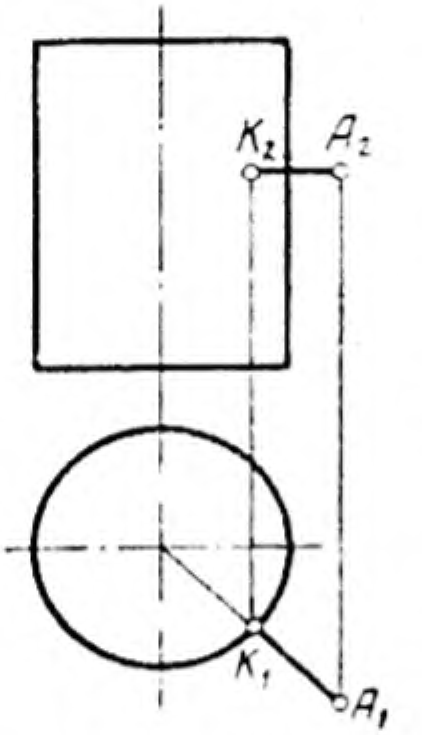


Fig. 470

The solution of the problem of determining the distance from point A to the surface of sphere is seen in Fig. 468. This distance is given by the length of the normal AK. Point K is the piercing point of line AC on the surface of the sphere. The construction is carried out using the method of revolution. $\overline{A}_2\overline{K}_2$ is the required distance.

The shortest distance from a given point A to the surface of a cone (Fig. 469) is also determined by drawing a normal to the conical surface.

Point A is rotated about the axis SC of the cone to the position $\overline{A}$. The radius of revolution CA then occupies the frontal position $C\overline{A}$. From point $\overline{A}_2$ we drop a perpendicular to the frontal projection S_2B_2 of the generatrix SB. Point $\overline{K}$, the intersection of the perpendicular and generatrix SB, is the point on the surface of the cone nearest to point $\overline{A}$, so $\overline{A}_2\overline{K}_2$ is the required distance. Lines A_1K_1 and A_2K_2 are the projections of the normal AK.

In Fig. 470 is shown the determination of the shortest distance from point A to surface of cylinder. This distance AB projects onto plane P_1 in its true length.

Problems and Exercises

1. How many planes can be passed through a given point tangent to a surface of a) a cylinder, b) a cone and c) a sphere?
2. Is it possible, in general, to pass a plane tangent to the surface of a) two cones, b) a cylinder and a cone?
3. Draw a tangent plane through a given point on the surface of an elliptical cone.
4. Draw a tangent plane to the surface of a torus passing through a given point on its surface.
5. Draw a tangent plane to an oblique helicoid passing through a given point on its surface.
6. Through point M taken outside an oblique cone draw a tangent plane to its surface.
7. Draw a plane α tangent to the surface of a right circular cone parallel to a line AB.

APPENDIX

The Use of Models in Descriptive Geometry

The special difficulty, which the beginner as a rule experiences, is to be able to conceive clearly the connection between the figures drawn on paper and the actual figures in space which they represent.

In other words, when reading the text where the constructions on a plane drawing are explained, it is necessary to imagine as taking place in space the various operations used to solve the problem on paper with the help of points, lines and plane figures.

This difficulty may be largely avoided by making use of models at the beginning of the course.

The drawings enabling models to be made for 24 constructions are given in the book. The problems are arranged so that by using the multiview drawings and the axonometric representation the student, without help of the teacher, can study the development of surfaces and the way to lay out patterns for them.

It has been found that by using models very clear and precise concepts are formed from the start and a correct method of working is acquired.

The drawings given include models of planes of projection, projecting planes and geometrical figures with lines and projections plotted on them.

The dotted lines indicate the edges to be cut and the allowances to be left along the various sides of the pattern for overlap and where to glue the projecting planes to the planes of projection. Each pair of planes of projection should be fixed at right angles with the help of narrow strips of paper as shown in the drawing for making Fig. 10. Sometimes, for example in the case of the model suggested for Fig. 80, the planes of projection may be fixed at right angles without using strips.

To represent horizontal-projecting lines match sticks may be used. If a match is not long enough two may be joined. A convenient way of doing this is to make a small cut in the end of one match while the other is sharpened into the form of a spatula. This is then dipped in glue and inserted into the cleft of the first match.

When matches are supposed to stand vertically the end to be glued to the horizontal plane of projection should be cut clean and square. The matches should be held in place until the glue hardens sufficiently to hold them in a vertical position.

Bases are not always necessary. Sometimes, as for example in the model for Figs. 57 and 59, the projection line may be fixed with the help of holes made in the planes of projection which are slightly lengthened.

The student may himself make other models for those constructions which seem most difficult. Nevertheless it should not be forgotten that excessive use of models may hinder him from visualizing projections on the drawings.

The student should endeavour to understand the drawings and be able to solve any problems involving projections, no matter how complex, by applying the methods of descriptive geometry without having often to resort to models.

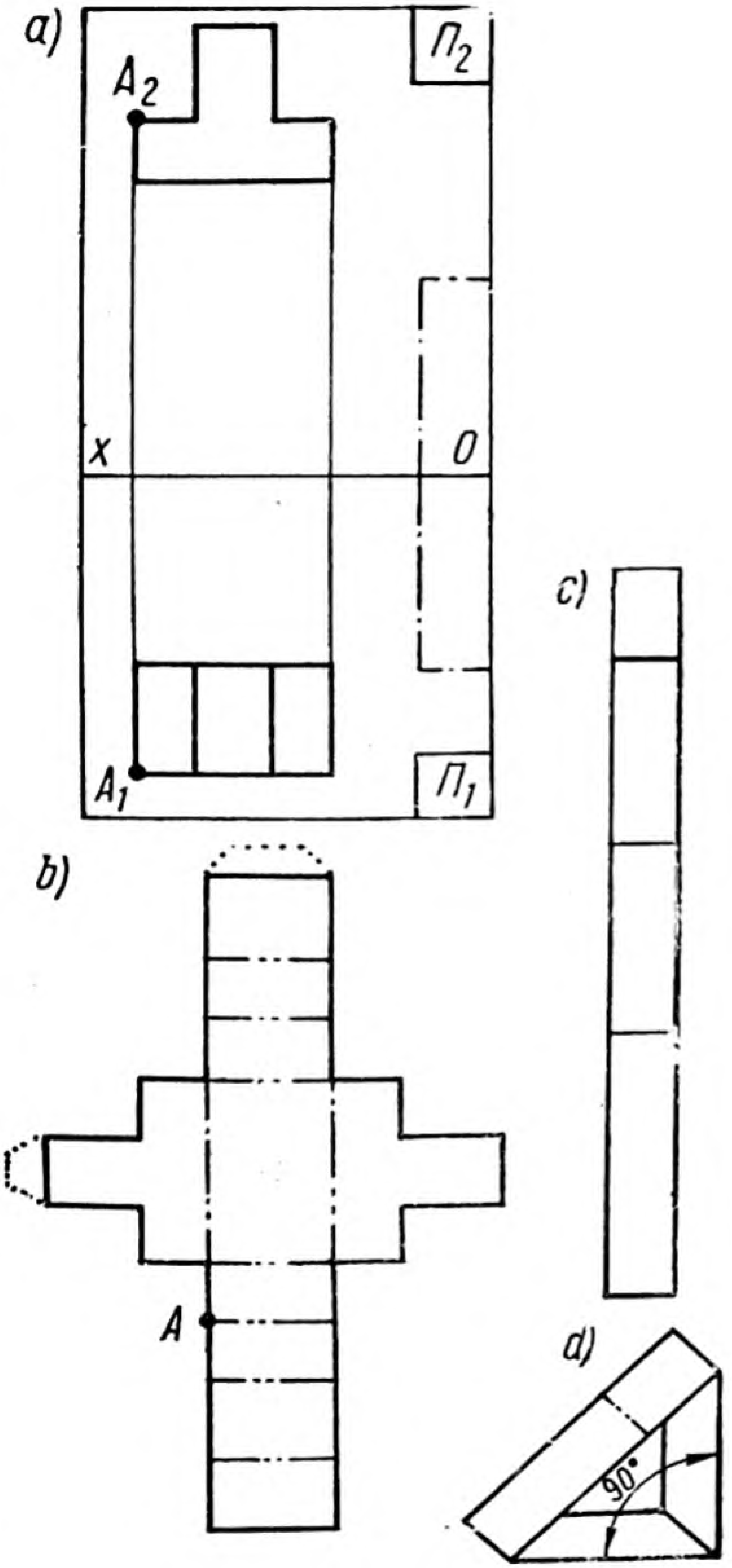

a)
A_2
$П_2$
x
0
A_1
$П_1$
b)
A
c)
d)
90°

Fig. 471 to Fig. 10

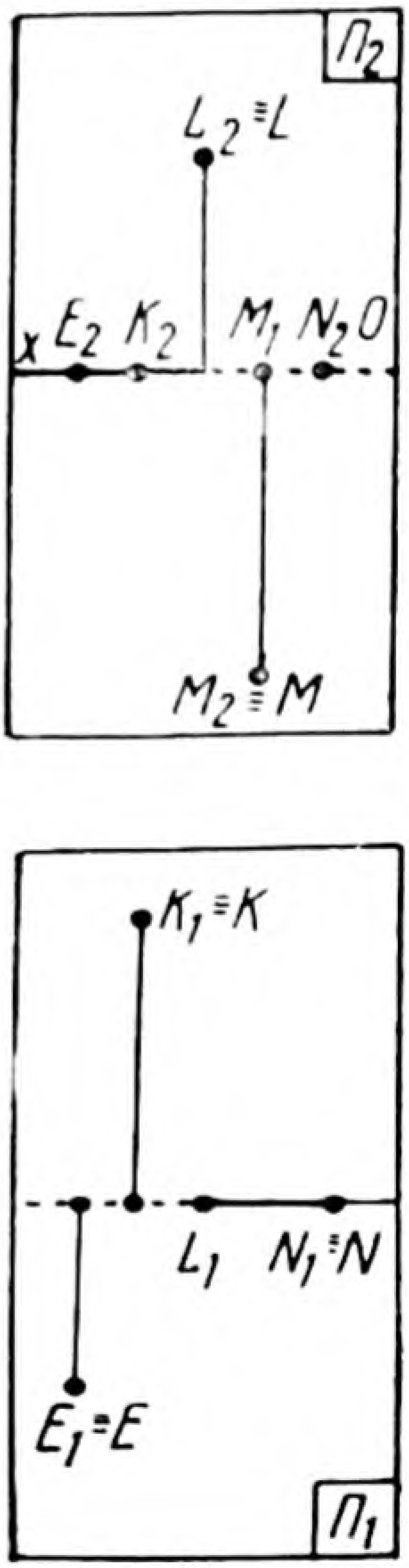
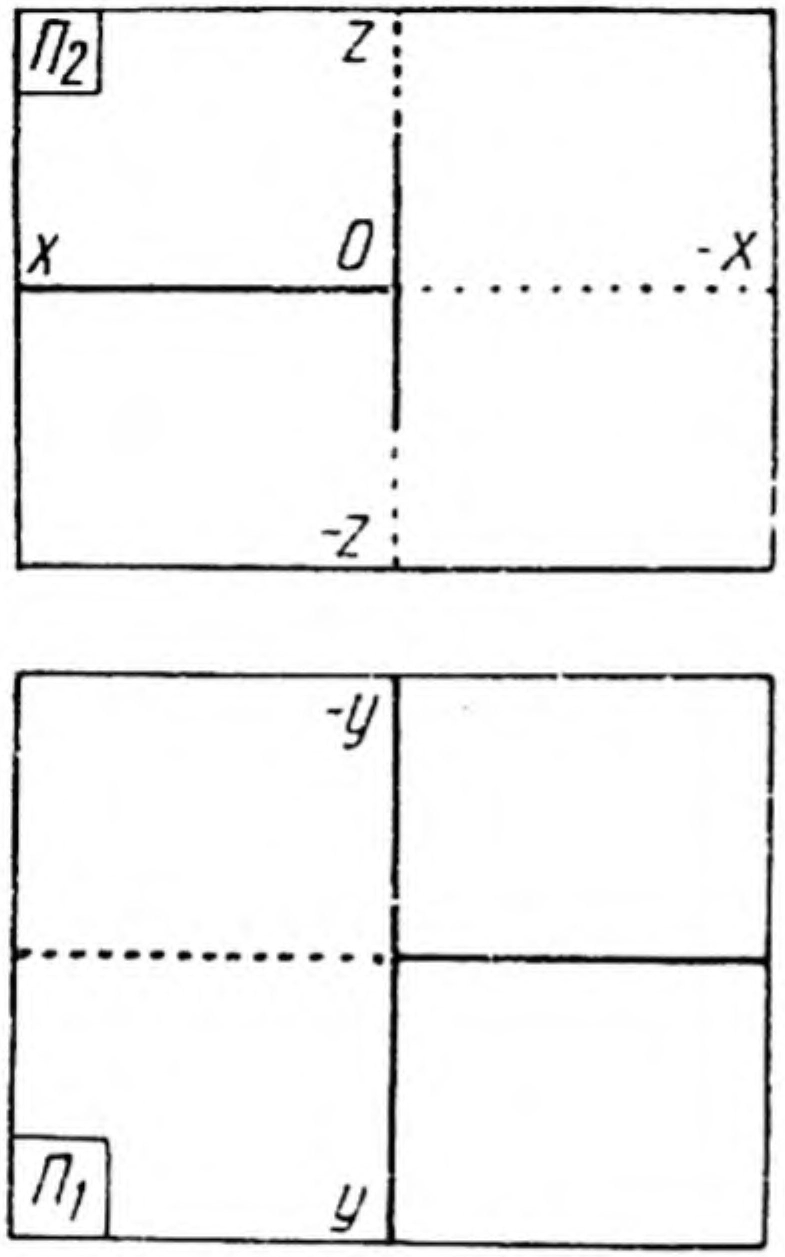

$П_2$
$L_2 \equiv L$
x
E_2
K_2
M_1
N_2
0
$M_2 \equiv M$
$K_1 \equiv K$
L_1
$N_1 \equiv N$
$E_1 \equiv E$
$П_1$

Fig. 472 to Fig. 20


П2
Z
X
0
-X
-Z
-y
П1
y

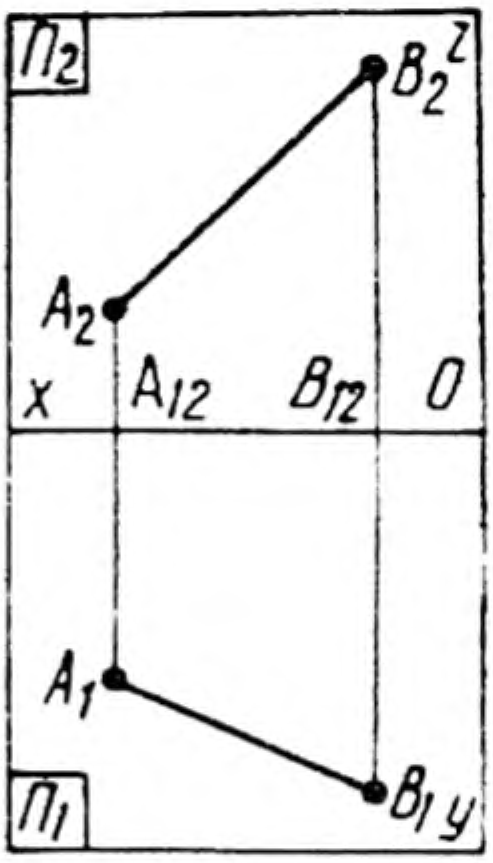

П2
B2
Z
A2
X
A12
B12
0
A1
B1
y
П1

Fig. 474 to Fig. 36

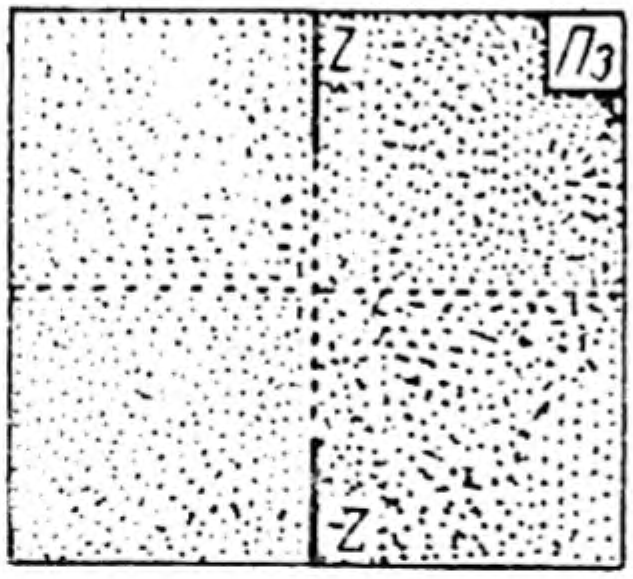


Fig. 473 to Fig. 21

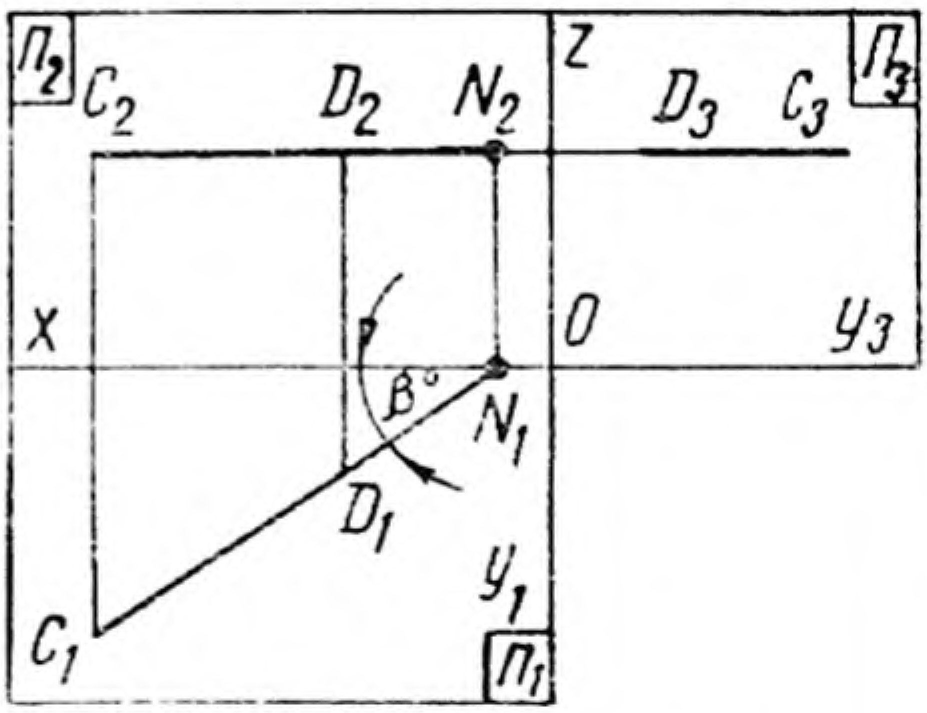

П2
C2
D2
N2
Z
D3
C3
П3
X
0
y3
β°
N1
D1
y1
C1
П1

Fig. 475 to Fig. 39

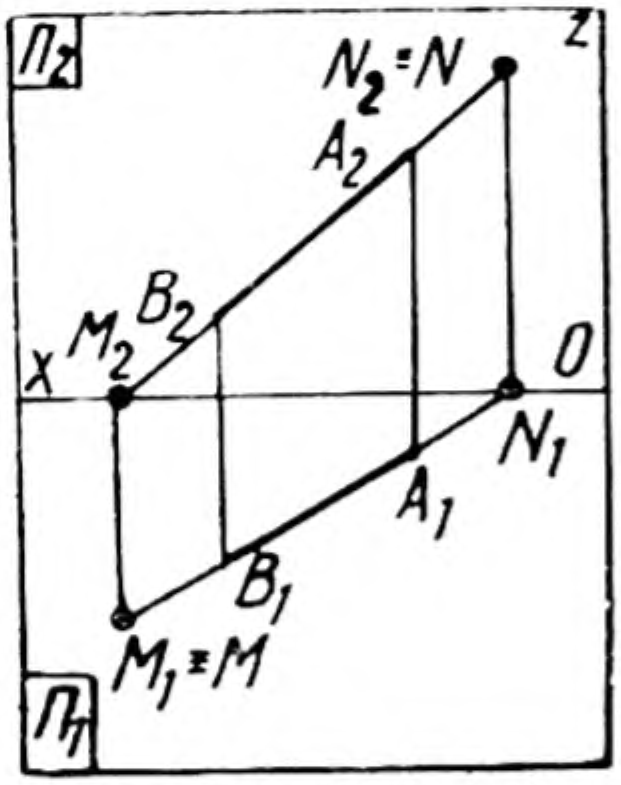

П2
z
N2≡N
A2
B2
x
M2
O
N1
A1
B1
M1≡M
П1

Fig. 476 to Fig. 57

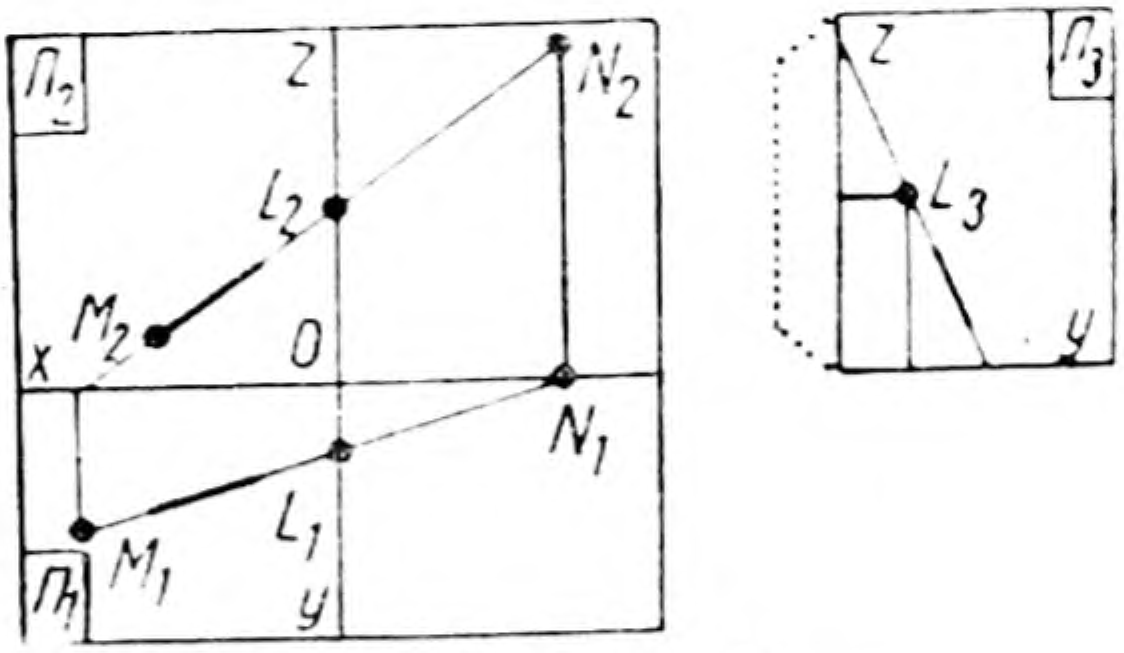

П2
z
N2
l2
M2
x
O
N1
L1
M1
П1
y
z
П3
L3
y

Fig. 477 to Fig. 59

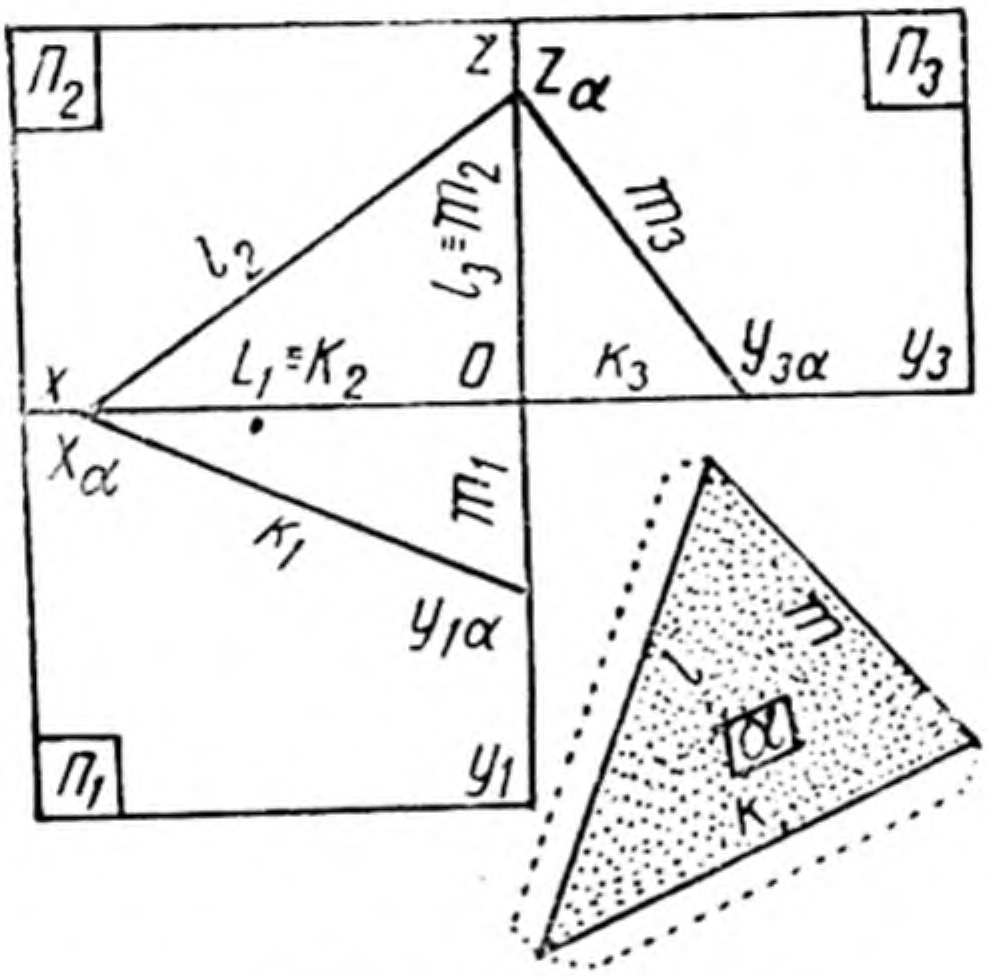

П2
z
zα
П3
l2
l3≡m2
m3
x
L1≡K2
O
K3
y3α
y3
Xα
m1
K1
y1α
П1
y1
m
l
α
K

Fig. 478, to Fig. 80

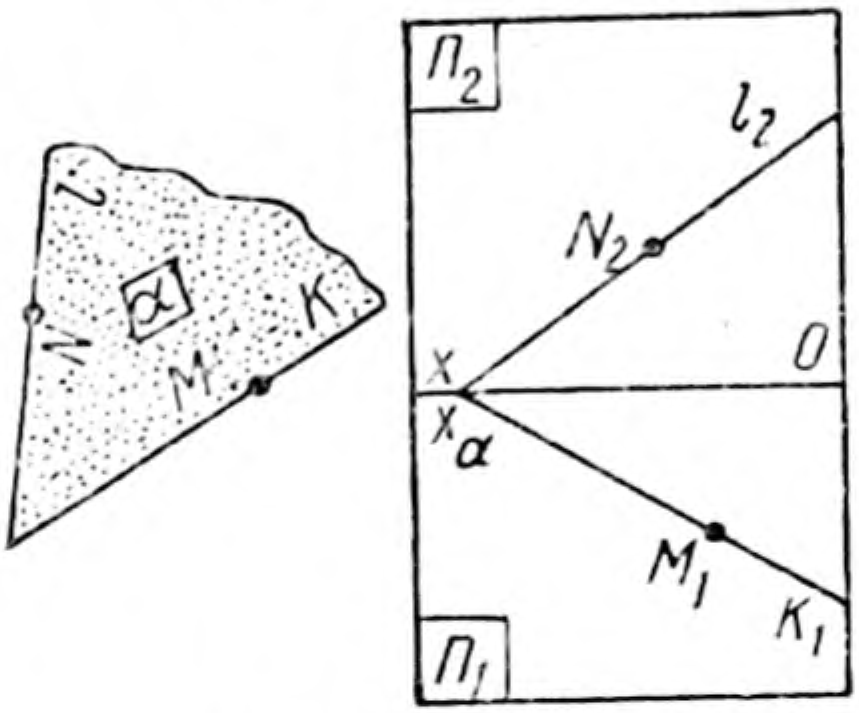

П2
l2
N2
x
O
Xα
M1
K1
П1
l
α
N
M
K

Fig. 479 to Fig. 86

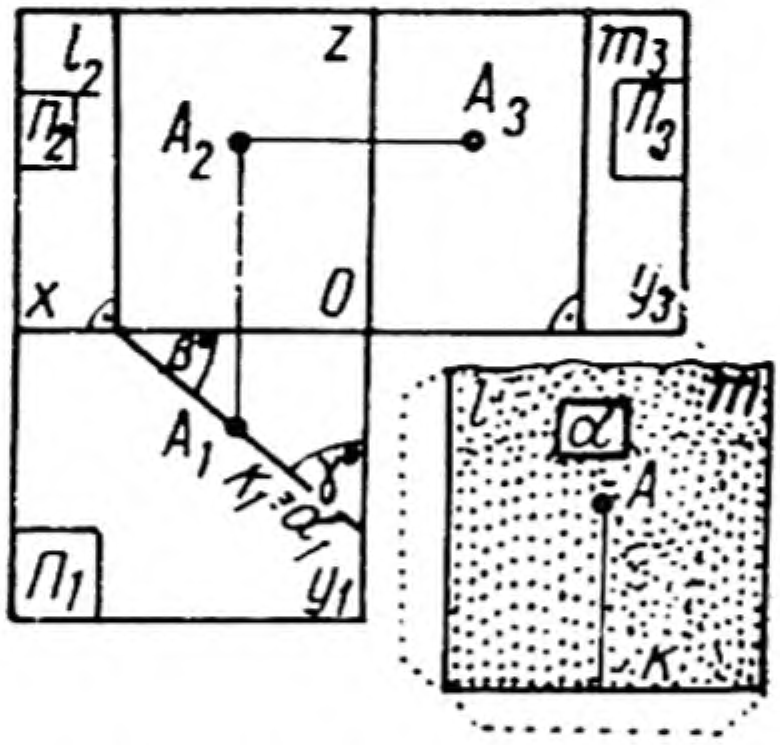

l2
z
m3
П2
A2
A3
П3
x
0
y3
β°
A1
K1≡a1
γ°
П1
y1
l
m
α
A
K

Fig. 480 to Fig. 90

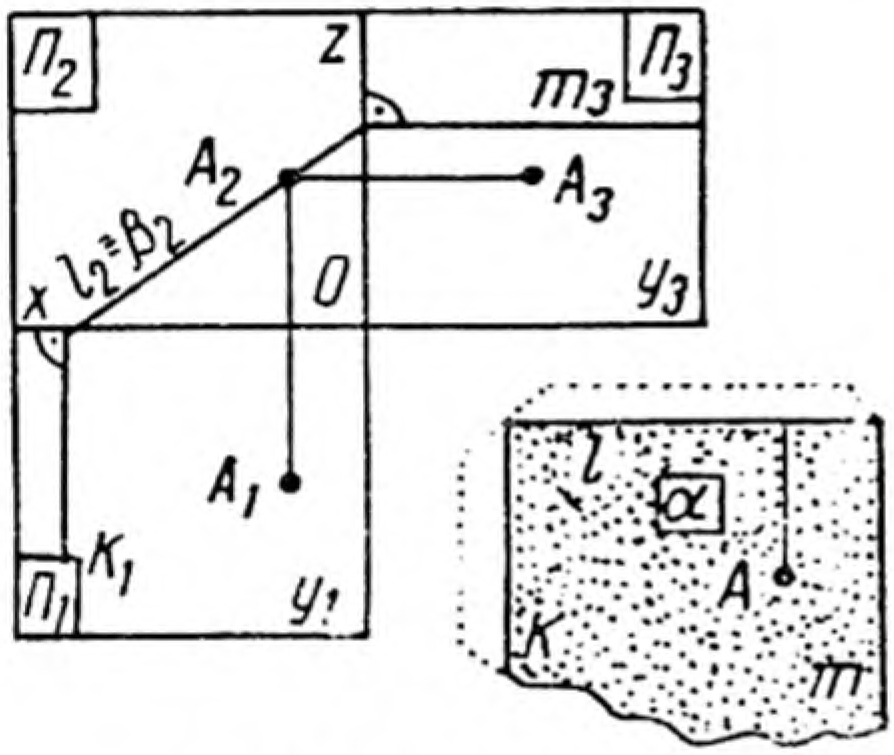

П2
z
П3
m3
A2
A3
x
l2≡β2
0
y3
A1
K1
П1
y1
l
α
A
K
m

Fig. 481 to Fig. 92

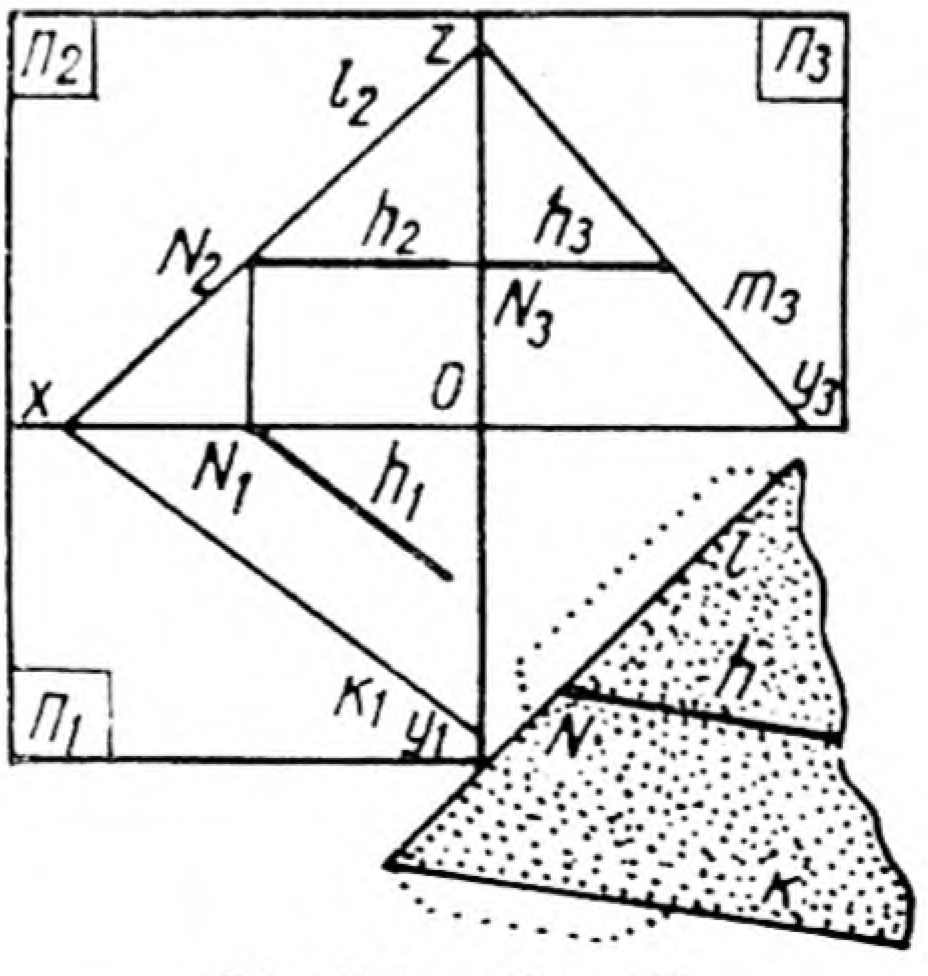

П2
z
П3
l2
h2
h3
N2
N3
m3
x
0
y3
N1
h1
K1
y1
П1
l
h
N
K

Fig. 482 to Fig. 104

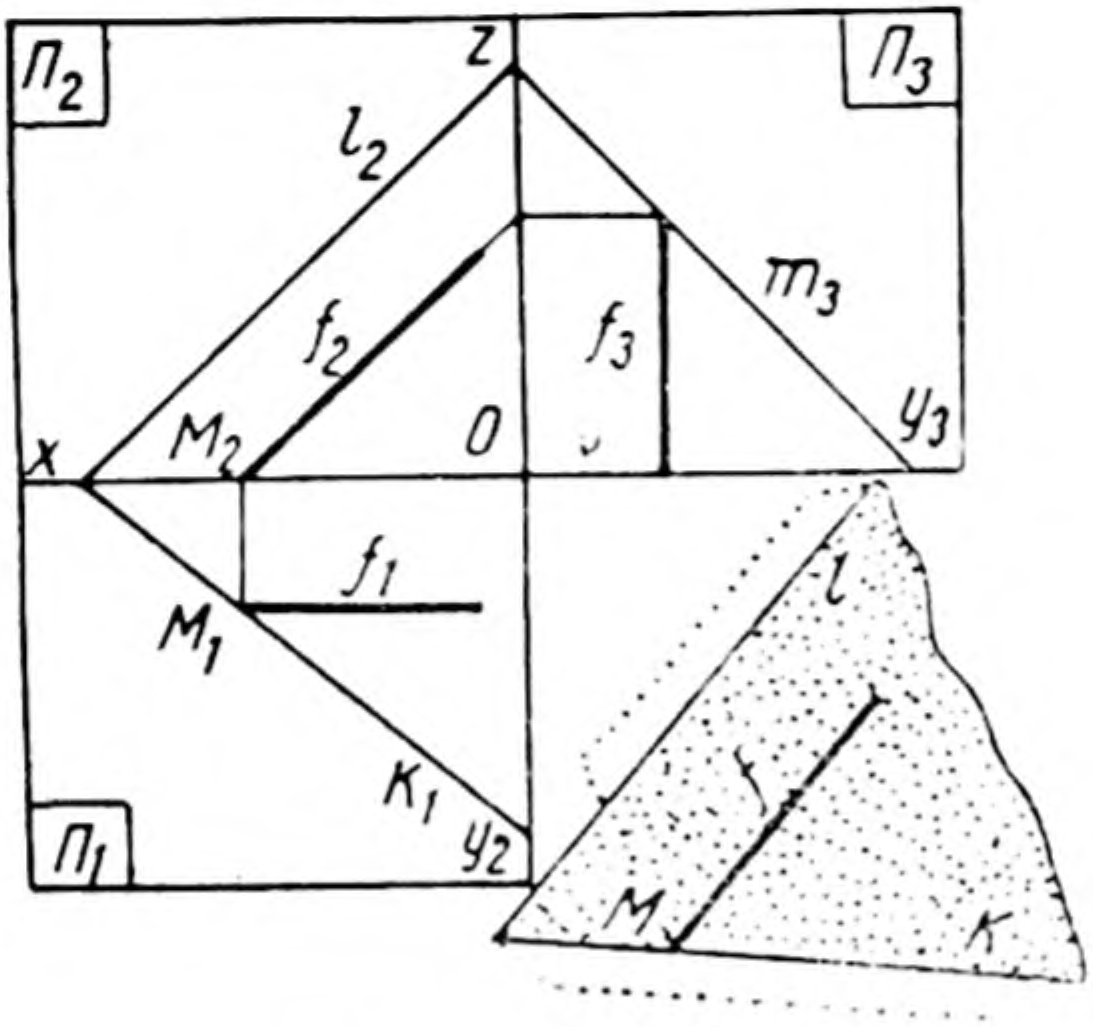

П2
П3
П1
z
x
O
y3
y2
l2
m3
f2
f3
f1
M2
M1
K1
l
f
M
K

Fig. 483 to Fig. 105

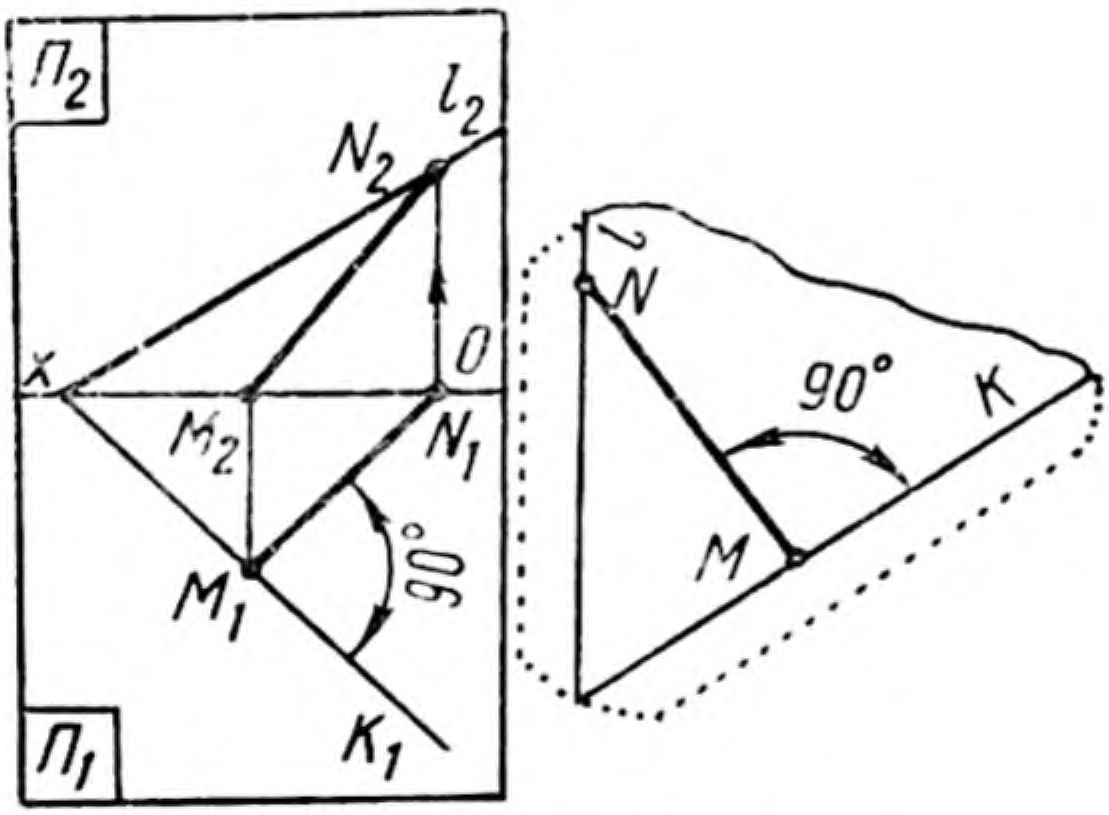

П2
П1
l2
N2
x
O
M2
N1
M1
K1
90°
l
N
90°
K
M

Fig. 484 to Fig. 116

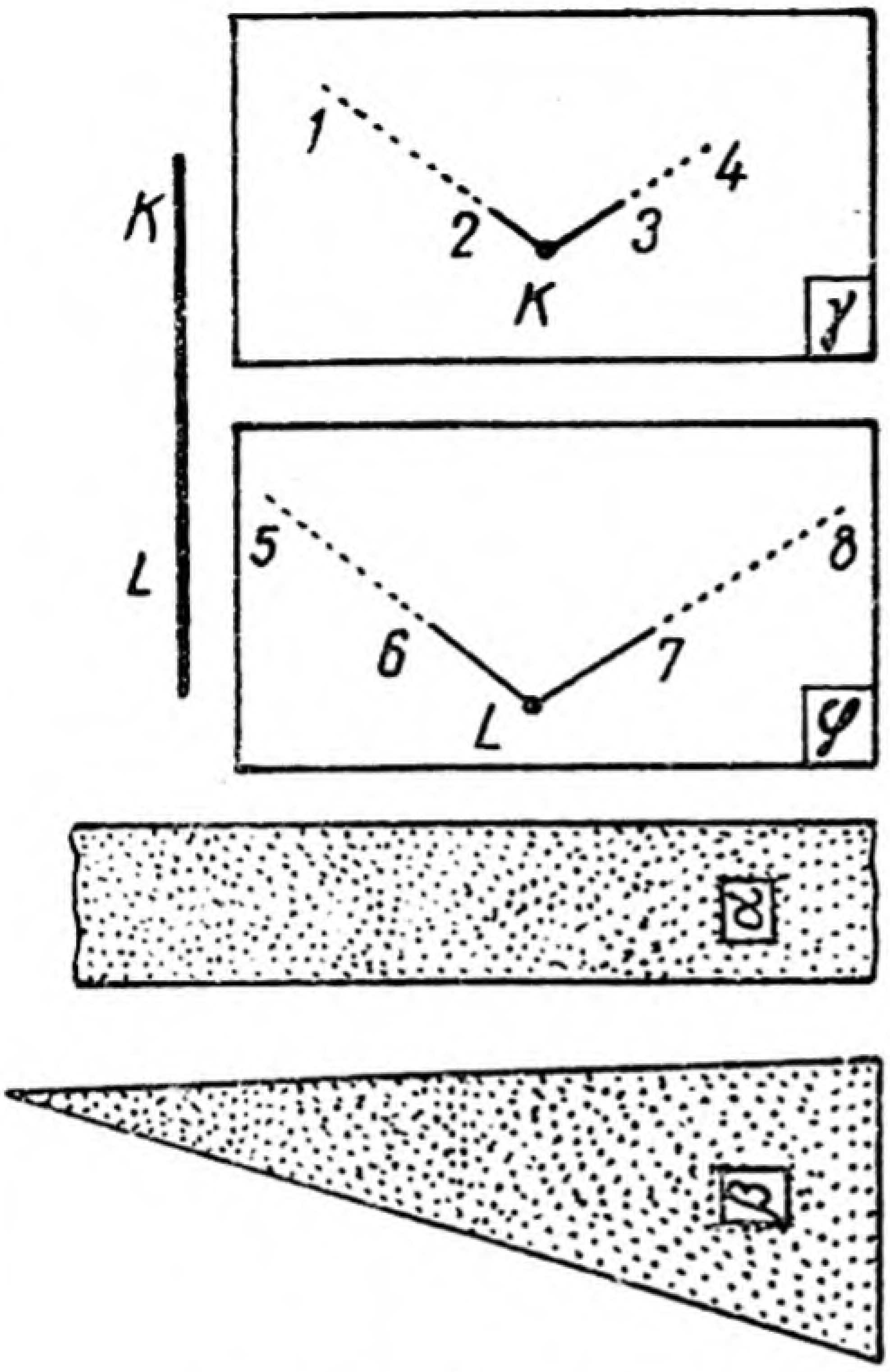

K
L
1
4
2
3
K
γ
5
8
6
7
L
φ
α
β

Fig. 485 to Fig. 149

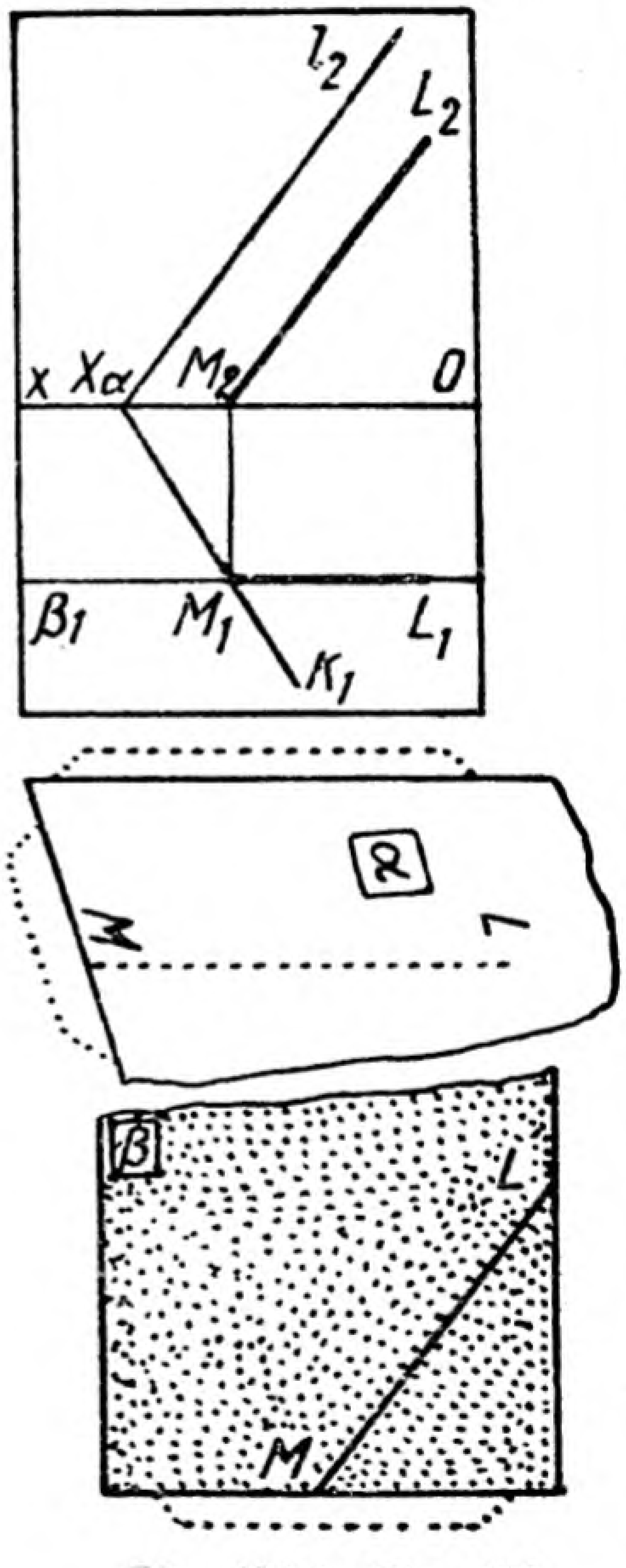

l_2
L_2
x
X_α
M_2
0
β_1
M_1
L_1
K_1
α
M
L
β
L
M

Fig. 486 to Fig. 154

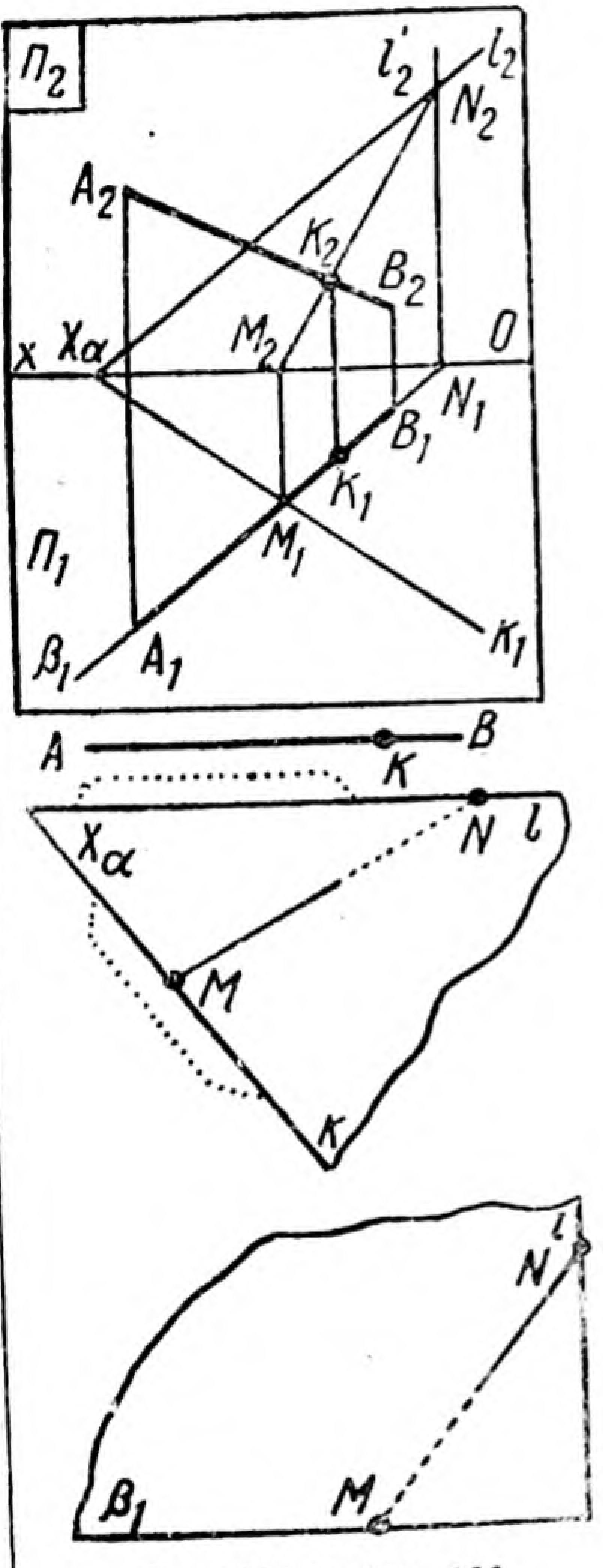

Π2
l'2
l2
N2
A2
K2
B2
x
Xα
M2
0
N1
B1
K1
Π1
M1
β1
A1
K1
A
K
B
Xα
N
l
M
K
l
N
β1
M

Fig. 487 to Fig. 166

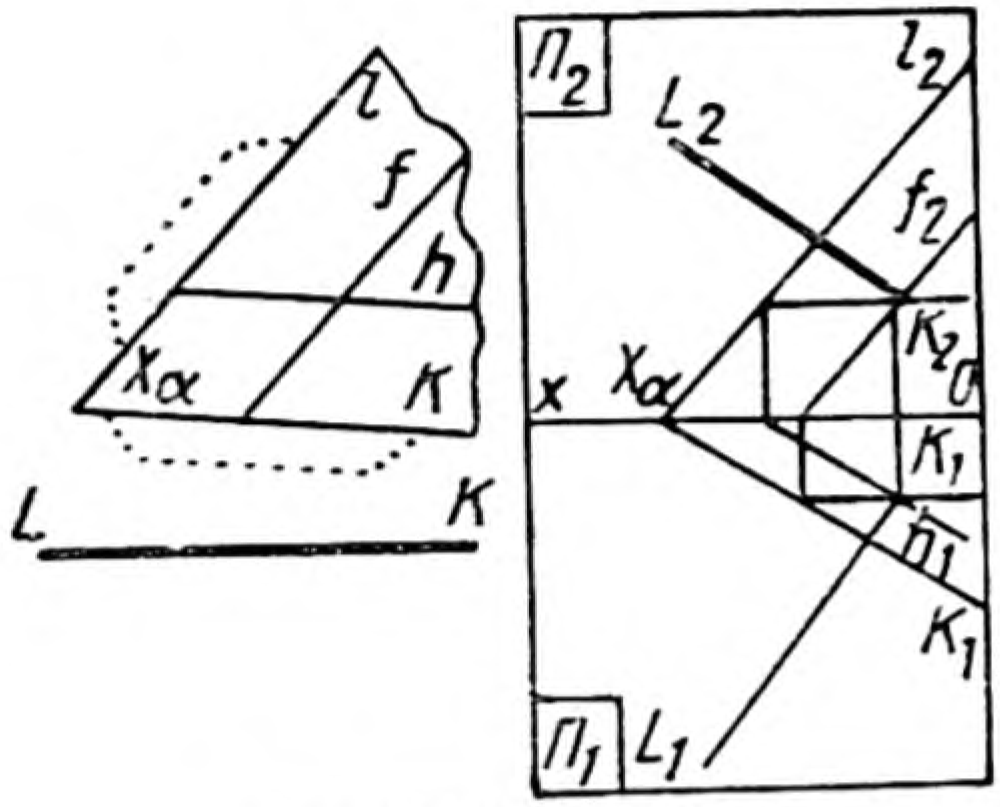

l
f
h
Xα
K
L
K
Π2
l2
L2
f2
x
Xα
K2
0
K1
h1
K1
Π1
L1

Fig. 488 to Fig. 177

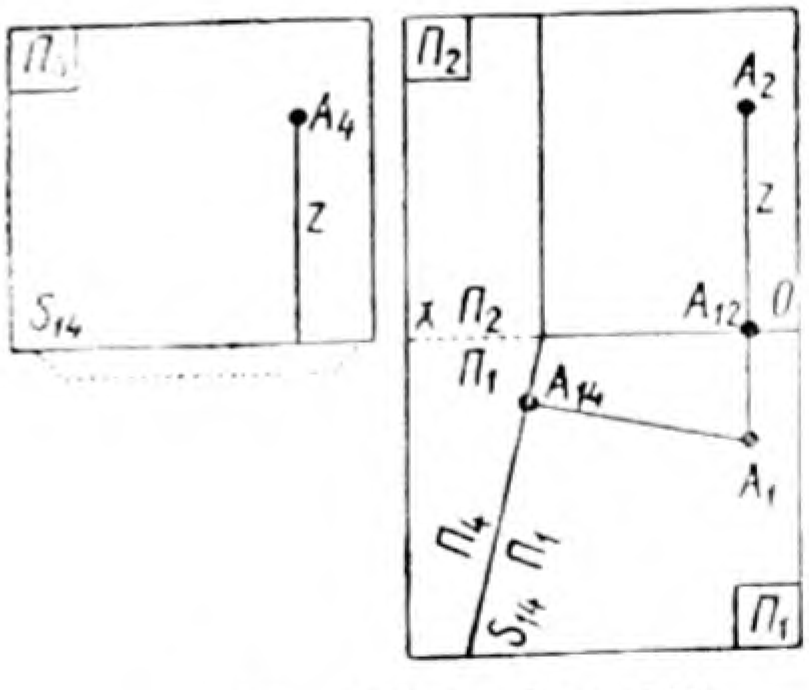
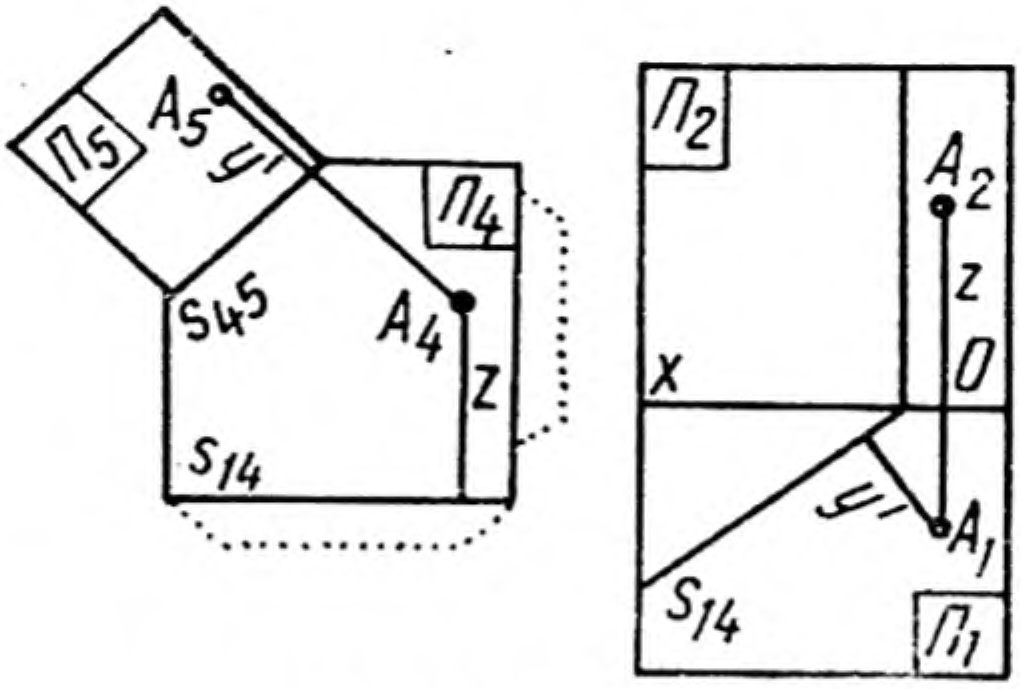
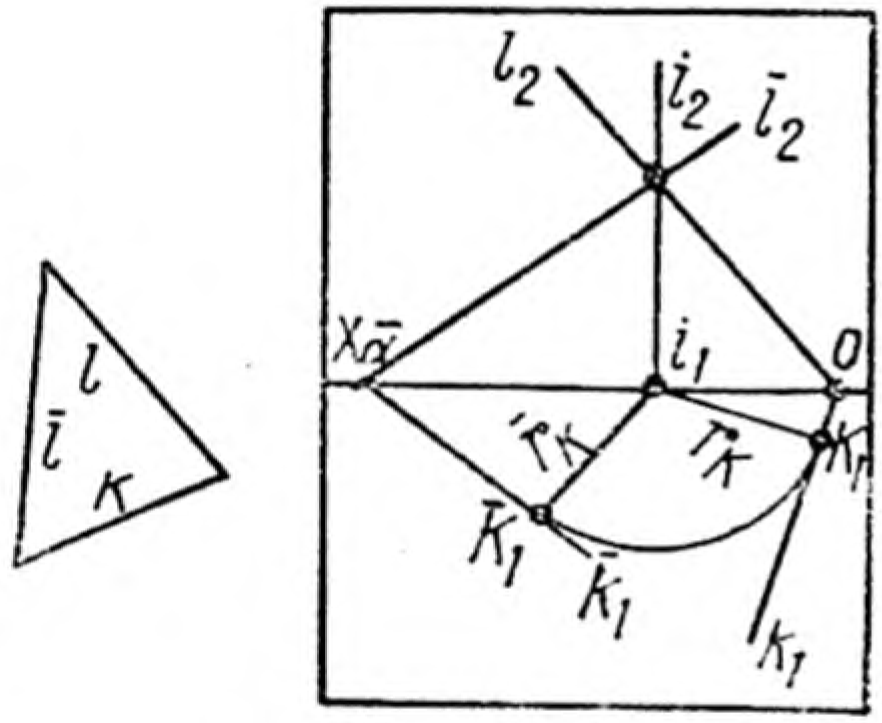

Π4
A4
z
S14
Π2
A2
z
x
Π2
A12
0
Π1
A14
A1
Π4
Π1
S14
Π1

Fig. 489 to Fig. 198



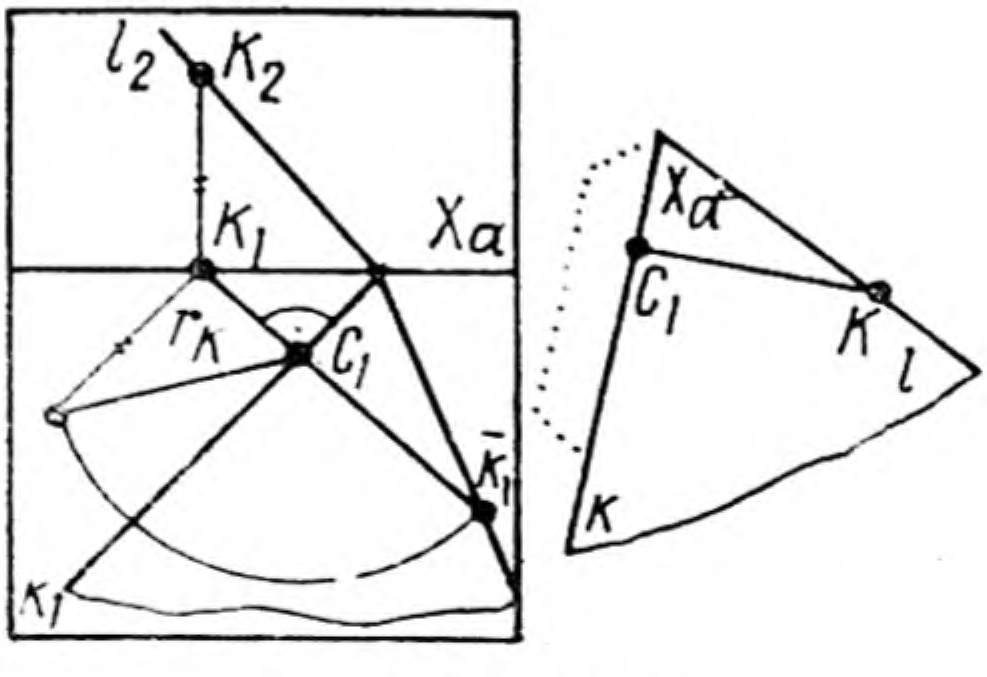
Π5
A5
y'
Π4
S45
A4
z
S14
Π2
A2
z
0
x
y'
A1
S14
Π1

Fig. 490 to Fig. 208



l
l̄
K
l2
i2
l̄2
Xᾱ
i1
0
rK
rK
K1
K̄1
k̄1
k1

Fig. 491 to Fig. 247



l2
K2
K1
Xα
rK
C1
k̄1
k1
Xα
C1
K
l
k

Fig. 492 to Fig. 269

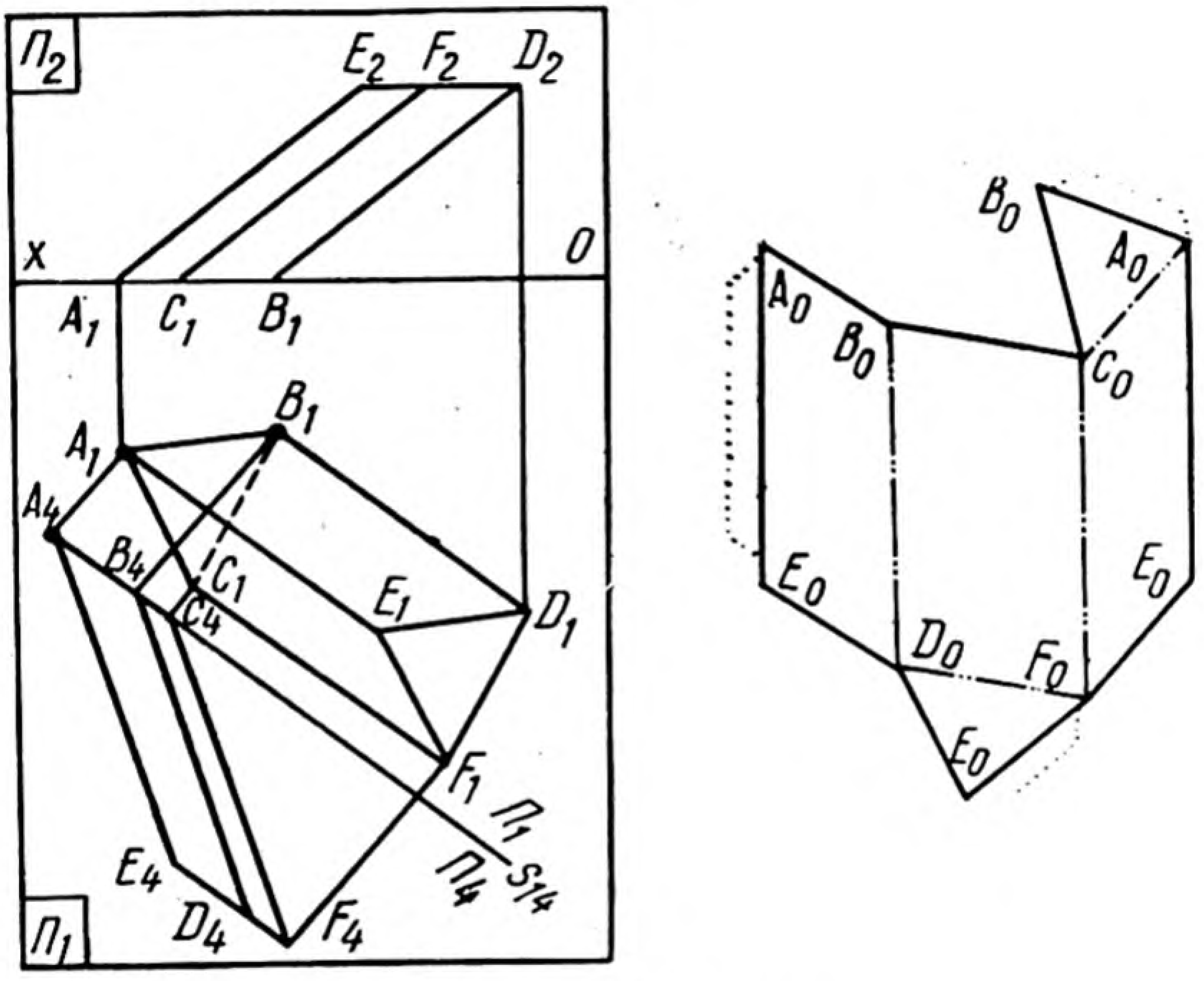

Π_2
E_2
F_2
D_2
x
0
A_1
C_1
B_1
B_1
A_1
A_4
B_4
C_1
C_4
E_1
D_1
F_1
Π_1
Π_4
S_{14}
E_4
D_4
F_4
Π_1
B_0
A_0
A_0
B_0
C_0
E_0
E_0
D_0
F_0
E_0

Fig. 493 to Fig. 292

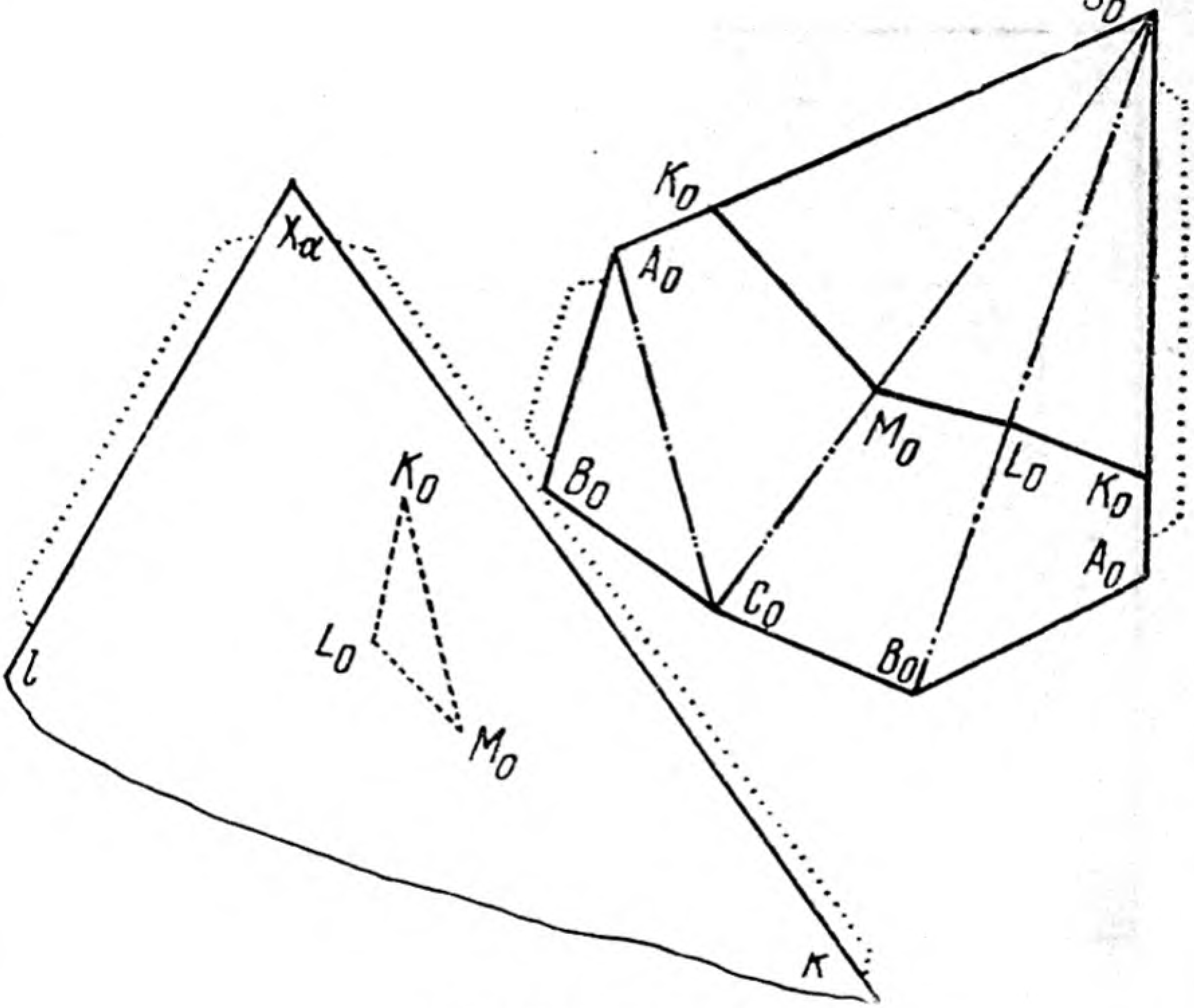

S0
K0
A0
M0
L0
K0
B0
A0
C0
B0
Xα
K0
L0
M0
l
K

Fig. 494 to Fig. 311

GIRLS COL
12154

А. Т. ЧАЛЫЙ

КУРС НАЧЕРТАТЕЛЬНОЙ ГЕОМЕТРИИ

(на английском языке)